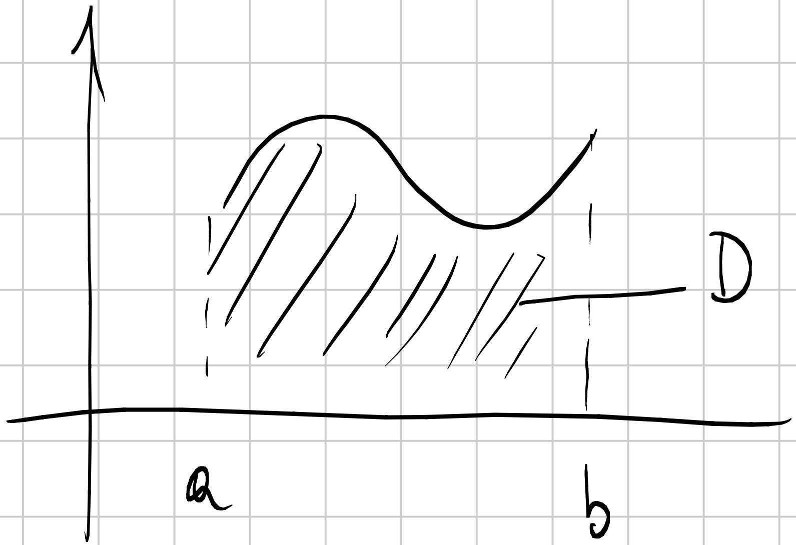
Prima parte lezione dell
24 Gennaio

Uso degli integrali definiti per il calcolo delle aree.

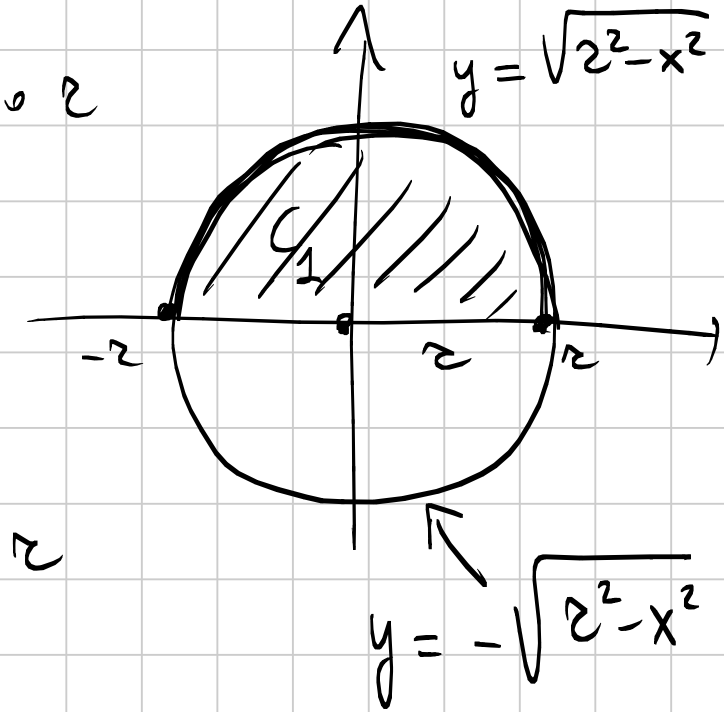
$f \geq 0$

$[a, b]$



$$\int_a^b f(x)\,dx = \text{Area}(D)$$

es. Area del cerchio di raggio r



$$x^2 + y^2 = r^2$$

$$y^2 = r^2 - x^2$$

$$y = \pm\sqrt{r^2 - x^2}, \quad -r \le x \le r$$

$$A(C) = 2A(C_1) = 2\int_{-r}^{r} \sqrt{r^2 - x^2}\, dx =$$

$$= 4\int_0^r \sqrt{r^2 - x^2}\, dx = 4r\int_0^r \sqrt{1 - \left(\frac{x}{r}\right)^2}\, dx =$$

$$\frac{x}{r} = \operatorname{sen} y \qquad x = r \operatorname{sen} y \qquad dx = r \cos y \, dy$$

