Lezione del 9 Novembre

Limiti di funzioni

Definizione topologica di limite di funzione (p. 119)

$$\lim_{x \to 0} \frac{1}{x} \qquad \lim_{x \to +\infty} x^2 \qquad \lim_{x \to -\infty} x^2$$

$f(x) : I \to \mathbb{R}$

$x \to c$ — c può essere un p.to in cui è definita la funzione oppure no

$c \in \mathbb{R}^*$

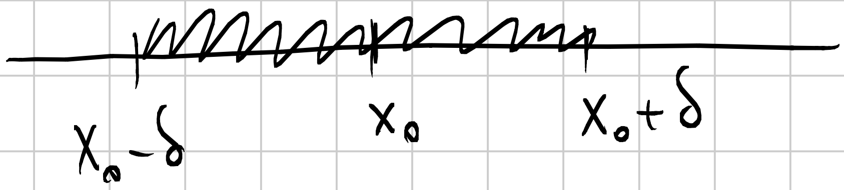
$$\lim_{x \to c} f(x) = \ell \qquad c \in \mathbb{R}^* \quad \ell \in \mathbb{R}^*$$

$$\lim_{n \to +\infty} a_n = \ell$$

<u>Definizione di intorno</u> $\qquad x_0 \in \mathbb{R}^*$

- $x_0 \in \mathbb{R}$ $\qquad U$ intorno di x_0 se

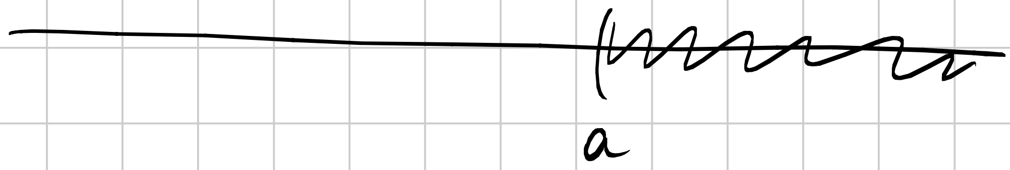
$$U = \{ x \in \mathbb{R} : |x - x_0| < \delta \} =$$



$$= (x_0 - \delta, \ x_0 + \delta)$$

$$U_{x_0}$$

- $x_0 = +\infty$



U intorno di $+\infty$

è un intervallo del tipo $(a, +\infty)$

$U = \{ x \in \mathbb{R} : x > a \}$

$U_{+\infty}$

- $x_0 = -\infty$ U intorno di $-\infty$ intervallo del tipo

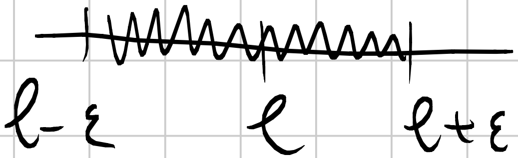
$(-\infty, b)$

$U = \{ x \in \mathbb{R} : x < b \}$

$U_{-\infty}$

Def. di limite di successione con gli intorni

$$\lim_{n\to+\infty} a_n = l \iff \forall \varepsilon>0\ \exists N : |a_n - l| < \varepsilon \quad \forall n > N$$

$|a_n - l| < \varepsilon \iff$  $\iff a_n \in U_l = (l-\varepsilon, l+\varepsilon)$ intorno di l

$n > N$  $\iff n \in V_{+\infty}$ intorno di $+\infty$

$$\lim_{n\to+\infty} a_n = l \iff \forall U_l \text{ intorno di } l\ \exists V_{+\infty} \text{ intorno di } +\infty \text{ t.c. } a_n \in U_l \quad \forall n \in V_\infty$$

