Lezione del 23 Novembre

$$f'(x_0) \doteq \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

$$f(x) = \operatorname{sen} x \Rightarrow f'(x) = \cos x$$

$\uparrow$ x_0

$$f'(x_0) = \cos x_0$$

$$f(x) = \cos x \Rightarrow f'(x) = -\operatorname{sen} x$$

$$f(x) = x^{\alpha} \Rightarrow f'(x) = \alpha x^{\alpha - 1}$$

$\alpha \in \mathbb{R}$

$$f(x) = e^{x} \Rightarrow f'(x) = e^{x}$$

$$f(x) = \log x \Rightarrow f'(x) = \frac{1}{x}$$

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Teo. Se f è derivabile in $x_0 \Rightarrow f$ è continua in x_0

Dim. $f(x_0+h) - f(x_0)$

continuità $\lim_{h \to 0} f(x_0+h) = f(x_0)$

voglio dim. che $f(x_0+h) - f(x_0) \xrightarrow[h \to 0]{} 0$

$$f(x_0+h) - f(x_0) = \frac{f(x_0+h) - f(x_0)}{h} \cdot h$$

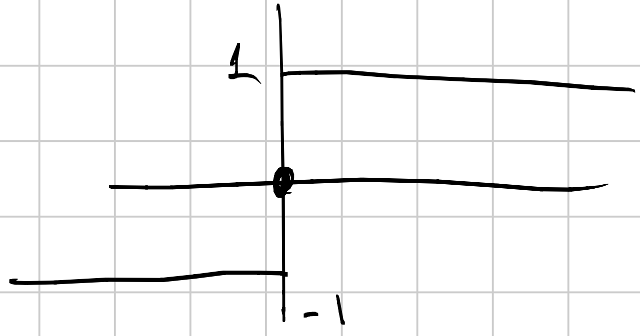
$h \to 0$

$f'(x_0) \in \mathbb{R}$

$\to 0$

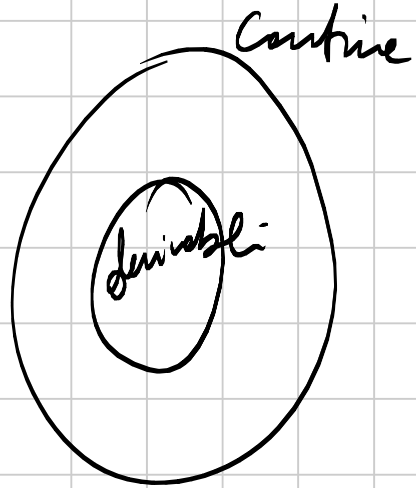
<u>oss</u> se f <u>non</u> è continua in x_0 $\Rightarrow$ f <u>non</u> è derivabile in x_0

<u>es.</u> $f(x) = \operatorname{sgn} x$



non è continua in $x = 0$

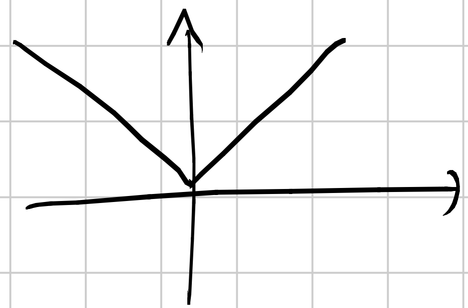
$\Rightarrow$ non è derivabile



oss. non vale il viceversa

f continua in x_0 $\not\Rightarrow$ f è derivabile in x_0

es. $f(x) = |x|$



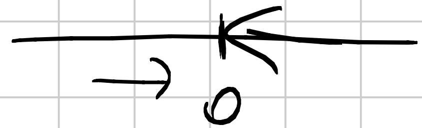
f continua $\forall x \in \mathbb{R}$

in particolare è continua in $x = 0$

ma $f(x) = |x|$ non è derivabile in $x = 0$

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{|h|}{h}$$

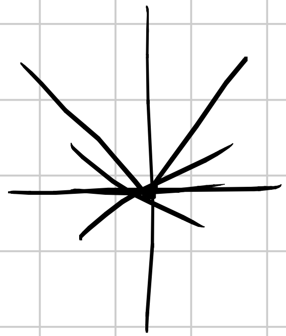
$$\lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{\cancel{h}}{\cancel{h}} = 1$$



$(h>0)$

$$\lim_{h\to 0^-} \frac{-h}{h} = -1$$

$\neq$ $\quad \nexists \lim_{h\to 0}$



$f(x)=|x|$ <u>non</u> è derivabile in $x=0$

f ha un p.to <u>angoloso</u> in $x=0$

In generale f ha un p.to angoloso in $x=x_0$ se f continua in x_0 e

$$f'_+(x_0) := \lim_{h\to 0^+} \frac{f(x_0+h)-f(x_0)}{h} = l_1 \neq \underbrace{\lim_{h\to 0^-} \frac{f(x_0+h)-f(x_0)}{h} = l_2}$$

derivata dx in x_0

$=: f'_-(x_0)$

$l_1, l_2 \in \mathbb{R}$

derivata ~~sx~~ in x_0

Algebra delle derivate

<u>Teo.</u> f, g derivabili in (a,b) allora $f \pm g$, $f \cdot g$, $\frac{f}{g}$ $(g \neq 0)$ sono derivabili in (a,b) e

$$(f \pm g)' = f' \pm g'$$

$$(f \cdot g)' = f' \cdot g + f g'$$

$$\left(\frac{f}{g}\right)' = \frac{f' g - f g'}{g^2}$$

Dim. solo il prodotto.