$$= 4r \int_0^{\pi/2} \sqrt{1 - \operatorname{sen}^2 y} \; r \cos y \, dy =$$

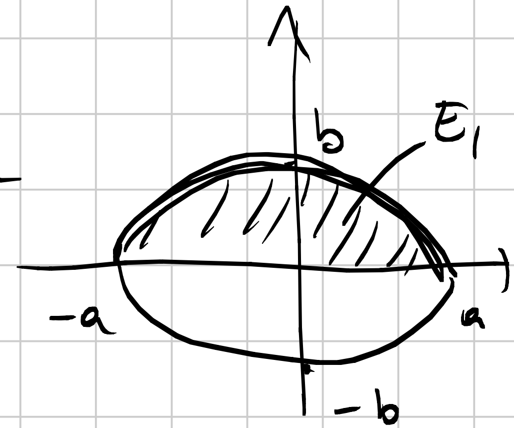
$$= 4r^2 \int_0^{\pi/2} |\cos y| \cos y \, dy = 4r^2 \int_0^{\pi/2} \cos^2 y \, dy =$$

$$\int \cos^2 y \, dy = \frac{\operatorname{sen} y \cos y + y}{2} + C \qquad \text{(già fatto)}$$

$$4r^2\left(\frac{\sin y\cos y + y}{2}\right)\Bigg|_{y=0}^{y=\pi/2} =$$

$$= \cancel{2} r^2 \frac{\pi}{\cancel{4}} = \pi r^2$$

PC. Area ellisse $\quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$A(\text{Ellisse}) = 2A(E_1)$$

$$y = b\sqrt{1-\frac{x^2}{a^2}}$$

$$A(E_1) = \int_{-a}^{a} b\sqrt{1 - \frac{x^2}{a^2}}\, dx \quad \ldots$$

$$R = \pi a \cdot b$$

Natura dei p.ti stazionari con le derivate successive di f in x_0.

Teo. f derivabile n volte in $x_0 \in (a,b)$.

$f'(x_0) = 0$

$f''(x_0) = 0 \quad \ldots \quad f^{(n-1)}(x_0) = 0, \quad f^{(n)}(x_0) \neq 0$

Allora n dispari $\Rightarrow$ x_0 non è p.to di estremo

n pari $\Rightarrow \begin{cases} f^{(n)}(x_0) > 0 & x_0 \text{ p.to di minimo locale} \\ f^{(n)}(x_0) < 0 & x_0 \text{ p.to di massimo locale} \end{cases}$

Es. $f(x) = x^4$ $\quad f'(x) = 4x^3 \longrightarrow$

$x=0$ p.to stazionario $\quad f''(x) = 12x^2$

$f'(0) = 0 \qquad f''(0) = 0$

$f'''(x) = 24x \qquad f'''(0) = 0$

$f^{IV}(x) = 24 \qquad f^{IV}(0) = 24 > 0 \quad n=4$

$x=0$ p.to di minimo

es. $f(x) = x \operatorname{sen} x - \cos 2x$

$f'(x) = \operatorname{sen} x + x\cos x + 2\operatorname{sen} 2x$

$x = 0$ $\quad f'(0) = 0$ $\quad$ p.to stazionario

$f''(x) = \cos x + \cos x - x \operatorname{sen} x + 4\cos 2x$

$f''(0) = 1 + 1 + 4 = 6 > 0$ $\quad n = 2$

$x = 0$ p.to di minimo relativo.

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Es. Trovare lo sviluppo di Mc Laurin di ordine 4 di

$$f(x) = \log\left(1+\operatorname{sen}^2 x\right) - \alpha x^2 + \operatorname{sen}\left(e^{x^2}-1\right) + e^{-1/x}$$

$x \to 0^+$

al variare di $\alpha \in \mathbb{R}$.

$$\log\left(1+\operatorname{sen}^2 x\right) = \operatorname{sen}^2 x - \frac{\operatorname{sen}^4 x}{2} + o\left(\operatorname{sen}^4 x\right)$$

$$\cancel{\log\left(1+\cos^2 x\right)}$$

$$\log(1+y) = y - \frac{y^2}{2} + \frac{y^4}{4} + o(y^4)$$

$$\sin^2 x = x^2 - \frac{1}{3}x^4 + o(x^4) \qquad \text{(già fatto).}$$

$$= \left(x - \frac{x^3}{3!} + o(x^3)\right)^2 \qquad \sin x = x - \frac{x^3}{3!} + o(x^3)$$

$$\log(1+\sin^2 x) = x^2 - \frac{1}{3}x^4 + o(x^4) - \frac{1}{2}x^4 =$$

$$= x^2 - \frac{5}{6}x^4 + o(x^4)$$

$$\sin x = x - \frac{x^3}{3!} + o(x^3)$$

$$\sin^2 x = \left(x - \frac{x^3}{3!} + o(x^3)\right)^2 = x^2 - \frac{2}{3!}x^4 + o(x^4).$$

$$\operatorname{sen}\left(e^{x^2}-1\right) = \left(e^{x^2}-1\right) - \frac{1}{6}\left(e^{x^2}-1\right)^3 + o\left(e^{x^2}-1\right)^3$$