Definizione di limite

Sia $c \in \mathbb{R}^*$, f definita (almeno) in un intorno di $x = c$

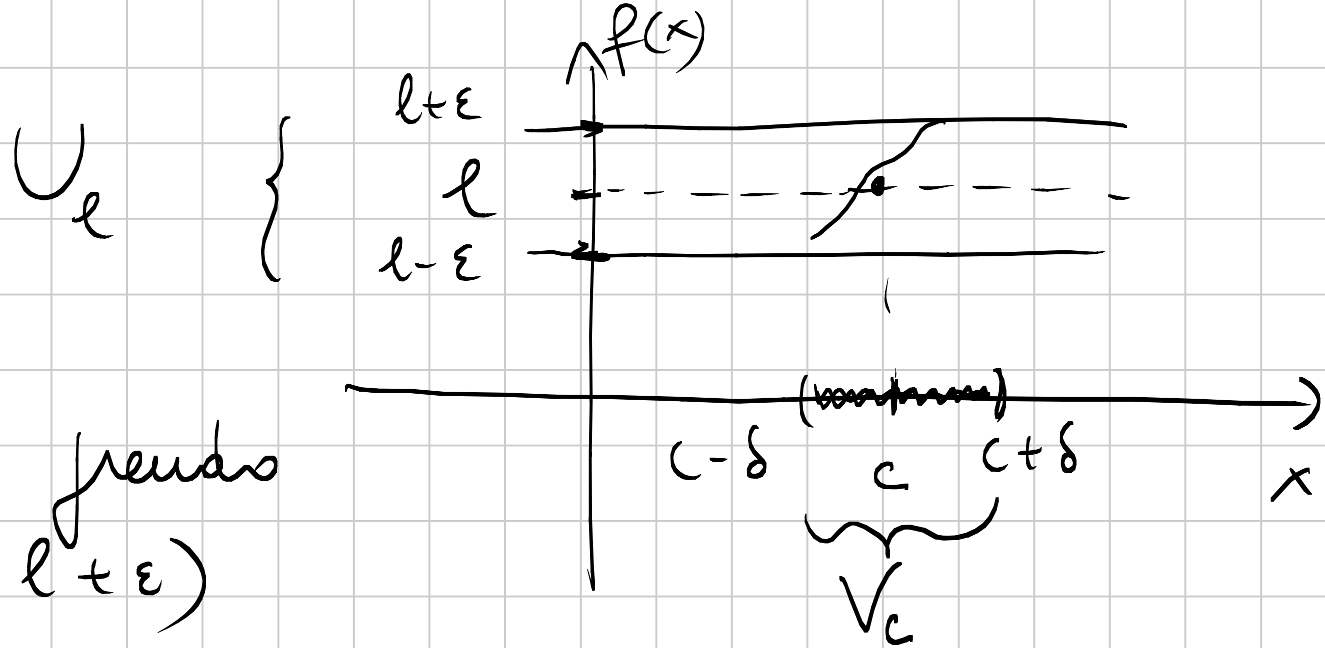
$\lim_{x \to c} f(x) = \ell \in \mathbb{R}^*$ significa

$\forall\, U_\ell$ intorno di ℓ esiste V_c intorno di c t.c.

$f(x) \in U_\ell$, $\forall\, x \in V_c$, $x \neq c$

1° caso

$c, \ell \in \mathbb{R}$ $\qquad \lim_{x \to c} f(x) = \ell$



In questo caso prendo

$$U_\ell = (\ell-\varepsilon, \ell+\varepsilon)$$

$$V_c = (c-\delta, c+\delta)$$

Quindi la definizione di limite in questo caso diventa

$l, c \in \mathbb{R}$

$$\lim_{x \to c} f(x) = l \iff \forall \varepsilon > 0 \quad \exists\, \delta > 0 \text{ t.c.}$$

$$|f(x) - l| < \varepsilon, \quad \forall x \quad |x - c| < \delta, \quad x \neq c$$

oss. 1

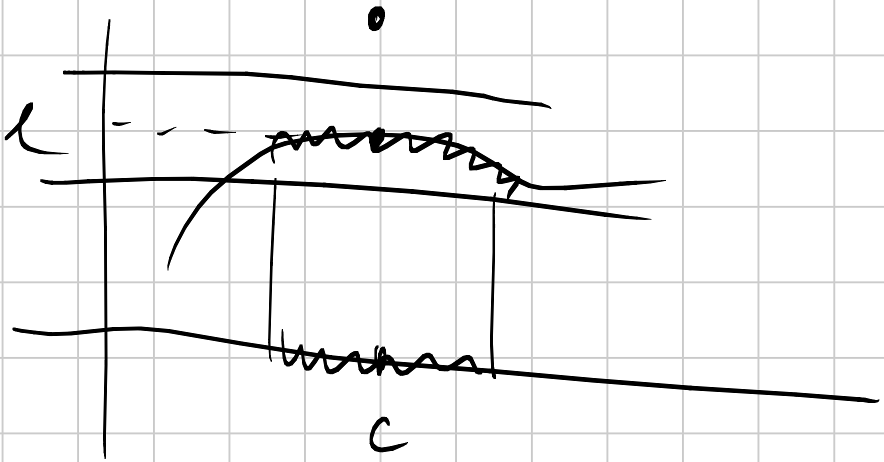
l'intorno di c dipende dalla scelta dell'intorno di l, cioè δ dipende da ε.

