Serie numeriche

$$\sum_{k=0}^{+\infty} a_k \qquad\qquad a_k$$

$$\sum_{n=0}^{+\infty} a_n \qquad \lim_n S_n$$

$\exists$ finito

$\exists \pm\infty$

$\nexists$

S_n = ridotte, somme parziali

$$S_n = \sum_{k=0}^{n} a_k$$

$$\sum_{k=0}^{+\infty} q^k \qquad q$$

$S_n = ?$

$$\sum_{k=1}^{+\infty} \frac{1}{k^\alpha} \begin{cases} \alpha > 1 & \text{converge} \\ \alpha \leq 1 & \text{diverge} \end{cases}$$

se $\sum a_k$ converge $\Rightarrow \lim_k a_k = 0$

se $\lim a_k \neq 0 \Rightarrow \sum a_k$ non converge.

Serie a termini positivi

$$\sum_{k=1}^{+\infty} a_k \quad , \quad a_k > 0$$

$$S_{n+1} = S_n + \underbrace{a_{k+1}}_{} > S_n$$

S_n è monotona crescente

$\lim_n S_n$ $\exists$ sempre $\begin{cases} \text{finito} \\ \text{infinito } +\infty \end{cases}$

1) Una serie a termini positivi o converge o diverge (a $+\infty$)

Es. $\sum_{k=1}^{+\infty} 2^{1/k}$

$a_k = 2^{1/k} > 0$

$\lim_k 2^{1/k} = 1 \neq 0 \implies$ la serie diverge a $+\infty$

$S_n = ?$

oss tutti questi risultati valgono bene anche se la serie è a termini negativi

Es $\sum \left(-2^{1/k}\right) \rightarrow -\sum 2^{1/k}$ $\quad a_k < 0$

$$\sum \lambda a_k = \lambda \sum a_k$$

Es. $\sum \frac{(-1)^k}{k}$ serie di segno variabile

Criterio del <u>confronto</u>

$0 \leq a_k \leq b_k$

1) $\sum b_k$ converge $\Rightarrow$ $\sum a_k$ converge

2) $\sum a_k$ diverge $\Rightarrow$ $\sum b_k$ diverge

<u>Dim.</u> 1) $t_n = \sum_{k=0}^{n} b_k$ successione delle ridotte

per ipotesi $\exists$ finito $\lim t_n$

$S_n = \sum_{k=0}^{n} a_k$ dobbiamo dim. che $\exists$ finito $\lim S_n$

$0 \leq S_n \leq t_n$ $\exists$ finito $\lim t_n$

$\exists$ limite S_n

posso dire che anche

$\lim_{n} s_n$ è <u>finito</u>!

$\Rightarrow \sum a_k$ converge.

$(s_n \to +\infty \Rightarrow t_n \to +\infty)$.

<u>Es.</u> $\displaystyle\sum_{k=1}^{\infty} \frac{(\operatorname{sen} k)^2}{k^2}$ serie a termini positivi $a_k \geq 0$

$$a_k = \frac{(\operatorname{sen} k)^2}{k^2} \xrightarrow[k \to +\infty]{} 0$$

$$a_k \leq \frac{1}{k^2} \; b_k$$

$\sum b_k$ converge $\Rightarrow$ $\sum a_k$ converge

$\sum \frac{1}{k^2}$ <u>converge</u> $\qquad \sum \frac{1}{k^\alpha}$

$\Rightarrow \sum a_k$ <u>converge</u> !

<u>oss.</u> Se $a_k \le b_k$

e $\sum b_k$ diverge $\Rightarrow$ non so niente su $\sum a_k$!

$a_k \le b_k$

oss. Basta che valga
$a_k \leq b_k$ da un certo k in poi.

Es. (caso 2) $\displaystyle \sum \underbrace{\frac{(\cos k + 2)^2}{\sqrt{k}}}_{b_k}$

$$b_k \geq \underbrace{\frac{1}{\sqrt{k}}}_{a_k} \qquad \cos k + 2 \geq 1$$

$\displaystyle \sum \frac{1}{\sqrt{k}}$ diverge

$\Longrightarrow \sum b_k$ diverge

Il criterio del confronto non si applica facilmente

$\sum \frac{1}{k^2}$ converge $a_k \to 0$

$\sum \frac{1}{\sqrt{k}}$ diverge $a_k \to 0$

$\frac{1}{k^2}$

Criterio del <u>confronto asintotico</u>

$a_k, b_k \geq 0$

1) $\lim_{k \to +\infty} \frac{a_k}{b_k} = L \ (\neq 0, \neq \pm\infty)$ allora

$\sum a_k$ e $\sum b_k$ hanno lo stesso carattere.