$$(f \cdot g)' = \lim_{h} \frac{f(x+h) \cdot g(x+h) - f(x)g(x)}{h} =$$

$$= \lim_{h} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} =$$

$$= \lim_{h \to 0} \underbrace{f(x+h)}_{\substack{\downarrow \\ f(x) \\ (f \text{ è continua})}} \underbrace{\left(\frac{g(x+h) - g(x)}{h} \right)}_{\substack{\downarrow h \to 0 \\ g'(x)}} + \underbrace{\frac{g(x)\left(f(x+h) - f(x)\right)}{h}}_{\substack{\downarrow \\ g(x) f'(x)}}$$

$$= f(x)\,g'(x) + g(x)\,f'(x)$$

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es $\quad \left(k\,f(x)\right)' = \cancel{0\,f(x)} + k\,f'(x) \qquad k \in \mathbb{R}$

$$\left(3x^2\right)' = 3 \cdot 2x$$

$$-\operatorname{sen} x = -\cos x$$

$$\left(\operatorname{sen} x + e^x\right)' = \cos x + e^x$$

$$\left(3x^3 + e^x \cdot \cos x\right)' = 3 \cdot 3x^2 +$$

$$+ e^x \cos x + e^x(-\operatorname{sen} x)$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} \qquad f = 1$$

$$\left(\frac{1}{g}\right)' = -\frac{g'}{g^2}$$

$$\left(\frac{1}{\operatorname{sen} x}\right)' = -\frac{\cos x}{\operatorname{sen}^2 x}$$

$$\frac{f'g - fg'}{g^2}$$

<u>es</u> $f(x) = tg\,x = \dfrac{\operatorname{sen} x}{\cos x} = \dfrac{f}{g}$

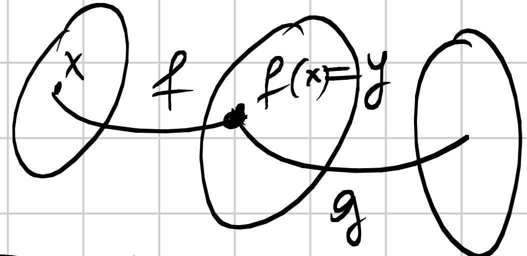
$$f'(x) = \frac{\cos x \cdot \cos x - \operatorname{sen} x\,(-\operatorname{sen} x)}{\cos^2 x} =$$

$$= \frac{\cos^2 x + \operatorname{sen}^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$= 1 + tg^2 x$$

$$(tg\,x)' = \begin{cases} \dfrac{1}{\cos^2 x} \\ 1 + tg^2 x \end{cases}$$

Derivata di una funzione composta (regola della catena)



Teo. $g \circ f$. f derivabile in x e g sia derivabile in $y = f(x)$ allora $g \circ f$ è derivabile in x e

$$(g \circ f)'(x) = g'(f(x)) \cdot f'(x)$$

es. $\text{sen}\,(x^2)$

$f(x) = x^2$

$g(y) = \text{sen}\, y$

$$\left(\operatorname{sen}(x^2)\right)' = \underbrace{\cos(x^2)}_{g'(f(x))} \cdot \underbrace{2x}_{f'(x)}$$

es. e^{-2x+1}

$$f(x) = -2x+1$$

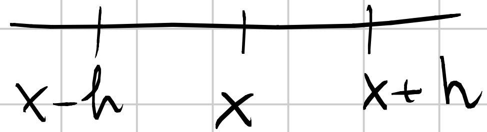
$$g(y) = e^y$$

$$\left(e^{-2x+1}\right)' = \underbrace{e^{-2x+1}}_{g'(f(x))} \cdot \underbrace{(-2)}_{f'(x)}$$

<u>Dim.</u> (caso facile) ipotesi che $f'(x) \neq 0$

$$g \circ f\,(x) = g(f(x))$$

rapporto incrementale



in un intorno di x $f(x+h) \neq f(x)$ (per h "piccolo")

$$\frac{g(f(x+h)) - g(f(x))}{h} =$$

$$= \frac{g(f(x+h)) - g(f(x))}{f(x+h) - f(x)} \cdot \underbrace{\frac{f(x+h) - f(x)}{h}}$$

$$h \to 0 \quad f(x+h) - f(x) \to 0 \qquad \downarrow g'(f(x)) \qquad \downarrow \underset{h\to 0}{} f'(x)$$

$$- \quad f(x) = y$$
$$f(x+h) = y_0$$

$$\frac{g(y_0) - g(y)}{y_0 - y} \xrightarrow[y \to y_0]{} g'(y)$$

$\#$

<u>oss.</u> Ho diviso per $f(x+h) - f(x)$ che è $\neq 0$ per h suff. piccolo. (per l'ipotesi $f'(x) \neq 0$).