oss. 2 $\quad \forall x \in V_c, \quad x \neq c$

non mi interessa quanto vale la $f(x)$ in $x = c$

non voglio vedere se $|f(c) - l| < \varepsilon$

es.



es. $\lim_{x \to 0} x^2 = 0$

$c = 0$

$l = 0$

x^2 in $x = 0$ vale 0

$\Updownarrow$

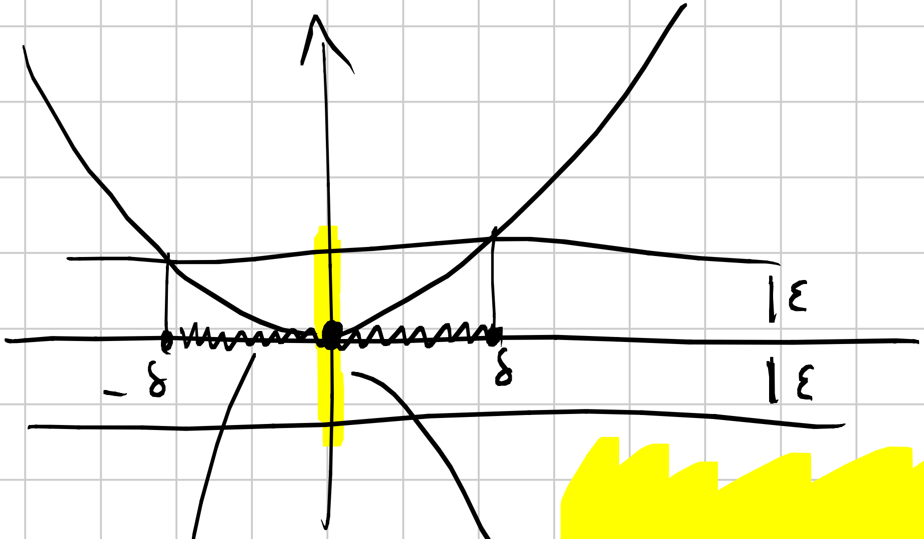
$\forall \varepsilon > 0 \ \exists \delta > 0$ t.c. $|x^2 - 0| < \varepsilon$ se $|x - 0| < \delta$
$x \neq 0$

$\forall \varepsilon > 0 \ \exists \delta > 0$ t.c. $x^2 < \varepsilon$ se $|x| < \delta$, $x \neq 0$.

$x^2 < \varepsilon \Rightarrow |x| < \sqrt{\varepsilon} = \delta$ prendo come δ

$$\delta = \sqrt{\varepsilon}$$

$$\lim_{x \to 0} x^2 = 0$$



l'intorno V_c

limite sbagliato

$c = 0$

$\ell = 1$

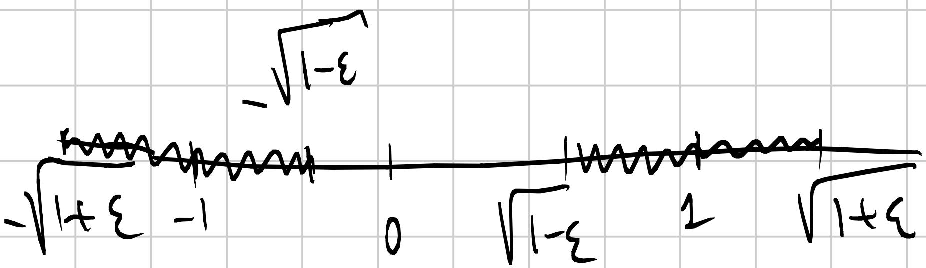
$$\lim_{x \to 0} x^2 = 1$$

$$\forall \varepsilon > 0 \ \exists \delta : \underbrace{|x^2 - 1| < \varepsilon}, \quad |x| < \delta$$

