Lezione del 7 Dicembre

esercizi

$$\lim_{x \to 0} \frac{2(\cos x - 1)}{x^2 - x^3} =$$

$$= \lim_{x \to 0} \frac{2\left(\cancel{1} - \frac{x^2}{2} + o(x^2) - \cancel{1}\right)}{x^2 + o(x^2)} =$$

$$= \lim_{x \to 0} - \frac{x^2 + o(x^2)}{x^2 + o(x^2)} = \lim_{x \to 0} \frac{-x^2}{x^2} = -1$$

es. per cose $\lim_{x \to 0} \frac{\operatorname{tg} x - x^2}{\operatorname{sen} x + x^3} =$ $\qquad \operatorname{tg} x = x + o(x)$

$$= \lim_{x \to 0} \frac{x + o(x)}{x + o(x)} = 1$$

$x \to 0$

$-x = y \quad y \to 0$

$$\underbrace{\operatorname{sen}(-x)} = (-x) - \frac{(-x)^3}{3!} + \left(\frac{-x^5}{5!}\right) + o(x^5)$$

$$= -x + \frac{x^3}{3!} - \frac{x^5}{5!} + o(x^5)$$

$x \to 0 \qquad y = x+1 \to 1$

$\operatorname{sen}(x+1)$

$\operatorname{sen} y$

$x \to 0 \qquad \sqrt{x} = y \to 0$

$$\cos\sqrt{x} = 1 - \frac{(\sqrt{x})^2}{2} + \frac{(\sqrt{x})^4}{4!} + o\left((\sqrt{x})^4\right) =$$

$$\underset{x \to 0^+}{=} 1 - \frac{x}{2} + \frac{x^2}{4!} + o(x^2)$$

$\cos\left(e^{-1/x}\right) \qquad x \to 0^+ \quad e^{-1/x} \to 0$

$$\cos\left(e^{-1/x}\right) = 1 - \frac{\left(e^{-1/x}\right)^2}{2} + \frac{\left(e^{-1/x}\right)^4}{4!} + o\left(\left(e^{-1/x}\right)^4\right) =$$

$$= 1 - \frac{e^{-\frac{2}{x}}}{2} + \frac{e^{-\frac{4}{x}}}{4!} + o\left(e^{-\frac{4}{x}}\right)$$

$x \to 0^+$

$$\cos\left(e^{1/x}\right)$$

$y \to +\infty$

$\cos y$

$$(\operatorname{sen} x)^2 \underset{x\to 0}{=} \left(x - \frac{x^3}{3!} + o(x^3)\right)^2 =$$

$$= x^2 + \frac{x^6}{(3!)^2} + o(x^6) - \frac{\cancel{2}}{\cancel{2}\cdot 3}x^4 + o(x^4)$$

$$= x^2 \underbrace{- \frac{1}{3}x^4 + o(x^4)}_{o(x^2)} \longrightarrow$$

es. $e^{\operatorname{sen} x}$

$x \to 0$

$\operatorname{sen} x = y \to 0$

sviluppo di McLaurin di una funzioni composta.

$$e^y \underset{y\to 0}{=} 1 + y + \frac{y^2}{2} + o\left(y^2\right) =$$

$$= 1 + \operatorname{sen} x + \frac{(\operatorname{sen} x)^2}{2} + o\left((\operatorname{sen} x)^2\right) =$$

$$= 1 + x - \frac{x^3}{3!} + o\left(x^3\right) + \frac{1}{2}\left(x^2 - \frac{x^4}{3} + o\left(x^4\right)\right)$$

$$+ o\left(x^2\right) =$$

$$= 1 + x + \frac{1}{2}x^2 + o\left(x^2\right) \qquad x \to 0.$$

es. $\lim_{x\to 0} \dfrac{\operatorname{sen} x - x}{\cos(2\sqrt{x}) - e^{-2x}}$

N. $\operatorname{sen} x - x = \cancel{x} + o(x) - \cancel{x} = o(x)$

$\operatorname{sen} x - x = \cancel{x} - \dfrac{x^3}{3!} + o(x^3) - \cancel{x} =$