<u>es.</u> $h(x) = a^x = e^{x \log a}$

$$h'(x) = \underbrace{e^{x\log a}}_{g'} \cdot \log a = a^x \cdot \log a$$

la regola della catena si può applicare a più di due funzioni.

$$h(x) = 2\left(\log\left(\operatorname{sen}\left(x^2\right)\right)\right)$$

$$h'(x) = 2\,\frac{1}{\operatorname{sen} x^2} \cdot \cos\left(x^2\right) \cdot 2x$$

es. $f(x) = \operatorname{senh} x = \dfrac{e^x - e^{-x}}{2}$

$$f'(x) = \frac{1}{2}\left(e^x - e^{-x} \cdot (-1)\right) = \frac{e^x + e^{-x}}{2}$$

$$= \cosh x$$

$$(\operatorname{senh} x)' = \cosh x$$

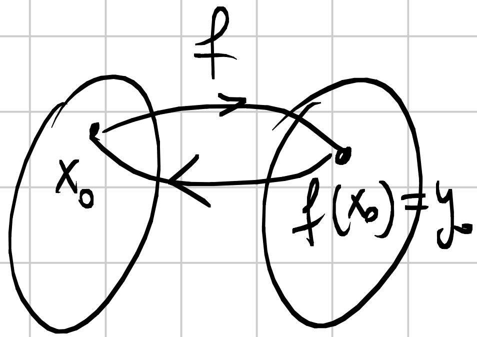
$$(\cosh x)' = \operatorname{senh} x$$

fate voi!

Derivata di funzione inversa

f continua e invertibile in (a,b) e derivabile in $x_0 \in (a,b)$, con $f'(x_0) \neq 0$

Allora $f^{-1}(y)$ è derivabile in $y_0 = f(x_0)$ e



$$\left(f^{-1}(y_0)\right)' = \frac{1}{f'(x_0)}$$

dove
$y_0 = f(x_0)$

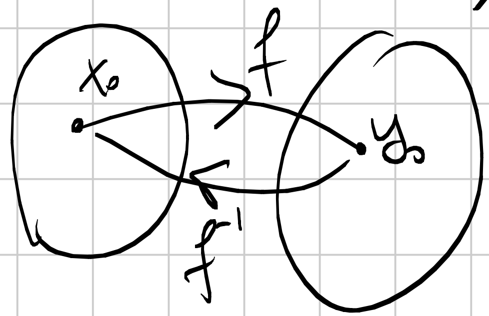
$$f(x) = e^x \qquad f^{-1}(y) = \log y$$

$$e^x = y \iff x = \log y$$

$$\left(f^{-1}(y)\right)' = (\log y)' = \frac{1}{e^x} = \frac{1}{y}$$

Dim. Mi calcolo $\underbrace{\left(f^{-1}(y_0)\right)'}_{\text{rapporto incrementale}}$

$y_0 = f(x_0)$
$x_0 = f^{-1}(y_0)$



$x_0 + k$

x_0

$$\frac{f^{-1}(y_0+h) - f^{-1}(y_0)}{h}$$

$f(x_0+k) = y_0+h$

$f^{-1}(y_0+h) = x_0+k$

se $h \to 0 \Rightarrow k \to 0$

perché f^{-1} è continua

$$= \frac{\cancel{x_0} + k - \cancel{x_0}}{f(x_0+k) - f(x_0)}$$

$h = f(x_0+k) - f(x_0)$

$$= \frac{k}{f(x_0+k) - f(x_0)}$$

se $h \to 0$

$k \to 0$

$$\to \frac{1}{f'(x_0)}$$

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es. $f(x) = \operatorname{sen} x$

$$f^{-1}(y) = \operatorname{arcsen} y = x$$

$$\Updownarrow$$

$$y = \operatorname{sen} x$$

$$\begin{cases} \left(f^{-1}(y_0)\right)' = \dfrac{1}{f'(x_0)} \\ y_0 = f(x_0) \end{cases}$$

$x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

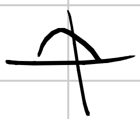
$y \in [-1, 1]$

$$\left(f^{-1}(y)\right)' = \left(\operatorname{arcsen} y\right)' = \frac{1}{\cos x}$$

$y = \operatorname{sen} x$

in funzione di $y = \operatorname{sen} x$

$$\operatorname{sen}^2 x + \cos^2 x = 1$$

$x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ 

$$\cos^2 x = 1 - \operatorname{sen}^2 x$$

$$\cos x = \pm\sqrt{1 - \operatorname{sen}^2 x}$$

$\Downarrow$ $\cos x \geq 0$ quindi si sceglie

$$\cos x = \sqrt{1-\sin^2 x} =$$

$$= \sqrt{1-y^2}$$

$$(\arcsin y)' = \frac{1}{\cos x} = \frac{1}{\sqrt{1-y^2}}$$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$$

P.C. $(\arccos y)' = -\dfrac{1}{\sqrt{1-y^2}}$