$$\Downarrow$$

$$1 - \varepsilon < x^2 < 1 + \varepsilon \iff \sqrt{1-\varepsilon} < x < \sqrt{1+\varepsilon}$$

$$-\sqrt{1+\varepsilon} < x < -\sqrt{1-\varepsilon}$$



non sono intorni di $x = 0$

Non riesco a trovare un intorno di $x=0$ e quindi un δ, nella def. di limite.

es. $\lim_{x \to 0} \operatorname{sen} x = 0$

$c=0$

$l=0$

$$\forall \varepsilon>0 \quad \exists \delta>0 : \quad |\operatorname{sen} x| < \varepsilon, \quad \forall\, |x|<\delta$$

$$-\varepsilon < \operatorname{sen} x < \varepsilon$$

$$\operatorname{arcsen}(-\varepsilon) < x < \operatorname{arcsen} \varepsilon$$

poiché siamo vicini a $x=0$

($\operatorname{arcsen} x$ è dispari)

$$-\arcsin(\varepsilon) < x < \arcsin \varepsilon$$

prendo $\delta = \arcsin \varepsilon$

$$-\delta < x < \delta \Rightarrow |x| < \delta$$

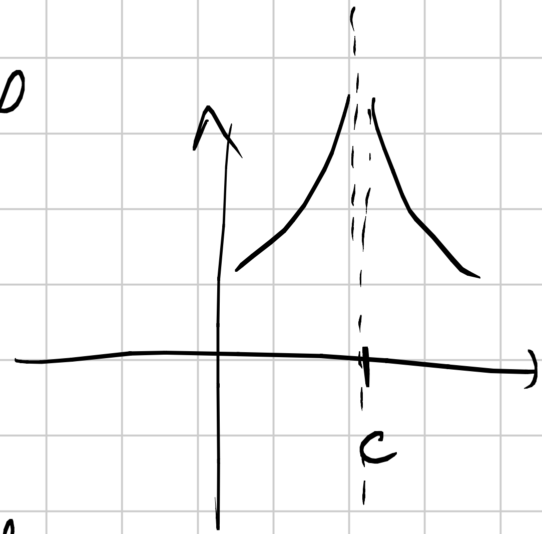
$$\lim_{x \to 0} x^2 = 0 = f(0)$$ x^2 calcolato in $x=0$

$$\lim_{x \to 0} \operatorname{sen} x = 0 = \operatorname{sen} 0$$

dico che le funzioni x^2 e $\operatorname{sen} x$ sono funzioni <u>continue</u> in $x=0$

perché $\lim\limits_{x \to c} f(x) = f(c)$

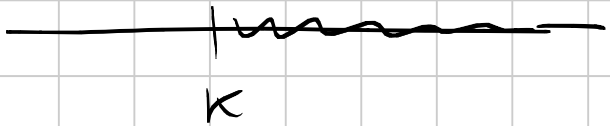
Caso 2 $\quad \lim\limits_{x \to c} f(x) = +\infty$