$= -\dfrac{x^3}{6} + o(x^3)$

D: $\Rightarrow \cos 2\sqrt{x} - e^{-2x}$

$$\cos(2\sqrt{x}) = 1 - \frac{1}{2}(2\sqrt{x})^2 + o\left((2\sqrt{x})^2\right) =$$

$$= 1 - 2x + o(x)$$ $\underline{2^\circ \text{ordine}}$

$$e^{-2x} = 1 - 2x + o(x) \rightarrow \underline{1^\circ \text{ordine}}$$

$$\cos 2\sqrt{x} - e^{-2x} = \cancel{1} - \cancel{2}x - \cancel{1} + \cancel{2}x + o(x)$$

non basta

$$\cos 2\sqrt{x} = 1 - \frac{1}{2}(2\sqrt{x})^2 + \frac{1}{4!}(2\sqrt{x})^4 + o\left((\sqrt{x})^4\right)$$

$$= 1 - 2x + \frac{\cancel{2}\cdot 2\cdot \cancel{2}\cancel{2}}{\cancel{2}\cdot 3\cdot \cancel{4}}x^2 + o(x^2) =$$

$$= 1 - 2x + \frac{2}{3}x^2 + o(x^2)$$

$$e^{-2x} = 1 - 2x + \frac{1}{2}(-2x)^2 + o(x^2) =$$

$$= 1 - 2x + 2x^2 + o(x^2)$$

$$\cos 2\sqrt{x} - e^{-2x} = \cancel{1} - \cancel{2x} + \frac{2}{3}x^2 - \cancel{1} + \cancel{2x} - 2x^2 + o(x^2)$$

$$= -\frac{4}{3}x^2 + o(x^2)$$

$$\frac{N.}{D.} = \lim \frac{-\frac{1}{6}x^3 + o(x^3)}{-\frac{4}{3}x^2 + o(x^2)} = \lim_{x \to 0} \frac{-\frac{1}{6}\cancel{x^3}}{-\frac{4}{3}\cancel{x^2}} \to 0.$$

---

es. calcolare al variare di $\alpha \in \mathbb{R}$

$$\lim_{x \to 0^+} \frac{x^\alpha (\operatorname{senh} x - x)}{\cos x}$$

$$\lim_{x \to 0^+} x^\alpha (\operatorname{senh} x - x) \quad = 0 \quad \alpha \geq 0$$

$\alpha < 0 \qquad \frac{0}{0}$

$$\operatorname{senh} x = x + \frac{x^3}{3!} + o(x^3)$$

$$x^\alpha(\operatorname{senh} x - x) = x^\alpha\left(\cancel{x} + \frac{x^3}{6} + o(x^3) - \cancel{x}\right)$$

$$= \frac{x^{\alpha+3}}{6} + o(x^{\alpha+3}) \begin{cases} \to 0 & \alpha > -3 \\ \to 1/6 & \alpha = -3 \\ \to +\infty & \alpha < -3 \end{cases}$$

$$\operatorname{senh} x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^5)$$

$$\cosh x = 1 + \frac{x^2}{2} + \frac{x^4}{4!} + o(x^4)$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2} x^2 + o(x^2)$$

$x \to 0$

$\uparrow$

$$\text{ls.} \quad \sqrt{1+x^8} = 1 + \frac{1}{2} x^8 + \frac{1}{2}\frac{1}{2}\left(-\frac{1}{2}\right) x^{16} + o(x^{16})$$

$x \to 0 \qquad x^8 \to 0$

$$= 1 + \frac{x^8}{2} - \frac{1}{8} x^{16} + o(x^{16})$$

$$\lim_{x\to 0^+} \frac{27x^5 + \log(1+x^7)}{\sqrt{1+x^8} - 1 + \alpha \operatorname{sen}^5 x} =$$

$$= \lim_{x\to 0^+} \frac{27x^5 + x^7 + o(x^7)}{\frac{x^8}{2} + o(x^8) + \alpha x^5 + o(x^5)}$$

$$(\operatorname{sen} x)^5 = (x + o(x))^5 = x^5 + o(x^5)$$

$$= \lim_{x \to 0^+} \frac{27 x^5 + o(x^5)}{\alpha x^5 + \alpha o(x^5) + \frac{x^8}{2} + o(x^8)}$$