2) $\lim_{k \to +\infty} \frac{a_k}{b_k} = 0$ $\qquad \left(a_k \underset{k \to +\infty}{=} o(b_k) \right)$

se $\sum b_k$ converge $\Rightarrow \sum a_k$ converge

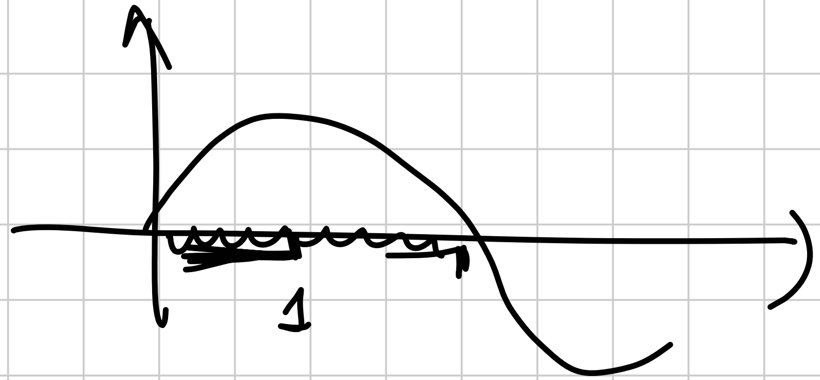
$\sum a_k$ diverge $\Rightarrow \sum b_k$ diverge

<u>Dim.</u> no!

Es.

$$\sum_{k=1}^{\infty} \underbrace{\operatorname{sen}\left(\frac{1}{k^2}\right)}_{a_k}$$

$a_k \to 0$ è verificata la condizione necessaria.



$\operatorname{sen}\left(\frac{1}{k^2}\right) > 0 \quad \forall k$

$k \to +\infty$

$\operatorname{sen} x \quad x \to 0$

$\frac{1}{k^2} = x \to 0$

$\frac{\operatorname{sen} x}{x} \to 1$
$x \to 0$

$\operatorname{sen} x \sim x$
$x \to 0$

$\sin \frac{1}{k^2} \sim \frac{1}{k^2}$

$$\frac{\overbrace{\sin \frac{1}{k^2}}^{a_k}}{\underbrace{\frac{1}{k^2}}_{b_k}} \to 1$$

$\sum \underbrace{\sin \frac{1}{k^2}}_{a_k}$ è convergente!

$\sum \underbrace{\frac{1}{k^2}}_{b_k}$ serie convergente

Es. $\sum \left(\cos\frac{1}{n^2} - 1\right) = \sum -\left(1-\cos\frac{1}{n^2}\right) =$

$= -\sum \underbrace{\left(1-\cos\frac{1}{n^2}\right)}_{\geq 0}$

$a_n \to 0$

$a_n \leq 0$

$-a_n = 1-\cos\frac{1}{n^2}$

$n \to +\infty$ $\quad \frac{1}{n^2} = x \quad \frac{1-\cos x}{x^2} \xrightarrow[x\to 0]{} \frac{1}{2}$

$x^2 = \frac{1}{n^4} \quad \frac{1-\cos\frac{1}{n^2}}{\frac{1}{n^4}} \xrightarrow[n\to\infty]{} 1/2$

$\sum \left(1 - \cos \frac{1}{n^2}\right)$ e $\sum \frac{1}{n^4}$ hanno lo stesso carattere!

$\Downarrow$ converge (da $\sum \frac{1}{n^4}$ converge)

<u>OSS.</u> Ho sempre confrontato a_k con $\frac{1}{k^\alpha}$ $\Rightarrow$ $\frac{a_k}{\left(\frac{1}{k}\right)^\alpha}$

α è l'ordine d'infinitesimo di a_k.

Scrivo il confronto asintotico con $b_k = \frac{1}{k^\alpha}$

$$\lim_{k} \frac{a_k}{\frac{1}{k^\alpha}} = l \begin{cases} l \neq +\infty, \ \alpha > 1 \Rightarrow \sum a_k \text{ converge} \\ l \neq 0, \ \alpha \leq 1 \Rightarrow \sum a_k \text{ diverge } (+\infty). \end{cases}$$

<u>Es.</u> $\sum \log\left(1 + \frac{1}{n}\right)$

$a_n \to 0$ $\quad \frac{1}{n} = x \to 0$ $\quad \log(1+x) \underset{x \to 0}{\sim} x$