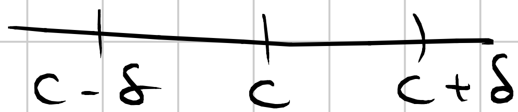
$c \in \mathbb{R}$, $\ell = +\infty$

si dice che $f(x)$ ha asintoto verticale in $x = c$

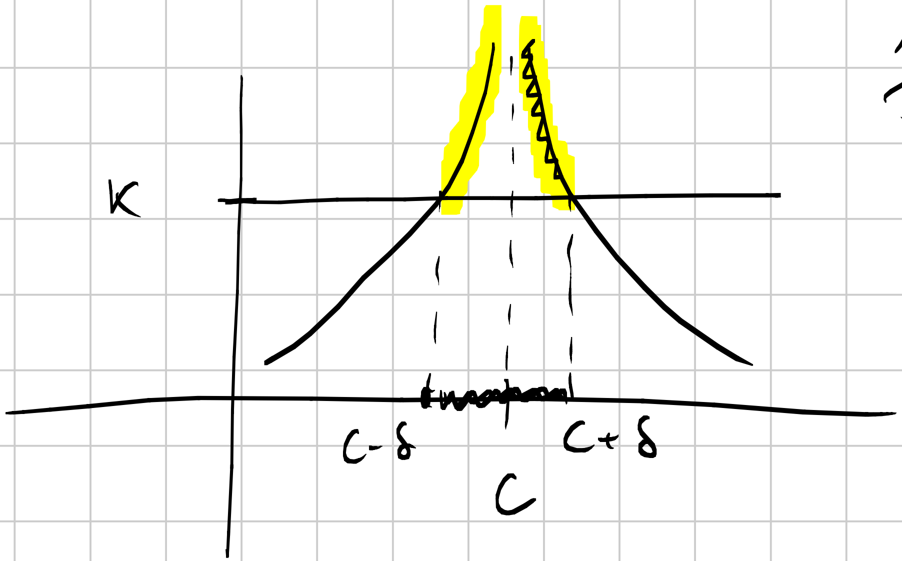
$U_\ell = U_{+\infty} = (K, +\infty)$



$V_c = (c-\delta, \; c+\delta)$



$$\lim_{x\to c} f(x) = +\infty \iff \forall K>0 \quad \exists\, \delta \text{ t.c. } f(x) > K, \ \forall x: |x-c|<\delta, \ x \neq c$$



es. $$\lim_{x\to 1} \frac{1}{(x-1)^2} = +\infty$$

$$\iff \forall K>0 \quad \exists\, \delta \text{ t.c. } \frac{1}{(x-1)^2} > K \ : |x-1|<\delta$$

$$(x-1)^2 < \frac{1}{K} \Rightarrow \underbrace{|x-1|}_{\text{fuendo}} < \frac{1}{\sqrt{K}} = \delta$$

