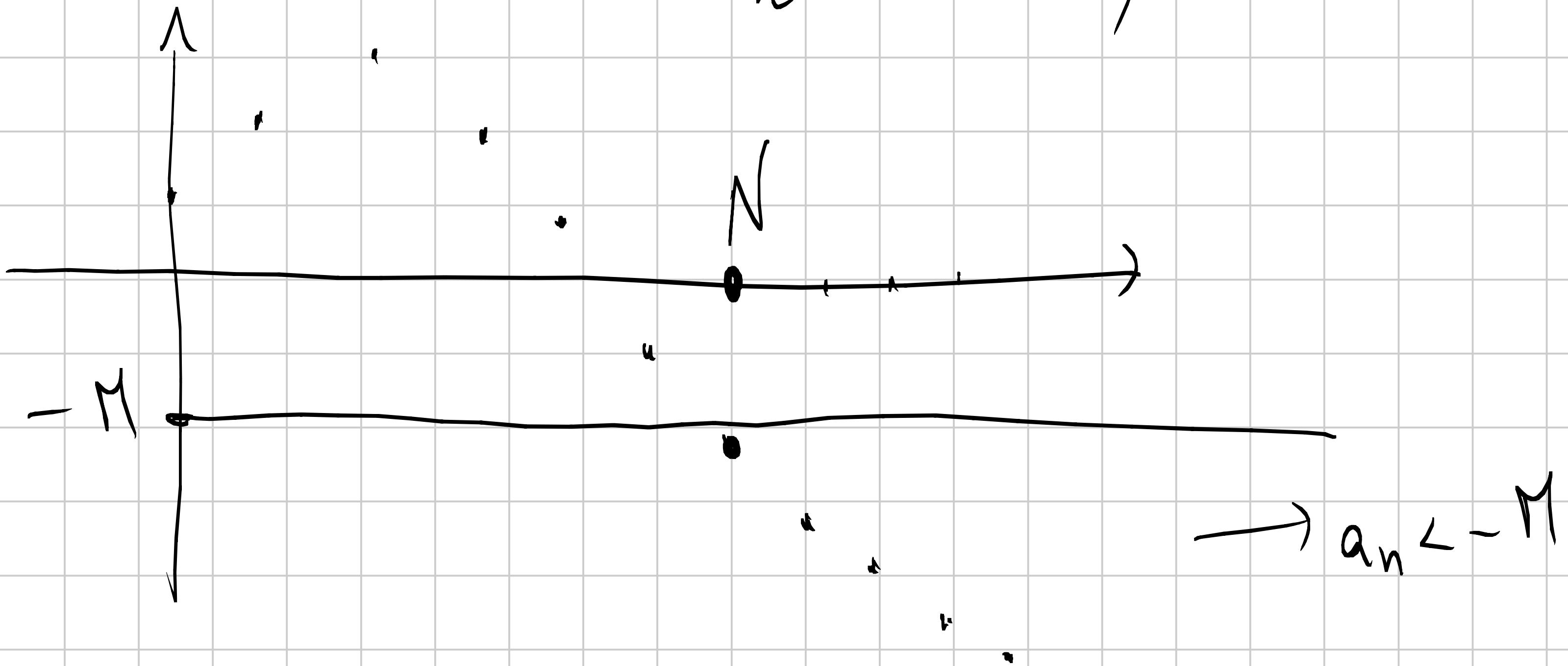
Lezione del 27 Ottobre

Analogamente

$$\lim_{n \to +\infty} a_n = -\infty \iff \forall M > 0 \ \exists N \text{ t.c. }$$
$$a_n < -M, \ \forall n > N$$



Successione divergente $\Longleftrightarrow \lim\limits_{n \to +\infty} a_n = \begin{matrix} +\infty \\ -\infty \end{matrix}$