$\alpha \neq 0$ $$\frac{27 x^5 + o(x^5)}{\alpha x^5 + o(x^5)} \to \frac{27}{\alpha}$$

$\alpha = 0$ $$\frac{27 x^5 + o(x^5)}{\frac{x^8}{2} + o(x^8)} \to +\infty$$

$$\lim_{x \to 0^+} \frac{x^5 + \lg(x^7)}{(\operatorname{sen} x)^5}$$

<u>Variante</u> $$\lim_{x \to 0^+} \frac{27x^5 + \log\left(1 + e^{-1/x}\right)}{\sqrt{1 + x^8} - 1 + \operatorname{sen}^5 x}$$

$$= \lim_{x \to 0^+} \frac{27x^5 + e^{-1/x} + o\left(e^{-1/x}\right)}{x^5 + o(x^5)}$$

? $e^{-1/x} = o(x^5)$ $\quad x \to 0$

$$\lim_{x\to 0^+} \frac{e^{-1/x}}{x^5} \overset{?}{=} 0 \qquad \frac{1}{x} = y$$

gerarchie infiniti

$$\Rightarrow \lim_{y\to +\infty} e^{-y} \cdot y^5 = 0$$

in generale

$$e^{-1/x} = o(x^\alpha) \qquad \forall \alpha > 0$$

$$= \lim_{x \to 0^+} \frac{27x^5 + o(x^5)}{x^5 + o(x^5)} = 27$$

Oss.

$$2^x = e^{x \log 2} = 1 + x \lg 2 + \frac{x^2 (\lg 2)^2}{2} + o(x^2)$$

$x \to 0 \qquad x \lg 2 \to 0 \qquad y = x \log 2$

Limiti di successioni con McLaurin.

$\alpha \in \mathbb{R}$

$$\lim_{n \to +\infty} \frac{1 - n^2 \log\left(1 + \frac{1}{n^2}\right) + \frac{\operatorname{sen} n}{n^3}}{n^\alpha \operatorname{arctg}\left(\frac{1}{n}\right)}$$

$$\log\left(1 + \frac{1}{n^2}\right) = \frac{1}{n^2} - \frac{1}{2}\left(\frac{1}{n^2}\right)^2 + o\left(\left(\frac{1}{n^2}\right)^2\right) =$$

$$\begin{matrix} x \to 0 \\ n \to +\infty \end{matrix} \qquad = \frac{1}{n^2} - \frac{1}{2}\frac{1}{n^4} + o\left(\frac{1}{n^4}\right)$$

$\operatorname{sen} n$ $\quad n \to +\infty$ <u>no!</u> NO lim

se ci fosse stato scritto $\operatorname{sen}\left(\frac{1}{n}\right)$ $\quad x \to 0$

$x \to 0$

$$\operatorname{arctg} x = x - \frac{x^3}{3} + \frac{x^5}{5} + o(x^5)$$

$$\operatorname{arctg}\left(\frac{1}{n}\right) = \frac{1}{n} - \frac{1}{3}\frac{1}{n^3} + o\left(\frac{1}{n^3}\right)$$

$$= \lim_{n \to +\infty} \frac{1 - n^2\left(\frac{1}{n^2} - \frac{1}{2n^4} + o\left(\frac{1}{n^4}\right)\right) + \frac{\operatorname{sen} n}{n^3}}{n^\alpha\left(\frac{1}{n} - \frac{1}{3n^3} + o\left(\frac{1}{n^3}\right)\right)}$$

$$= \frac{\cancel{1} - \cancel{1} + \frac{1}{2n^2} + o\left(\frac{1}{n^2}\right) + \frac{\operatorname{sen} n}{n^3}}{n^{\alpha - 1} + o\left(\frac{1}{n^{1-\alpha}}\right)}$$

$N.\quad \frac{1}{2n^2} + o\left(\frac{1}{n^2}\right) + \frac{\operatorname{sen} n}{n^3}$

