Esercizi su limiti con sviluppi di McLaurin

- $\operatorname{sen} x = x - \frac{x^3}{3!} + o(x^3) \qquad x \to 0$

$(\operatorname{sen} x)^2 = x^2 - \frac{1}{3} X^4 + o(x^4) \qquad x \to 0$

$x \to 0$

- $\log(1 + \operatorname{sen}^2 x)$

$\log(1 + y) = y - \frac{y^2}{2} + o(y^2) \qquad y \to 0$

$\log(1 + \operatorname{sen}^2 x) = \operatorname{sen}^2 x - \frac{1}{2} \operatorname{sen}^4 x + o(\operatorname{sen}^4 x) =$

$= x^2 - \frac{1}{3} x^4 + o(x^4) - \frac{1}{2} (x + o(x))^4 + o(x^4) =$

$$\underset{x\to 0}{=} x^2 - \frac{1}{3}x^4 + o(x^4) - \frac{1}{2}x^4 \underset{x\to 0}{=} x^2 - \frac{5}{6}x^4 + o(x^4)$$

$$\log(1+\sin^2 x) - x^2 \underset{x\to 0}{=} -\frac{5}{6}x^4 + o(x^4)$$

$$\_\_\_\_\_\_ \cdot \_\_\_\_\_\_$$

$$\log(1+\cos^2 x)$$
$$x \to 0$$

$$\log\left(1+(x^2+x)\right) \qquad x \to 0$$

$$y = x^2 + x \to 0$$

$$\log(1+y) = y - \frac{y^2}{2} + o(y^2)$$

$$o\left((x^2+x)^2\right) \qquad x \to 0$$

$$\log\left(1+(x^2+x)\right) = x^2 + x - \frac{1}{2}(x^2+x)^2 + o(x^2) =$$

$$= x^2 + x - \frac{1}{2}(x^2) + o(x^2) =$$

$$= x + \frac{1}{2}x^2 + o(x^2)$$

Trovare ordine di infinitesimo di

$x \to 0$

$$\log\left(1+(x^2+x)\right) - \log(1+x) = x + \frac{1}{2}x^2 + o(x^2) -$$

$$-\left(x - \frac{1}{2}x^2 + o(x^2)\right) = x^2 + o(x^2)$$

l'ordine è 2.

$$\lim_{x\to 0^+} \frac{\operatorname{sen}(x^\alpha) - x + e^{-1/x^2}}{(\operatorname{tg} x)^2} \cdot \cos x \qquad \alpha > 0$$

$$e^{-1/x^2} = o(x^\beta) \quad \forall \beta > 0$$

$$(\operatorname{tg} x)^2 = \left(x + \frac{x^3}{3} + o(x^3)\right)^2 =$$

$$= x^2 + \frac{2}{3}x^4 + o(x^4)$$

$$x^{\alpha}\, o(x^{\beta}) = o(x^{\alpha+\beta})$$

$x^{\alpha} = y \qquad y \to 0$

$$\operatorname{sen} x^{\alpha} = x^{\alpha} - \frac{1}{6} x^{3\alpha} + o(x^{3\alpha})$$

$\operatorname{sen} y = y - \frac{1}{3!} y^3 + o(y^3)$

$$\lim_{x \to 0^+} \left(\qquad \right) = \lim_{x \to 0^+} \frac{x^{\alpha} - \frac{1}{6} x^{3\alpha} + o(x^{3\alpha}) - x + o(x^3)}{x^2 + \frac{2}{3} x^4 + o(x^4)}$$

$\forall \beta > 0$

$$\alpha > 1 \qquad = \lim_{x \to 0^+} \frac{-x + o(x)}{x^2 + o(x^2)} = -\infty$$

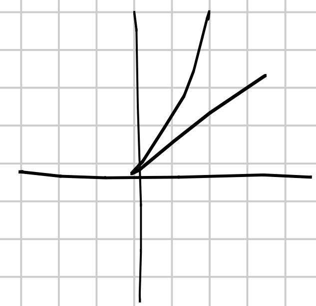
$\alpha < 1 \qquad = \lim_{x \to 0} \dfrac{x^{\alpha} + o(x^{\alpha})}{x^2 + o(x^2)} = +\infty$

$\alpha = 1 \qquad = \lim_{x \to 0} \dfrac{-\frac{1}{6}x^3 + o(x^3)}{x^2 + o(x^2)} = 0$

$$\frac{x^{\alpha} - x^2 + x^3}{\quad}$$

Si consideri

$$a_n = \frac{\frac{1}{n^\alpha}\left(\operatorname{senh}\frac{1}{n} - \frac{1}{n}\right)}{\cos\frac{1}{n}}$$



trovare l'ordine d'infinitesimo e dire per quali α la serie $\sum a_n$ converge.

$$\operatorname{senh}\left(\frac{1}{n}\right) \underset{n\to+\infty}{=} \frac{1}{n} + \frac{1}{6}\frac{1}{n^3} + o\left(\frac{1}{n^3}\right) \qquad \operatorname{senh} x = x + \frac{x^3}{6} + o(x^3)$$

N.

