Seconda parte dell 4/11

a_n, b_n infinite

$$\frac{a_n}{b_n} \longrightarrow 1$$

a_n e b_n sono infiniti dello stesso ordine

In questo caso si dice che a_n e b_n sono asintotiche e $a_n \sim b_n$

$$a_n = n^2 + \frac{3}{n} + n$$

$$b_n = n^2$$

Alcune proprietà

- se $a_n \sim b_n \Rightarrow$ le due successioni hanno lo stesso comportamento

- se $a_n \sim a'_n$, $b_n \sim b'_n$ $\Rightarrow$ $a_n \cdot b_n \sim a'_n \cdot b'_n$

  ma non $a_n + b_n \sim a'_n + b'_n$

- se $\dfrac{a_n}{b_n} \to \ell \neq 0 \Rightarrow \dfrac{a_n}{\ell b_n} \to 1$

  a_n e b_n non sono asintotiche ma lo sono a_n e ℓb_n

Esercizi

$$\lim_{n} \frac{\log n + n^2 + 5^n}{n! + n^{100}} = \frac{5^n\left(1 + \frac{\log n}{5^n} + \frac{n^2}{5^n}\right)}{n!\left(1 + \frac{n^{100}}{n!}\right)} = 0$$

$$\lim_{n} \left(\frac{n! + e^{5n}}{e^{6n} - n \operatorname{sen} n} \right) \qquad e^{5n} = (e^{5})^{n}$$

$$\frac{n!\left(1 + \frac{(e^{5})^{n}}{n!}\right)}{(e^{6})^{n}\left(1 - \frac{n \operatorname{sen} n}{(e^{6})^{n}}\right)} \longrightarrow +\infty$$

$$\frac{n \operatorname{sen} n}{(e^{6})^{n}} \longrightarrow 0$$

$$\lim_{n} \frac{\log\left(1+n^{2}\right)}{3^{n+1}} = \frac{\log\left(n^{2}\left(1+\frac{1}{n^{2}}\right)\right)}{3\cdot 3^{n}} =$$

$$= \frac{\log n^{2} + \log\left(1+\frac{1}{n^{2}}\right)}{3\cdot 3^{n}} = \underbrace{\frac{2\log n}{3\cdot 3^{n}}}_{\downarrow \atop 0} + \underbrace{\frac{\log\left(1+\frac{1}{n^{2}}\right)}{3\cdot 3^{n}}}_{\downarrow \atop 0}$$

$$\lim_{n} \log\left(1+e^{n}\right) - n \qquad \infty - \infty$$

$$\log\left(1+e^{n}\right) - n = \log\left(e^{n}\left(1+\frac{1}{e^{n}}\right)\right) - n =$$

$$= \log e^{n} + \log\left(1+\frac{1}{e^{n}}\right) - n = \cancel{n} + \lg\left(1+\frac{1}{e^{n}}\right) \cancel{-n}$$

$$\downarrow 0$$

$$\longrightarrow 0$$

. Calcolare al variare del parametro $a > 0$

$$\lim_{n} \frac{a^{n+1} + n^2 + (-1)^{n+1}}{\pi^n - 2n^3 - 2\operatorname{sen} n}$$

1) $a > 1$

$$\frac{a^{n+1}\left(1 + \frac{n^2}{a^{n+1}} + \frac{(-1)^{n+1}}{a^{n+1}}\right)}{\pi^n\left(1 - \frac{2n^3}{\pi^n} - \frac{2\operatorname{sen} n}{\pi^n}\right)} =$$

$$= \frac{a\, a^n}{\pi^n} \frac{(1 + \dots)}{(1 + \dots)} = a\left(\frac{a}{\pi}\right)^n \cdot \frac{(1 + \dots)}{(1 + \dots)}$$

$$\lim_n \left(\qquad \right) = \lim_n a\left(\frac{a}{\pi}\right)^n \frac{(1+\dots)}{(1+\dots)} =$$

$$= \begin{cases} +\infty & a > \pi \\ 0 & 1 < a < \pi \\ \pi & a = \pi \end{cases}$$