? $\frac{\operatorname{sen} n}{n^3} \overset{!}{=} o\left(\frac{1}{n^2}\right)$ verifico

$$\frac{\frac{\operatorname{sen} n}{n^3}}{\frac{1}{n^2}} = \cancel{n^2}\,\frac{\operatorname{sen} n}{n^{\cancel{3}}} \xrightarrow[n\to+\infty]{} 0$$

il mio limite $\lim_{n} \frac{\frac{1}{2n^2} + o\left(\frac{1}{n^2}\right)}{n^{\alpha-1} + o\left(\frac{1}{n^{1-\alpha}}\right)} =$

$$= \lim_{n} \frac{1}{2 h^{2} n^{\alpha-1}} = \lim_{n} \frac{1}{2 n^{\alpha+1}}$$

$\alpha+1<0$: $+\infty$

$\alpha+1=0$: $= \frac{1}{2}$

$\alpha+1>0$: $=0$

__Es.__ (per casa) $\alpha > 0$

$$e^{-1/x^2} = o(x^\alpha)$$

$$\lim_{x \to 0^+} \frac{\operatorname{sen}(x^\alpha) - x + e^{-1/x^2}}{(\operatorname{tg} x)^2} \cdot \cancel{\cos x}$$

$$\operatorname{tg} x = x + \frac{x^3}{3} + \frac{x^5}{5} + o(x^5)$$

$$(\operatorname{tg} x)^2 = \left(x + \frac{x^3}{3} + o(x^3)\right)^2 \overset{?}{\underset{?}{=}} x^2 + \frac{2}{3}x^4 + o(x^4)$$

R.

$\alpha > 1 \qquad -\infty$

$\alpha < 1 \qquad +\infty$

$\alpha = 1 \qquad 0$

es.

$$\lim_{x \to 0^+} \frac{\log\left(1 + (x^2 + x)\right) - \log(1+x) - \alpha x^2}{\underbrace{\underbrace{\operatorname{senh}\left(e^{-1/x}\right)}_{e^{-1/x} + o\left(e^{-1/x}\right)} + x^2}_{\text{D. } x^2 + o(x^2)}}$$