$\log\left(1 + \frac{1}{n}\right) \sim \frac{1}{n}$ $\quad n \to +\infty$

$$\sum \log\left(1+\frac{1}{n}\right) \qquad \sum \frac{1}{n}$$

$\Downarrow$ diverge

<u>Es.</u> $\displaystyle\sum \frac{1}{e^k}$

$a_k \to 0$

$$\frac{1}{e^k} = o\left(\frac{1}{k^\alpha}\right) \quad \forall \alpha$$

$$\lim_{k \to +\infty} \frac{\frac{1}{e^k}}{\frac{1}{k^\alpha}} = \lim_{k \to +\infty} \frac{k^\alpha}{e^k} = 0$$

$b_k = \frac{1}{k^2}$ $\qquad \frac{1}{e^k} = o\left(\frac{1}{k^2}\right)$

$\sum \frac{1}{k^2}$ converge $\Rightarrow \sum \frac{1}{e^k}$ converge

Es. $\sum_n \underbrace{\frac{n}{n^2+1}}_{a_n}$ $\qquad a_n > 0$

1) $a_n \longrightarrow 0$

2) $a_n = \frac{\cancel{n}}{n^{\cancel{2}}\left(1+\frac{1}{n^2}\right)} = \frac{1}{n\left(1+\frac{1}{n^2}\right)} \sim \frac{1}{n}$

$\frac{\cancel{n}}{\cancel{n}\left(1+\frac{1}{n^2}\right)} \longrightarrow 1$

$\sum a_n$ $\qquad \sum \frac{1}{n}$

diverge!

Es.

$$\sum \frac{e^n + 3^n}{n^3 2^n + 2^{2n}} \qquad a_n > 0$$

$$a_n$$

$$\lim_n \frac{e^n + 3^n}{n^3 2^n + 2^{2n}} = \lim_n \frac{3^n\left(\left(\frac{e}{3}\right)^n + 1\right)}{2^{2n}\left(\frac{n^3 \cancel{2^n}}{\cancel{(2^n)^2}} + 1\right)}$$

$$= \lim \frac{3^n}{4^n} = \lim_n \left(\frac{3}{4}\right)^n \longrightarrow 0$$

$a_n \sim \left(\frac{3}{4}\right)^n \quad n \to +\infty$

$\sum a_n \qquad \sum \left(\frac{3}{4}\right)^n$ serie geometrica

$q = \frac{3}{4}$

converge $\Rightarrow \sum a_n$ convergente

ma la somma è diversa da

$\sum \left(\frac{3}{4}\right)^n$.

Es. $\sum \dfrac{n + e^{-n} + \operatorname{sen} n + \log n}{n^5}$

$$a_n = \frac{n + e^{-n} + \operatorname{sen} n + \log n}{n^5} \quad \begin{matrix} \to \infty \\ \to \infty \end{matrix}$$

$a_n > 0$

$$\frac{\cancel{n}\left(1 + \frac{e^{-n}}{n} + \frac{\operatorname{sen} n}{n} + \frac{\log n}{n}\right)}{n^{\cancel{5}4}} \sim \frac{1}{n^4}$$

$a_n \sim \dfrac{1}{n^4}$ $\quad n \to \infty$

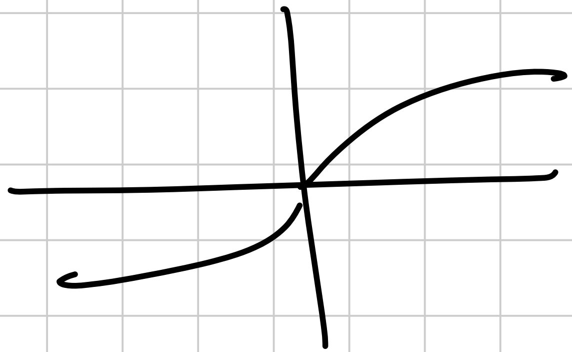
$\underbrace{\sum a_n}$ $\quad \underbrace{\sum \frac{1}{n^4}}$

converge!

<u>Es.</u> $\sum n^{\alpha} \operatorname{arctg} \frac{1}{n^3}$

dire per quali α la serie converge

$\operatorname{arctg} x$



$n \to \infty \quad \frac{1}{n^3} \to 0$

$x = \frac{1}{n^3} \to 0$

$$\lim_{x \to 0} \frac{\operatorname{arctg} x}{x} = 1$$

$\operatorname{arctg} x \sim x$
$x \to 0$

$\operatorname{arctg} \frac{1}{n^3} \sim \frac{1}{n^3} \quad n \to +\infty$

$$a_n = n^\alpha \operatorname{arctg} \frac{1}{n^3} \sim n^\alpha \frac{1}{n^3} = \frac{1}{n^{3-\alpha}}$$