$+\infty \; , \; -\infty$ si può estendere $\mathbb{R}$

$\mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$

$-\infty$ $\qquad\qquad\qquad\qquad$ $+\infty$

$\forall x \in \mathbb{R} \qquad x < +\infty$

$\forall x \in \mathbb{R} \qquad x > -\infty$

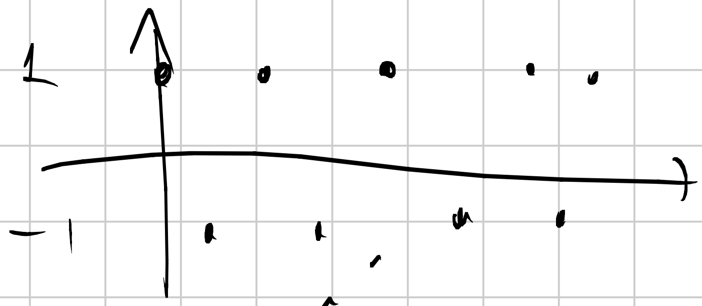
es. $\lim_{n \to +\infty} \log\left(\frac{1}{n}\right) = -\infty$

$\forall M > 0 \ \exists N$ t.c. $\log\left(\frac{1}{n}\right) < -M, \ \forall n > N$

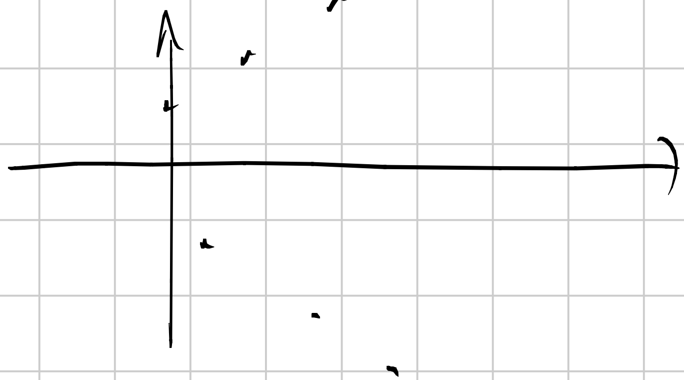
$\frac{1}{n} < e^{-M}$ $\qquad$ $n > \frac{1}{e^{-M}} = e^{M} \to N$

Def. Una successione che non è né convergente né divergente si dice <u>indeterminata</u> (o irregolare) e si dice che il limite non esiste es.

$a_n = (-1)^n$



$a_n = (-2)^n$



oss. $\{a_n\}$ limitata $\neq$ $\{a_n\}$ che ammette limite

$a_n = (-1)^n$ è limitata ma non ammette limite

Prop. Se una successione $\{a_n\}$ è convergente $\Rightarrow$ $\{a_n\}$ è limitata

Dim. Hp. $\lim_n a_n = l \in \mathbb{R}$

Ts. $\exists\, m, M : \underline{m} < a_n < \underline{M}, \quad \forall n \in \mathbb{N}$

$\forall \varepsilon > 0 \quad \exists N : \quad |a_n - l| < \varepsilon, \quad \forall n > N$

$l - \varepsilon < a_n < l + \varepsilon, \quad \forall n > N$

$a_0, a_1, \dots a_n$ sono n numeri reali

e quindi trovo m e M t.c. $m < a_n < M, \forall n \in \mathbb{N}$

$$\lim_{n \to +\infty} a_n = \begin{cases} l \in \widetilde{\mathbb{R}} & \text{convergente} \\ \pm\infty & \text{divergente} \\ \nexists & \text{indeterminato} \end{cases}$$

Calcolo dei limiti

Teorema $a_n \to a$ e $b_n \to b$, $a, b \in \mathbb{R}$

- $a_n \pm b_n \to a \pm b$
- $a_n \cdot b_n \to a \cdot b$
- $\dfrac{a_n}{b_n} \to \dfrac{a}{b}$, $b_n, b \neq 0$

$\cdot \quad (a_n)^{b_n} \to a^b \quad , \qquad a_n , a > 0$

<u>Dim</u> $\quad {}^{Ts}\ a_n + b_n \to a + b$

<u>Hp.</u> $\quad a_n \to a$ e $b_n \to b$

$\forall \varepsilon > 0 \quad \exists\ N_1 : \quad |a_n - a| < \varepsilon , \quad \forall n > N_1$

$\forall \varepsilon > 0 \quad \exists\ N_2 : \quad |b_n - b| < \varepsilon , \quad \forall n > N_2$