$$\log\left(1+(x^2+x)\right) - \log(1+x) = x^2 - x^3 + o(x^3)$$

Studio del grafico di

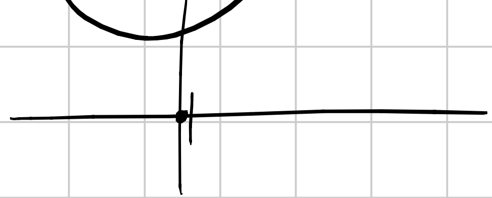
$x-1 \to x$

$$f(x) = (x-1)^2 \log\left(1+\frac{2}{|x-1|}\right)$$

$$f(x) = x^2 \log\left(1+\frac{2}{x}\right)$$

pari

$x>0$

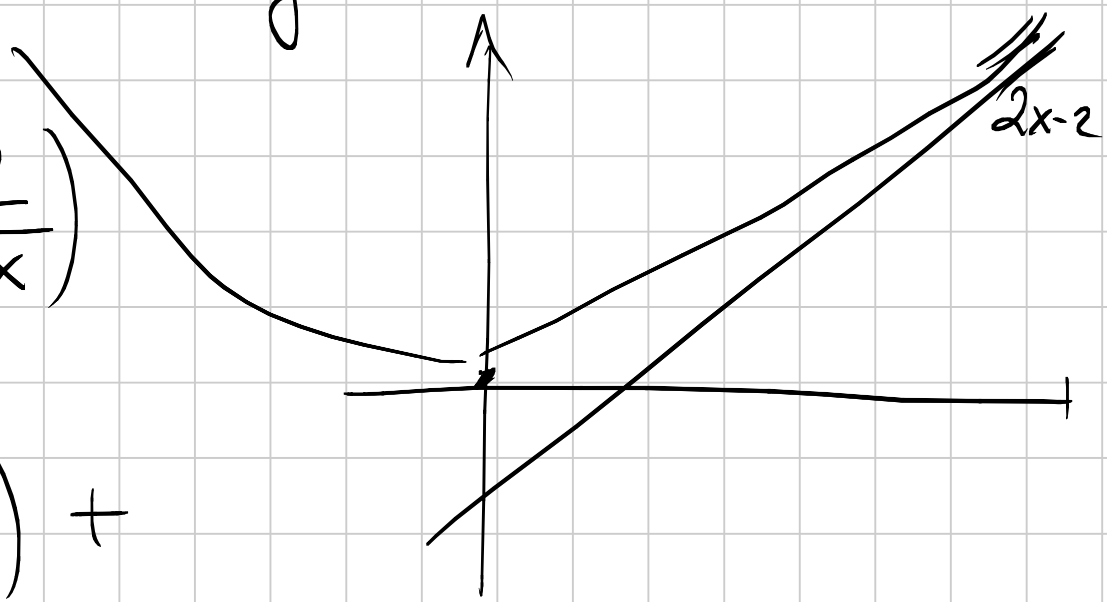


$\mathbb{R}\setminus\{0\}$

$$\lim_{x\to 0^+} f(x) = 0$$

$$\lim_{x \to +\infty} f(x) = +\infty$$

c'è asintoto obliquo $\quad y = 2x - 2$



$$f(x) = x^2 \log\left(1 + \frac{2}{x}\right)$$

$$f'(x) = 2x \log\left(1 + \frac{2}{x}\right) +$$

$$+ \cancel{x^2} \frac{1}{1 + \frac{2}{x}} \cdot \left(-\frac{2}{\cancel{x^2}}\right) = 2x \log\left(1 + \frac{2}{x}\right) - \frac{2x}{2+x}$$

$$f'(x) = 2x\Big(\underbrace{\log\left(1+\frac{2}{x}\right) - \frac{1}{2+x}}_{g(x)}\Big) > 0$$

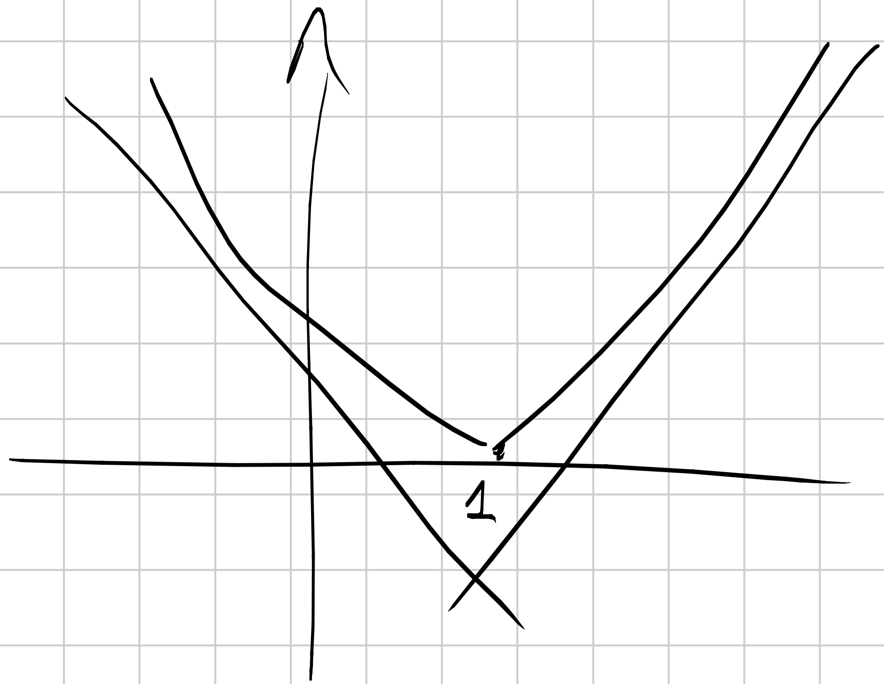
$g > 0$

$$g(x) = \log\left(1+\frac{2}{x}\right) - \frac{1}{2+x}$$

$$g'(x) = \frac{1}{1+\frac{2}{x}} \cdot \left(-\frac{2}{x^2}\right) + \frac{1}{(2+x)^2} =$$