$\sum \frac{1}{n^{3-\alpha}}$ converge $\iff$ $3-\alpha > 1$

$$\Downarrow$$

$$\alpha < 2$$

$\sum a_n$ converge $\forall\, \alpha < 2$

Criterio del <u>rapporto</u> e della <u>radice</u>

$\sum a_k \qquad a_k \geq 0$

$$\lim_{k} \frac{a_{k+1}}{a_k} = \ell$$

- $\ell < 1$ la serie è CONVERGENTE
- $\ell > 1$ la serie è divergente
- $\underline{\underline{\ell = 1}}$ non si può dire nulla

<u>es.</u> $\sum \frac{1}{k!} \qquad a_k \to 0 \qquad a_k = \frac{1}{k!}$

$$\frac{a_{k+1}}{a_k} = \frac{1}{(k+1)!} \cdot k! = \frac{\cancel{k!}}{\cancel{k!}(k+1)} \to 0$$

$\ell = 0 < 1$

$\Rightarrow \sum \frac{1}{k!}$ <u>converge</u>!

<u>es. 2</u> $\sum \frac{1}{k!}$

$\frac{1}{k!} = o\left(\frac{1}{k^2}\right)$ (verificare)
$k \to +\infty$

$\sum \frac{1}{k^2}$ converge $\Rightarrow \sum \frac{1}{k!}$ converge

Es. $\underbrace{\sum \frac{1}{n^2}}$

$$a_n = \frac{1}{n^2}$$

$$\frac{a_{n+1}}{a_n} = \frac{1}{(n+1)^2} n^2 \xrightarrow[n]{} 1 = \ell$$

es. $\sum \frac{n!}{n^n}$ $\qquad a_n = \frac{n!}{n^n}$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{(n+1)}} \cdot \frac{n^n}{n!} = \frac{\cancel{n!}\,\cancel{(n+1)}\; n^n}{(n+1)^n\,\cancel{(n+1)}\cdot\cancel{n!}} =$$

$$= \left(\frac{n}{n+1}\right)^n = \frac{1}{\left(1+\frac{1}{n}\right)^n} \to \frac{1}{e} < 1$$

$\ell < 1$

(elle)

$\Rightarrow$

$\sum \frac{n!}{n^n}$ converge !

Criterio della <u>radice</u>

$\sum a_k \quad , \quad a_k \geq 0$

$$\lim_k \sqrt[k]{a_k} = \ell \begin{cases} \ell < 1 & \sum a_k \text{ CONV.} \\ \ell > 1 & \sum a_k \text{ DIV.} \\ \ell = 1 & ? \end{cases}$$

$$\left(\lim_k \sqrt[k]{a_k} = \lim \frac{a_{k+1}}{a_k} \right)$$

<u>Es.</u> $\sum \frac{1}{k^k}$ $\qquad a_k = \frac{1}{k^k}$

$$\sqrt[k]{a_k} = \frac{1}{k} \to 0 < 1$$

$\Rightarrow \sum \frac{1}{k^k}$ converge!

(provare anche con il rapporto!).

<u>Es.</u> Dire per quali $b \geq 0$ la serie converge

$$\sum \frac{b^n}{n^n}$$

↓ altrimenti la serie non ha termini tutti dello stesso segno.

radice

$$\lim_{n} \sqrt[n]{a_n} = \lim_{n} \frac{b}{n} = 0 < 1$$

$\Rightarrow$ la serie <u>converge</u>!

es. $\sum 3^n \left(1 - \frac{1}{n^{3/2}}\right)^n$

$$\sqrt[n]{a_n} = 3\left(1 - \frac{1}{n^{3/2}}\right) \to 3 > 1$$

$\Rightarrow$ la serie diverge!

Es. $\sum \left(1 - \frac{1}{2n}\right)^{5n^2}$

$$a_n = \left(\left(1 - \frac{1}{2n}\right)^{5n}\right)^n$$

$$\sqrt[n]{a_n} = \left(1 - \frac{1}{2n}\right)^{5n \cdot \frac{2}{2}} =$$

$$= \left[\left(1 - \frac{1}{2n}\right)^{2n}\right]^{5/2} \longrightarrow \frac{1}{e^{5/2}} < 1$$

la serie converge!

Es.

$$\sum \frac{\sqrt{n+1} - \sqrt{n}}{n} \qquad a_n \geq 0$$

$$a_n = \left(\frac{\sqrt{n+1} - \sqrt{n}}{n}\right) \frac{(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} =$$

$$= \frac{1}{n\left(\sqrt{n+1} + \sqrt{n}\right)} = \frac{1}{n\sqrt{n}\left(1 + \sqrt{1 + \frac{1}{n}}\right)}$$

$$\sim \frac{1}{n^{3/2} \cdot 2} \Rightarrow \sum \frac{1}{2\, n^{3/2}} \text{ converge}$$

$$\Rightarrow \sum a_n \text{ converge.}$$