$n > \max\{N_1 , N_2\}$

$|(a_n + b_n) - (a+b)| = |a_n - a + b_n - b| \leq$

$$\leq \underbrace{|a_n - a|}_{<\varepsilon} + \underbrace{|b_n - b|}_{<\varepsilon} < \varepsilon + \varepsilon = 2\varepsilon = \tilde{\varepsilon}$$

$$\forall n > \max\{N_1, N_2\}$$

<u>es.</u> $\lim_n \frac{1}{n} = 0$

$$\frac{1}{n^2} = \underbrace{\frac{1}{n}}_{a_n} \cdot \underbrace{\frac{1}{n}}_{b_n}$$

$$\lim_n \frac{1}{n^2} = 0$$

$$\lim_n \frac{2}{n} = 0 \qquad \frac{2}{n} = \frac{1}{n} + \frac{1}{n}$$

<u>es.</u> $\lim_n 3^{1/n} = 3^0 = 1$

$$(a_n)^{b_n} \to a^b$$

$a_n = 3 \to 3$

$b_n = \frac{1}{n} \to 0$

Teorema della permanenza del segno

$a_n \to a$ e $a > 0$

allora $a_n > 0$, definitivamente

($\exists M$ t.c. $a_n > 0$, $\forall n > M$)

Dim. Hp. $a_n \to a$, $a > 0$

$\forall \varepsilon > 0 \quad \exists\, M$ t.c. $\underbrace{a-\varepsilon < a_n} < a+\varepsilon, \quad \forall n > M$

$a_n > a - \varepsilon > 0$

scelgo ε t.c. $a - \varepsilon > 0$

$\rightarrow \quad a_n > 0 \quad \forall\, n > M$ #

$a_n \to a, \quad a > 0 \Rightarrow a_n > 0$ da un certo n in poi.

$a_n \to a, \quad a \geq 0 \;\cancel{\Rightarrow}\; a_n \geq 0$ NO!

se si prende $a=0$

$a_n \to 0 \Rightarrow a_n \geq 0$

es. $a_n = -\dfrac{1}{n}$

$a_n \to a, \quad a>0 \Longrightarrow a_n > 0$ da un certo n in poi

! se $a_n \to a$ e $a_n > 0 \not\Rightarrow a > 0$?

NO

es. $a_n = \dfrac{1}{n} > 0$ ma $a_n \to 0$ che non è > 0.

Il teorema della permanenza del segno si può enunciare anche così

se $a_n \to a$, $a_n \geq 0 \Rightarrow a \geq 0$

Infatti se fosse $a < 0 \Rightarrow$ teo. della permanenza del segno, appena dim.

$$\left(\begin{array}{ll} a > 0 \Rightarrow a_n > 0 & \text{definitiv.} \\ a < 0 \Rightarrow a_n < 0 & \text{"} \end{array} \right) \longrightarrow$$

ci dice

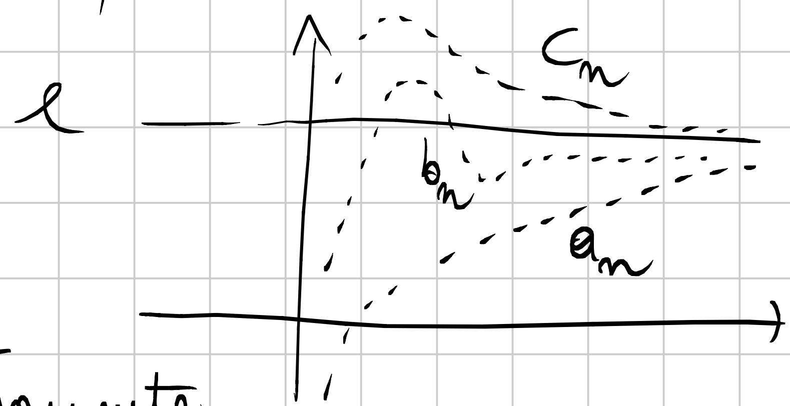
$a_n < 0$ definitivamente

Teorema del confronto (dei "carabinieri")