$\left(\frac{a}{\pi}\right)^n \longrightarrow +\infty$

$\frac{a}{\pi} > 1$

$0 < \left(\frac{a}{\pi}\right) < 1$

Riassunto

$$\lim_n \left(\frac{a^{n+1} + n^2 + (-1)^{n+1}}{\pi^n - 2n^3 - 2\operatorname{sen} n} \right)$$

$a \leq 1$

$$= \frac{n^2 \left(\frac{a^{n+1}}{n^2} + 1 + \frac{(-1)^{n+1}}{n^2} \right)}{\pi^n \left(1 - \frac{2n^3}{\pi^n} - \frac{2\operatorname{sen} n}{\pi^n} \right)} \to 0$$

Quindi

$$\lim_n \left(\qquad \right) = \begin{cases} 0 & a \leq 1 \\ 0 & 1 < a < \pi \\ \pi & a = \pi \\ +\infty & a > \pi \end{cases}$$

Es.

$$\lim_{n} \frac{2^n}{e^{n^2}}$$

se fosse stato

$$\frac{2^n}{e^n} = \left(\frac{2}{e}\right)^n \to 0$$

< 1

$$\frac{2^n}{e^{n \cdot n}} = \frac{2^n}{(e^n)^n} = \left(\frac{2}{e^n}\right)^n$$

0^∞

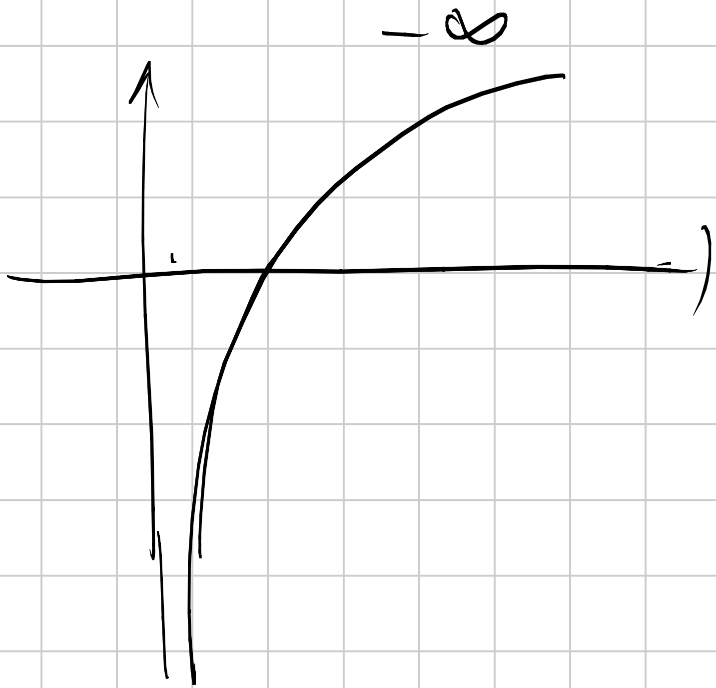
$$(a_n)^{b_n} = e^{\log (a_n)^{b_n}} = e^{b_n \log a_n}$$

$$x = e^{\log x}$$

$$\left(\frac{2}{e^n}\right)^n = e^{\log\left(\frac{2}{e^n}\right)^n} = e^{\underbrace{\overset{+\infty}{n}\,\overset{-\infty}{\log\left(\frac{2}{e^n}\right)}}_{-\infty}} \to 0$$

$$\left(\frac{2}{e^n}\right)^n \longrightarrow 0$$

$$n \underbrace{\log\left(\frac{2}{e^n}\right)}_{}$$



$$\left(\frac{2}{e^{n}}\right)^{n}$$

$$= \lim_{n} \frac{e^{2n}}{e^{n^2}} = \lim_{n} \frac{(e^2)^n}{(e^n)^n} =$$

$$= \lim_{n} \left(\frac{e^2}{e^n}\right)^n = \lim_{n} \left(\frac{1}{e^{n-2}}\right)^n =$$