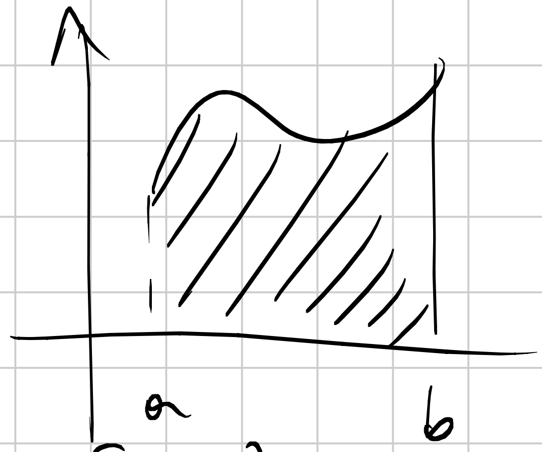
$$= \frac{-2x}{(x+2)\,x^2} + \frac{1}{(2+x)^2} = \frac{-2x(x+2)+x^2}{x^2(2+x)^2}$$

$$= \frac{-x^2 - 4x}{x^2(2+x)^2} < 0$$



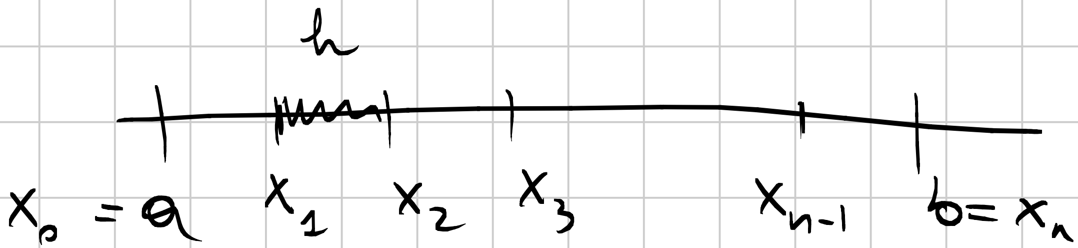
Integrali definiti

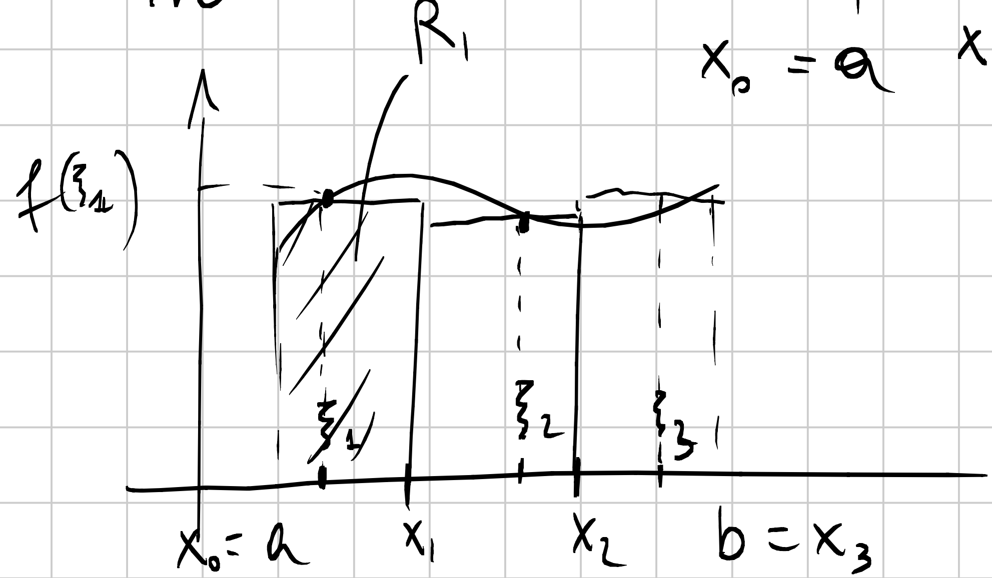
$f : [a,b] \to \mathbb{R}$ limitata

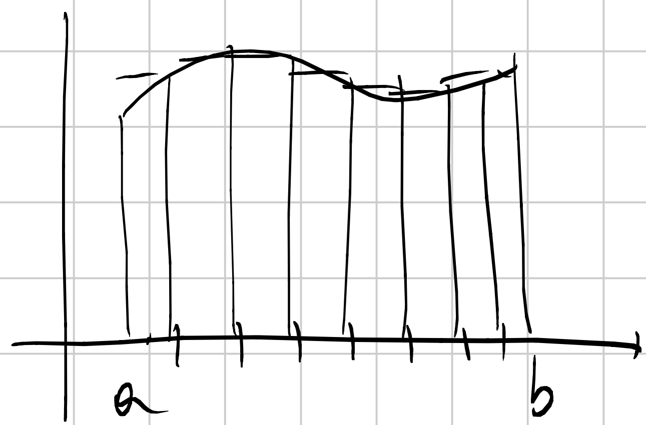


consideriamo una suddivisione di $[a,b]$

$$h = \frac{b-a}{n}$$







$$(x_1 - x_0) f(\xi_1) = A(R_1)$$

$$(x_2 - x_1) \cdot f(\xi_2) = A(R_2)$$

$$(x_3 - x_2) f(\xi_2) = A(R_3)$$

$$S_3 = \sum_{j=1}^{3} f(\xi_j)(x_j - x_{j-1}) = \text{somma delle 3 aree}$$

In generale