Se $a_n \le b_n \le c_n$ definitivamente

e $a_n \to \ell$, $c_n \to \ell$, $\ell \in \mathbb{R}$

$\Rightarrow b_n \to \ell$



Dim $\forall \varepsilon > 0 \quad |a_n - \ell| < \varepsilon$ definitivamente

$\forall \varepsilon > 0 \quad |c_n - \ell| < \varepsilon$ definitivamente

$\ell - \varepsilon < a_n < \ell + \varepsilon \qquad \ell - \varepsilon < c_n < \ell + \varepsilon$

$$\ell - \varepsilon < a_n \leq b_n \leq c_n < \ell + \varepsilon$$ definitivamente

$$\ell - \varepsilon < b_n < \ell + \varepsilon$$ definitivamente

$$|b_n - \ell| < \varepsilon \implies \#$$

<u>Come si usa</u> il teorema del confronto

- $|b_n| \leq c_n$ (per n suff. grande)

Se $c_n \to 0$ allora $b_n \to 0$

- $\left(-\underset{\downarrow 0}{c_n} \leq b_n \leq \underset{\downarrow 0}{c_n} \right) \implies b_n \to 0$

es. $\lim_{n} \frac{\text{sen } n}{n}$ b_n

$$b_n = \frac{\text{sen } n}{n} = (\text{sen } n) \cdot \frac{1}{n}$$

$$-1 \leq \text{sen } n \leq 1 \iff |\text{sen } n| \leq 1$$

$$|b_n| = \left| \frac{\text{sen } n}{n} \right| \leq \frac{1}{n} \to 0$$

$$|b_n| \leq c_n$$

anche $\frac{\text{sen } n}{n} \to 0$

<u>Applicazione</u> prodotto di due successioni: una che tende a zero e l'altra solo limitata (b_n)

c_n

se b_n è limitata

$|b_n| \le K$

$$|c_n \cdot b_n| \le \underbrace{|c_n|}_{\downarrow 0} K \to 0$$

una successione che tende a zero la chiamo <u>infinitesima</u>

$$\frac{1}{n}$$

Ho dim. che il prodotto di una successione infinitesima per una successione limitata è infinitesima

$$\frac{\operatorname{sen} n}{n} = \underbrace{(\operatorname{sen} n)}_{\text{limitata}} \underbrace{\left(\frac{1}{n}\right)}_{\text{infinitesima}} \longrightarrow 0$$

$$\frac{(-1)^n}{n^2} = \underbrace{(-1)^n}_{\text{limitata}} \cdot \underbrace{\frac{1}{n^2}}_{\text{infinitesima}} \longrightarrow 0$$

Con la definizione di limite si verifica che

$$\lim_{n \to +\infty} n^{\alpha} = \begin{cases} +\infty & \alpha > 0 \\ 1 & \alpha = 0 \\ 0 & \alpha < 0 \end{cases}$$

es

$$\lim_{n} \frac{n^{1/2} + n^{2} + 8}{n^{3/2} + n^{5}} = \lim_{n} \frac{\cancel{n^{2}}\left(1 + \frac{1}{n^{3/2}} + \frac{8}{n^{2}}\right)}{n^{\cancel{5}\,3}\left(1 + \frac{1}{n^{5-\frac{3}{2}}}\right)} = 0$$

$$1 + \frac{1}{n^{3/2}} + \frac{8}{n^{2}} \to 1$$

$$1 + \frac{1}{n^{\frac{1}{2}}} \to 1$$

Operazioni dei limiti nel caso infinito

- $a_n \to a$
  $b_n \to +\infty$

  $a_n + b_n \to +\infty$ $\qquad a + \infty = +\infty$
  $a_n - b_n \to -\infty$ $\qquad a - \infty = -\infty$

- $a_n \to +\infty$
  $b_n \to +\infty$

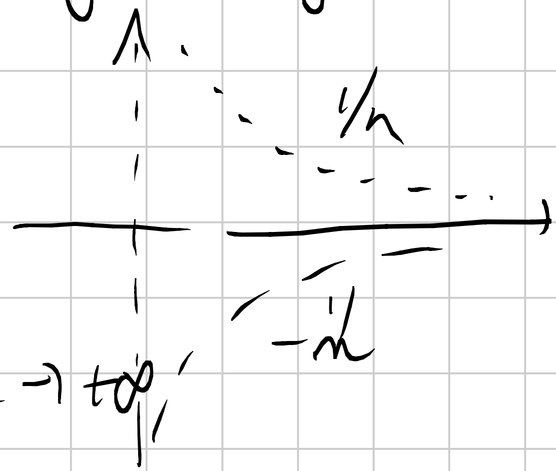
  $a_n + b_n \to +\infty$ $\qquad +\infty + \infty = +\infty$
  $-a_n - b_n \to -\infty$ $\qquad -\infty - \infty = -\infty$