$$= \lim_{n} \frac{1}{(e^{n-2})^n} = 0$$

si poteva anche fare così;

$$\lim \frac{e^{2n}}{e^{n^2}} = \lim_{n} e^{2n-n^2}$$

$$\lim_n 2n - n^2 = \lim_n -n^2\left(\frac{2n}{-n^2} + 1\right) = -\infty$$

$$e^{2n-n^2} \longrightarrow 0$$

Es. $$\lim_n \frac{4e^n - (\cosh n)^2}{\cos n + \underbrace{n^{3n} + e^{n^2}}_{?}}$$

$$\frac{n^{3n}}{e^{n \cdot n}} = \left(\frac{n^3}{e^n}\right)^n \longrightarrow 0$$

$$\underbrace{n \log\left(\frac{n^3}{e^n}\right)}_{\to -\infty}$$

<u>Denom.</u> $\cos n + n^{3n} + e^{n^2} = e^{n^2}\left(1 + \frac{\cos n}{e^{n^2}} + \frac{n^{3n}}{e^{n^2}}\right)$

<u>Numer.</u> $4e^{n} - (\cosh n)^2$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$(\cosh x)^2 = \frac{(e^x + e^{-x})^2}{4} = \frac{e^{2x} + e^{-2x} + 2}{4}$$

$$4e^{n} - (\cosh n)^2 = 4e^{n} - \frac{e^{2n} + e^{-2n} + 2}{4} =$$

$$= \frac{16e^{n} - e^{2n} - e^{-2n} - 2}{4} =$$

$$= -e^{2n}\left(-\frac{16}{e^{n}} + 1 + \frac{1}{e^{4n}} + \frac{2}{e^{2n}}\right)$$

$$\lim_{n} \frac{4e^{n} - (\cosh n)^2}{\cos n + n^{3n} + e^{n^2}} = \lim_{n} \frac{-e^{2n}\,\underbrace{(1 + \dots)}_{\to 1}}{e^{n^2}\,\underbrace{(1 + \dots)}_{\to 1}}$$

$= 0$

$\downarrow 0$ (vedi es. precedente)

b. $\lim \dfrac{n^{\sqrt{n}}}{\left(\sqrt{n}\right)^{n}}$ $\qquad n = \sqrt{n} \cdot \sqrt{n}$

$$\frac{n^{\sqrt{n}}}{\left(\sqrt{n}\right)^{\sqrt{n}\cdot\sqrt{n}}} = \left(\frac{n}{\left(\sqrt{n}\right)^{\sqrt{n}}}\right)^{\sqrt{n}} = \left(\frac{\left(\sqrt{n}\right)^{2}}{\left(\sqrt{n}\right)^{\sqrt{n}}}\right)^{\sqrt{n}} =$$

$$= \left(\frac{1}{\left(\sqrt{n}\right)^{\sqrt{n}-2}}\right)^{\sqrt{n}} = \frac{1}{\left(\left(\sqrt{n}\right)^{\sqrt{n}-2}\right)^{\sqrt{n}}} \to 0$$

PC. $\lim_{n} \dfrac{n^{\sqrt{n}} - \sqrt{n} + 2^{n}}{(\sqrt{n})^{n} + n^{3} - 3^{n}}$ $\left(\overset{R.}{=} 0\right)$

PC. $\lim_{n} \alpha n - \sqrt{n^2+3}$

$\alpha \in \mathbb{R}$

$\alpha < 0$ $\quad \lim = -\infty$

$\alpha = 0$ $\quad \lim = -\infty$

$$\alpha n - \sqrt{n^2+3} = \frac{\left(\alpha n - \sqrt{n^2+3}\right)\left(\alpha n + \sqrt{n^2+3}\right)}{\alpha n + \sqrt{n^2+3}} =$$

$$= \frac{\alpha^2 n^2 - n^2 - 3}{\alpha n + \sqrt{n^2+3}} = \frac{n^2(\alpha^2-1) - 3}{\alpha n + \sqrt{n^2+3}}$$