fuendo $\delta = 1/\sqrt{K}$

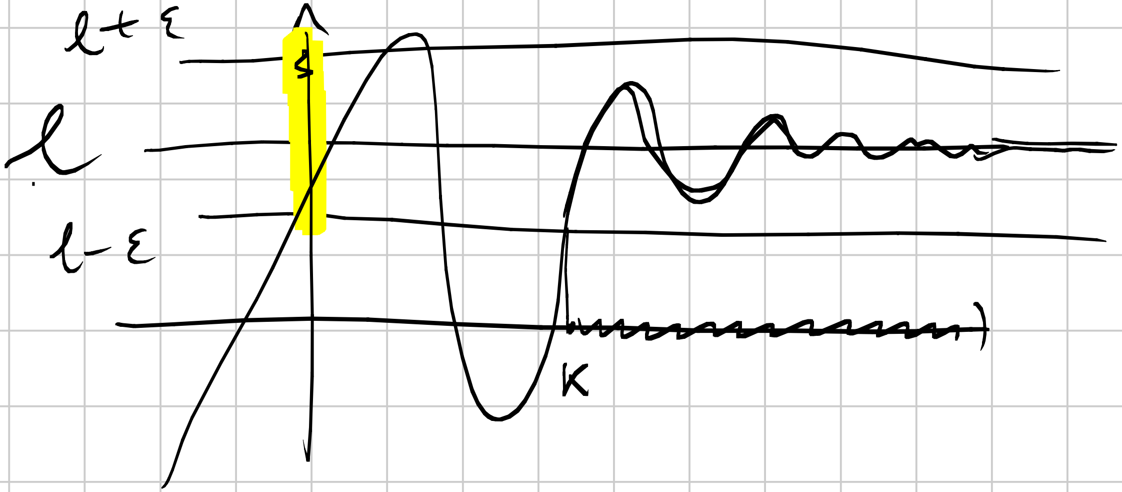
PC. $\lim_{x \to c} f(x) = -\infty$

Caso 3 $\lim_{x \to +\infty} f(x) = l$

$c = +\infty$
$l \in \mathbb{R}$

$$U_l = (l-\varepsilon,\ l+\varepsilon)$$
$$V_{+\infty} = (K,\ +\infty)$$

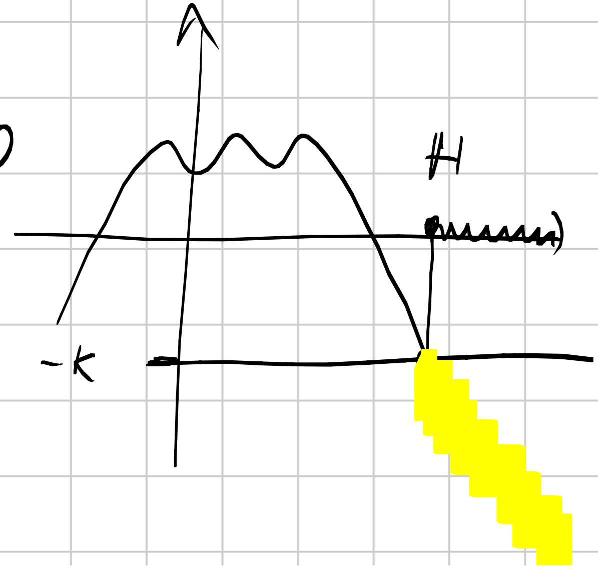
$$\lim_{x \to +\infty} f(x) = \ell \iff \forall \varepsilon > 0 \ \exists K > 0 \text{ t.c. } |f(x) - \ell| < \varepsilon, \ \forall x > K$$



$f(x)$ ha un ASINTOTO orizzontale

Caso 4

$$\lim_{x \to +\infty} f(x) = -\infty$$



$c = +\infty \qquad V_c = (H, +\infty)$

$\ell = -\infty \qquad U_\ell = (-\infty, -K)$

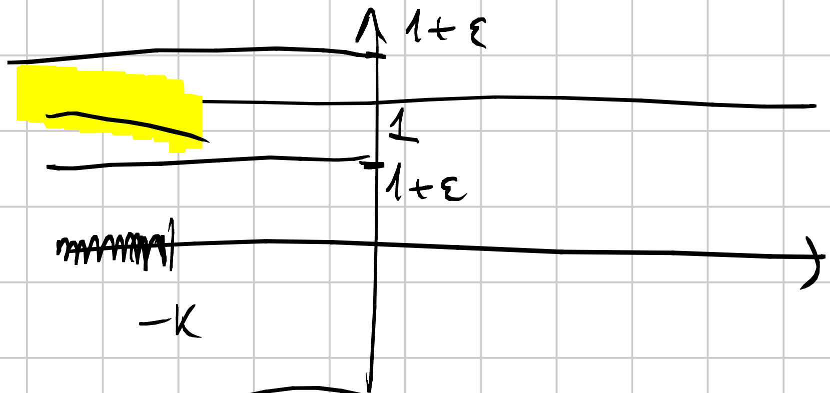
$$\lim_{x \to +\infty} f(x) = -\infty \iff \forall K > 0 \ \exists H > 0 \ \text{t.c.}$$
$$f(x) < -K, \quad \forall x > H$$

PC.

$$\lim_{x \to -\infty} f(x) = \ell$$

$$\lim_{x \to -\infty} f(x) = -\infty$$
$$= +\infty$$

es. $\lim_{x \to -\infty} 2^{1/x} = 1$



asintoto orizzontale per $x \to -\infty$

$\forall \varepsilon > 0 \quad \exists K > 0$ t.c. $\left| 2^{1/x} - 1 \right| < \varepsilon$

$\forall x < -K$

$1 - \varepsilon < 2^{1/x} < 1 + \varepsilon$

prendo $x < 0$ (perché faccio il limite $x \to -\infty$)

$\downarrow$ se $x<0$ $\quad\searrow$ vero $\forall x<0$

$$2^{1/x} < 1 < 1+\varepsilon$$

$$2^{1/x} > 1-\varepsilon \qquad \frac{1}{x} > \underbrace{\log_2(1-\varepsilon)}_{<0}$$

$\Downarrow$

$\varepsilon \geq 1$ sempre vero , se $\varepsilon < 1$

<0

$$x < \frac{1}{\log_2(1-\varepsilon)}$$

<0

$$\frac{1}{\log_2(1-\varepsilon)} = -K$$

è il mio $-K$