- $a_n \to a \neq 0$
  $b_n \to +\infty$

  $a_n \cdot b_n \to \pm\infty$ a seconda del segno di a
  $a \cdot (+\infty)$

- $b_n \to 0$ $\qquad \dfrac{1}{b_n} \to \pm\infty$ a seconda del segno di b_n per n grande

- $b_n \to +\infty$ $\qquad \dfrac{1}{b_n} \to 0$



$b_n = \dfrac{1}{n} \to 0 \qquad b_n > 0 \qquad \dfrac{1}{b_n} = n \to +\infty$

$b_n = -\dfrac{1}{n} \to 0 \qquad b_n < 0 \qquad \dfrac{1}{b_n} = -n \to -\infty$

- no <u>dim.</u>

mancano dalla lista le regole per calcolare

$+\infty - \infty$, $0 \cdot \infty$, $\frac{0}{0}$, $\frac{\infty}{\infty}$

(forme di indecisione)

$$\lim_{n} \frac{n^{1/2} + n^{2} + 8}{n^{3/2} + n^{5}} \qquad \begin{matrix} a_n \\ b_n \end{matrix}$$

$a_n \to +\infty$

$b_n \to +\infty$

$\frac{\infty}{\infty}$

altre forme di indecisione

$$e^{\log x} = x$$

$$\underbrace{a_n}_{x}^{b_n} = e^{\log\left(a_n^{b_n}\right)} = e^{\underbrace{b_n \cdot \log a_n}}$$

$$\underbrace{b_n} \cdot \underbrace{\log a_n}$$

Forme di indecisione

$$1^{\pm\infty}, \quad 0^0, \quad (+\infty)^0$$

Esempi:

$n^3 + 5 \to +\infty$

$n^2 + 2n \to +\infty$

$$\lim_{n \to +\infty} \underset{+\infty}{n^3} \underset{-\infty}{- n^2} + 5 - 2n =$$

$$= \lim_{n \to +\infty} n^3 \underbrace{\left(1 - \frac{1}{n} + \frac{5}{n^3} - \frac{2}{n^2}\right)}_{\to 1} = +\infty$$

es.

$$\lim_{n} \underbrace{n}_{+\infty} + \underbrace{\operatorname{sen} n}_{\nexists} + \underbrace{\frac{1}{n}}_{\to 0} + \underbrace{3^{1/n}}_{1} + \underset{5}{5} =$$

$$= \lim_{n} n \left(1 + \underset{\substack{\downarrow \\ 0}}{\frac{\operatorname{sen} n}{n}} + \underset{\substack{\downarrow \\ 0}}{\frac{1}{n^2}} + \underset{\substack{\downarrow \\ 0}}{\frac{3^{1/n}}{n}} + \underset{\substack{\downarrow \\ 0}}{\frac{5}{n}} \right)$$

$$= +\infty$$

$$\lim_n \sqrt{n} - \sqrt{n+1} \qquad a^2-b^2=(a-b)(a+b)$$

$$\infty - \infty$$

$$\sqrt{n} - \sqrt{n+1} = \frac{(\sqrt{n}-\sqrt{n+1})(\sqrt{n}+\sqrt{n+1})}{(\sqrt{n}+\sqrt{n+1})} =$$

$$= \frac{\cancel{n} - (\cancel{n}+1)}{\sqrt{n}+\sqrt{n+1}} = \frac{-1}{\underbrace{\sqrt{n}+\sqrt{n+1}}_{\to +\infty}} \to 0$$