$$\therefore \operatorname{sen} y = y - \frac{y^3}{3!} + o\left(y^3\right)$$

$$e^y = 1 + y + \frac{y^2}{2} + o(y^2)$$

$$e^{x^2} - 1 = x^2 + \frac{x^4}{2} + o\left(x^4\right)$$

$$\operatorname{sen}\left(e^{x^2}-1\right) = x^2 + \frac{x^4}{2} + o\left(x^4\right) - \frac{1}{6}\left(x^2 + \frac{x^4}{2} + o\left(x^4\right)\right)^3 +$$

$$o(x^4) \leftarrow + o\left(\quad\right)^3$$

$$= x^2 + \frac{x^4}{2} + o(x^4)$$

$$e^{-1/x} = o(x^4) \quad \text{perché} \quad e^{-1/x} = o(x^\beta)$$

$x \to 0^+$ $\qquad\qquad\qquad\qquad$ $\forall \beta$.

$$\lim_{x \to 0^+} \frac{e^{-1/x}}{x^4} = 0$$

$$f(x) = x^2 - \frac{5}{6} x^4 + o(x^4) - \alpha x^2 + x^2 +$$

$$+\frac{x^4}{2} = (2-\alpha)\,x^2 - \frac{1}{3}x^4 + o(x^4)$$

$$-\frac{5}{6}+\frac{1}{2}=\frac{-5+3}{6}$$

- Determinare α t.c.

$$f(x) = o(x^3) \quad , \quad x \to 0^+$$

$$f(x) = (2-\alpha)\,x^2 - \frac{1}{3}x^4 + o(x^4)$$

$\alpha \neq 2$ $\quad f(x)$ è infinitesimo di ordine 2

$\alpha = 2 \qquad f(x) = -\frac{1}{3}x^4 + o(x^4)$

$$= o(x^3)$$

es. $\lim_{x \to +\infty} \alpha e^x - \sqrt{e^{2x} + 3} =$

$\alpha \in \mathbb{R}$

$= \lim_{y \to +\infty} \alpha y - \sqrt{y^2 + 3} \qquad e^x = y$

$\alpha \leq 0 \Rightarrow \lim (\ ) = -\infty$

$\alpha > 0$

$$\frac{\left(\alpha y-\sqrt{y^2+3}\right)\left(\alpha y+\sqrt{y^2+3}\right)}{\left(\alpha y+\sqrt{y^2+3}\right)}=$$

$$=\frac{\alpha^2y^2-y^2-3}{\alpha y+\sqrt{y^2+3}}=\frac{y^2\left(\alpha^2-1\right)-3}{\alpha y+\sqrt{y^2+3}}$$

$\alpha^2-1\neq 0$ $\qquad$ $y\to+\infty$

$\lim\limits_{y\to+\infty}\left(\quad\right)=+\infty$ se $\alpha>1$

$=-\infty$ se $0<\alpha<1$

$\alpha^2-1=0$ $\quad$ $\lim\left(\quad\right)=0$

$$\alpha y - \sqrt{y^2+3} = y\left(\alpha - \sqrt{1+\frac{3}{y}}\right)$$

$$y\left(\alpha - 1\right)$$

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es.

$$\int_{1}^{+\infty} \frac{e^{1/x}}{x^2}\left(1-\cos\frac{1}{x}\right)dx$$

- dire se converge
- In caso affermativo calcolarlo.

$$f(x) = \frac{e^{1/x}}{x^2}\left(1 - \cos\frac{1}{x}\right) \underset{x\to+\infty}{\sim} \frac{1}{x^2}\left(\frac{1}{2x^2}\right)$$

$$\cos\frac{1}{x} \underset{x\to+\infty}{=} 1 - \frac{1}{2x^2} + o\left(\frac{1}{x^2}\right)$$

$$f(x) \underset{x\to+\infty}{\sim} \frac{1}{2x^4}$$

poiché $\frac{1}{x^4}$ è integrabile in $[1,+\infty)$

anche f lo è!

$$\int_1^{+\infty} \frac{e^{1/x}}{x^2}\left(1-\cos\frac{1}{x}\right)dx$$

$$\frac{1}{x}=y$$

$$\int \frac{e^{1/x}}{x^2}\left(1-\cos\frac{1}{x}\right)dx =$$

$$-\frac{1}{x^2}dx=dy$$

$$= -\int e^{y}(1-\cos y)\,dy$$

$$\int_1^{k} f(x)\,dx \qquad \lim_{k\to+\infty}\Big(\qquad\qquad \Big) =$$