$$S_n = \sum_{j=1}^{n} \underbrace{f(\xi_j)}_{\text{altezza rettangolo}} \underbrace{(x_j - x_{j-1})}_{\text{ampiezza dell'intervallo (base del rettangolo)}}$$

somme di Cauchy - Riemann

ξ_j p.to arbitrario sull'intervallino (x_j, x_{j-1})

Def. $f: [a,b] \to \mathbb{R}$ limitata è *integrabile* in $[a,b]$

se esiste $\lim_{n \to +\infty} S_n$ e tale limite

non dipende dalla scelta degli ξ_i

$$\text{e} \quad \lim_n S_n =: \int_a^b f(x)\,dx$$

x = variabile d'integrazione è "muta"

$$\int^{b} f(t)\,dt \qquad \int_{a}^{b} f(y)\,dy$$

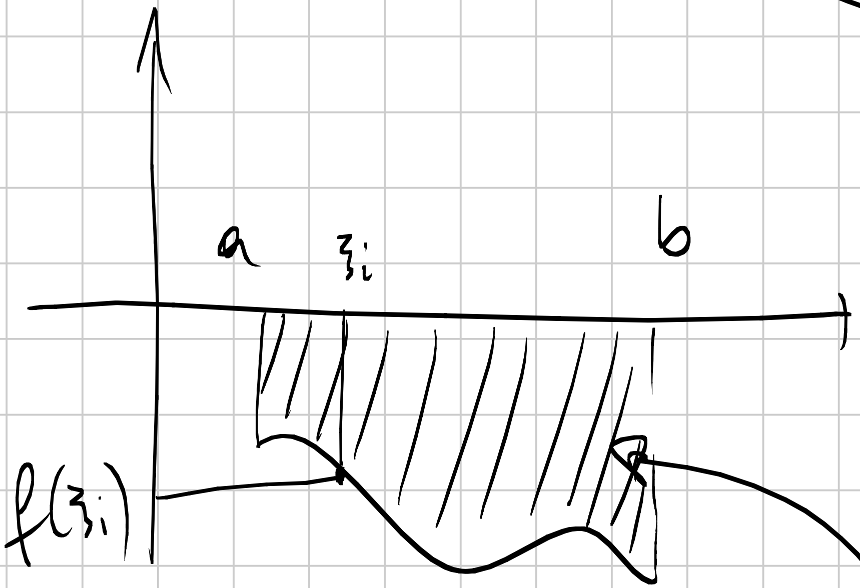
$$\sum_{k=1}^{3} \frac{1}{k} \qquad \sum_{j=1}^{3} \frac{1}{j}$$

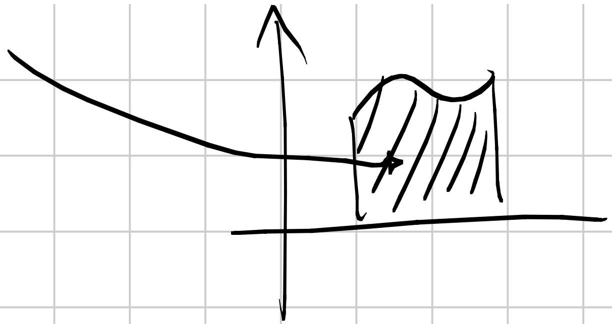
interpretazione geometrica $\quad f \geq 0$ continua