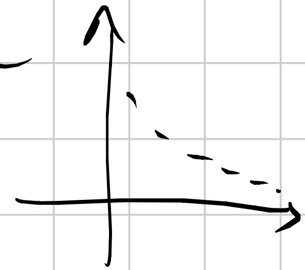
$a = \sqrt{n}$
$b = \sqrt{n+1}$

es. $$\lim_n \frac{1}{\underbrace{\sqrt{n}-\sqrt{n+1}}_{b_n<0}} = -\infty$$

$$\frac{1}{0}$$

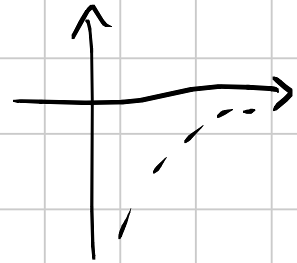
se $b_n \to 0$ e $b_n > 0$ per n grande

$$\frac{1}{b_n} \to +\infty \qquad (b_n \to 0^+)$$



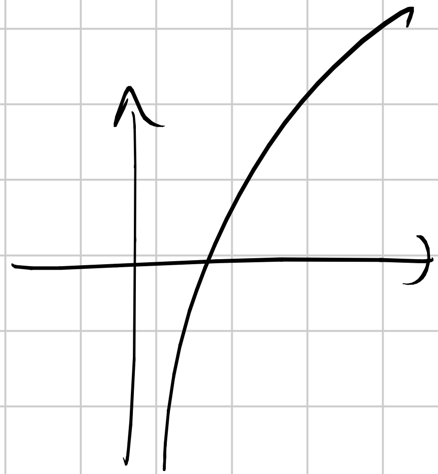
$b_n \to 0$ e $b_n < 0$ per n grande

$$\frac{1}{b_n} \to -\infty \qquad (b_n \to 0^-)$$



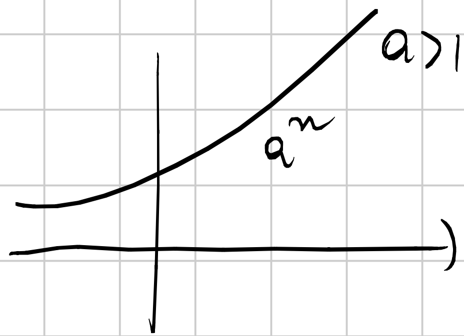
$\lim\limits_{n} a_n = 0$ infinitesimo

$\lim\limits_{n} a_n = \pm\infty$ infinito



Sono infiniti

- $\log_a n$ $(a>1)$
- n^{α} $(\alpha>0)$
- a^{n} $(a>1)$



$$\lim_{n} \frac{\log_5 n}{3^n} \qquad \lim_{n} \frac{2^n}{n^2}$$

Gerarchia degli infiniti

$a > 1$, $\alpha > 0$

$$\lim_{n} \frac{\log_a n}{n^\alpha} = 0$$

$$\frac{\infty}{\infty}$$

le potenze vanno più velocemente ad ∞ dei logaritmi

$$\lim_{n} \frac{n^{\alpha}}{a^{n}} = 0$$

l'esponenziale va più velocemente ad ∞ delle potenze

<u>es.</u> $$\lim_{n} \frac{\log n}{n^{3}} = 0$$

$$\lim_{n} \frac{\log n}{3^{n}} = 0$$

$\log_a n$, n^α , a^n

esercizio $\lim_{n} \dfrac{\log_3 n + n^2 + e^n}{e^{5n} + n^{100}}$

$\dfrac{+\infty}{+\infty}$

$$= \lim_{n} \frac{e^n \left(1 + \frac{\log_3 n}{e^n} + \frac{n^2}{e^n}\right)}{(e^5)^n \left(1 + \frac{n^{100}}{(e^5)^n}\right)} =$$

$\frac{\log_3 n}{e^n} \to 0$, $\frac{n^2}{e^n} \to 0$, $\frac{n^{100}}{(e^5)^n} \to 0$

$$= \lim_{n} \underbrace{\left(\frac{\cancel{e}}{\cancel{e^5}}\right)^n}_{\left(\frac{1}{e^4}\right)^n} \frac{\overbrace{(1+\dots)}^{\to 1}}{\underbrace{(1+\dots)}_{\to 1}} \to 0\cdot 1 = 0$$

<u>Per casa</u> Verificare con la definizione di limite,

- $\displaystyle\lim_{n\to+\infty} \frac{n}{n+1} = 1$ • $\displaystyle\lim_{n\to+\infty} 3^n+5 = +\infty$

- $\displaystyle\lim_{n\to+\infty} 1-n^2 = -\infty$ • $\displaystyle\lim_{n\to+\infty} \left(\frac{1}{3}\right)^n = 0$

$$\lim_{n \to +\infty} \frac{\cos n + e^{n} + e^{5n}}{1 + n^2 + \log n}$$