$$\lim_{n} S_n = \text{area del trapezoide sotto il grafico di } f$$

n è il numero dei sottointervalli

$f \leq 0$

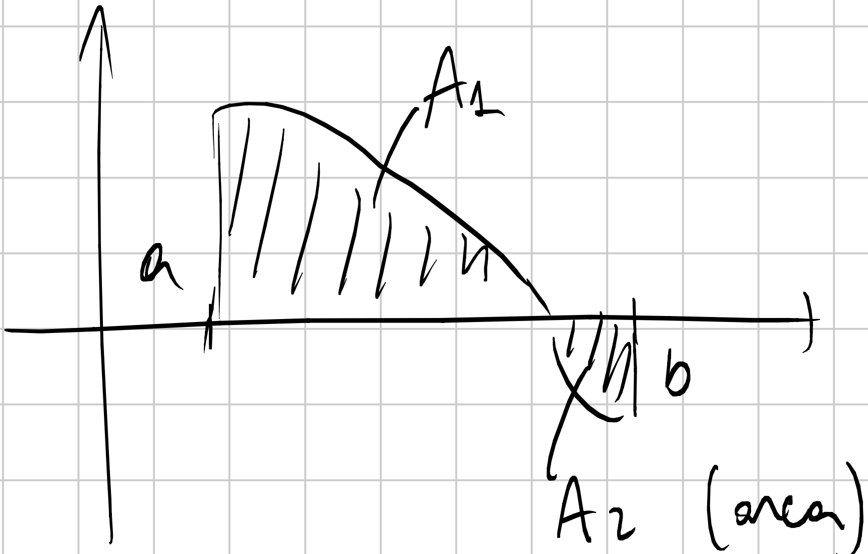




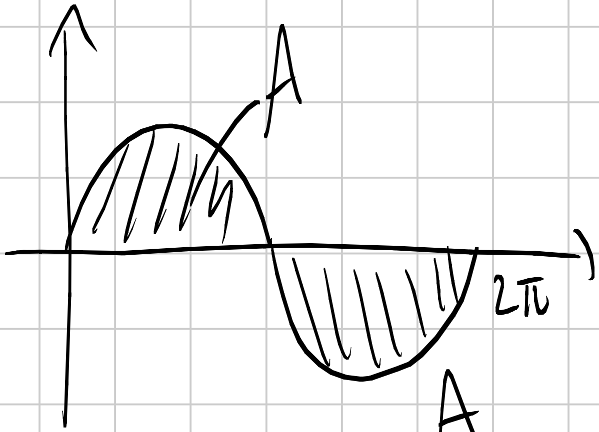
$S_n \leq 0$

$f(\xi_i) \leq 0$

$$\int_a^b f(x)\, dx = -\,(\text{area della regione}) = \text{area con il segno.}$$



$$\int_a^b f(x)\,dx = A_1 - A_2$$

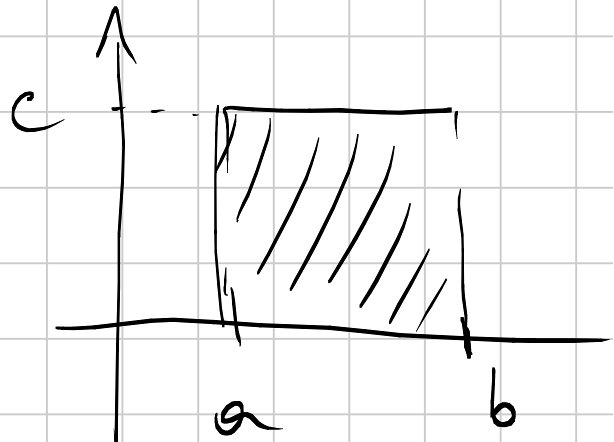


$$\int_a^b f(x)\,dx = A - A = 0$$

:

caso particolare $\quad f(x) = c \quad$ in $[a, b]$

$$\int_a^b f(x)\,dx \overset{?}{=} (b-a)\cdot c$$



$$S_n = \sum_{j=1}^{n} f(\xi_j)(x_j - x_{j-1}) = \sum_{j=1}^{n} c\left(\frac{b-a}{n}\right)$$

non dipende da j

$$= \cancel{n}\cdot c\,\frac{(b-a)}{\cancel{n}}$$

$$\int_a^b c \, dx \doteq \lim_n S_n = \lim_n c(b-a) = c(b-a)$$