$$\frac{1}{n^\alpha}\left(\operatorname{senh}\frac{1}{n} - \frac{1}{n}\right) = \frac{1}{n^\alpha}\left(\frac{1}{6}\frac{1}{n^3} + o\left(\frac{1}{n^3}\right)\right) =$$

$$= \frac{1}{6}\frac{1}{n^{3+\alpha}} + o\left(\frac{1}{n^{3+\alpha}}\right)$$

$$a_n = \frac{1}{6}\frac{1}{n^{3+\alpha}} + o\left(\frac{1}{n^{3+\alpha}}\right)$$

ordine di infinitesimo è $3+\alpha$

$$\sum a_n$$

per quali α converge?

serie a termini positivi

Si usa il criterio del confronto <u>asintotico</u>

$\sum a_n$ si comporta come $\sum \frac{1}{6 n^{3+\alpha}}$

converge $3+\alpha > 1$

$\alpha > -2$ —— $\sum \frac{1}{n^{3+\alpha}}$

$$\lim_{x\to 0^+} \frac{27x^5 + \log(1+x^7)}{\sqrt{1+x^8} - 1 + \alpha \operatorname{sen}^5 x} = \lim_{x\to 0^+} \frac{27x^5 + x^7 + o(x^7)}{\frac{1}{2}x^8 + o(x^8) + \alpha x^5 + o(x^5)}$$

$$\log(1+x^7) = x^7 + o(x^7)$$

$$(1+x)^\beta = 1 + \beta x + o(x) \qquad x \to 0$$

$$(1+x^8)^{1/2} = 1 + \frac{1}{2}x^8 + o(x^8)$$

$$\operatorname{sen} x = x + o(x) \qquad \operatorname{sen}^5 x = x^5 + o(x^5)$$

$$\lim_{x \to 0^+} \frac{27x^5 + o(x^5)}{\alpha x^5 + o(\alpha x^5) + \frac{1}{2}x^8 + o(x^8)}$$

$\alpha \neq 0 \qquad \lim\left(\quad\right) = \dfrac{27}{\alpha}$

$\alpha = 0 \qquad \lim\left(\quad\right) = +\infty$

<u>Es.</u> $$\lim_{n\to+\infty} \frac{1 - n^2 \log\left(1+\frac{1}{n^2}\right) - \frac{\operatorname{sen} n}{n^3}}{n^\alpha \operatorname{arctg}\left(\frac{1}{n}\right)} \qquad \alpha\in\mathbb{R}$$

$$\log(1+y) = y - \frac{1}{2}y^2 + o(y^2)$$

$$\operatorname{arctg} y = y - \frac{y^3}{3} + o(y^3)$$

$$\log\left(1+\frac{1}{n^2}\right) = \frac{1}{n^2} - \frac{1}{2}\frac{1}{n^4} + o\left(\frac{1}{n^4}\right)$$

$$\operatorname{arctg}\left(\frac{1}{n}\right) = \frac{1}{n} - \frac{1}{3}\frac{1}{n^3} + o\left(\frac{1}{n^3}\right)$$

$$= \lim_{n\to+\infty} \frac{1 - n^2\left(\frac{1}{n^2} - \frac{1}{2}\frac{1}{n^4} + o\left(\frac{1}{n^4}\right)\right) + \frac{\operatorname{sen} n}{n^3}}{n^\alpha\left(\frac{1}{n} - \frac{1}{3}\frac{1}{n^3} + o\left(\frac{1}{n^3}\right)\right)} =$$

$$= \lim_{n\to+\infty} \frac{\cancel{1} - \cancel{1} + \frac{1}{2}\cdot\frac{1}{n^2} + o\left(\frac{1}{n^2}\right) + \frac{\operatorname{sen} n}{n^3}}{\frac{1}{n^{1-\alpha}} + o\left(\frac{1}{n^{1-\alpha}}\right)}$$

$$? \quad \frac{\operatorname{sen} n}{n^3} = o\left(\frac{1}{n^2}\right) \qquad \frac{\operatorname{sen} n}{n^{\cancel{3}}} \cdot \cancel{n^2} = \frac{\operatorname{sen} n}{n} \to 0$$

$$= \lim \frac{\frac{1}{2}\frac{1}{n^2} + o\left(\frac{1}{n^2}\right)}{\frac{1}{n^{1-\alpha}} + o\left(\frac{1}{n^{1-\alpha}}\right)} = \lim \frac{1}{2}\frac{n^{1-\alpha}}{n^2} =$$

$$= \lim \frac{1}{2}\frac{1}{n^{1+\alpha}}$$

$$1+\alpha > 0 \qquad \lim (\quad) = 0$$

$1+\alpha < 0 \quad \lim (\quad) + \infty$

$1+\alpha = 0 \quad \ln (\quad) = \frac{1}{2}$

<u>Integrali:</u>

$\int_a^b f(x)\,dx$ = numero reale

f ha una primitiva G

$$\int_a^b f(x)\,dx = G(b) - G(a)$$

$f(x) = \operatorname{sen} x$

f continua

$$\int_c^x f(t)\,dt = F(x)$$

$f(x) = \dfrac{1}{1+x^2}$ $\qquad F(x) = \operatorname{arctg} x$

$f(x) = x \operatorname{sen} x$ $\qquad$ trovo la primitiva ?

$f(x) = (\operatorname{sen} x)^2$ $\qquad$ " ?

Integrazione per parti

f, g derivabili in $[a, b]$

$$(f \cdot g)' = f'g + fg'$$

$$fg' = (fg)' - f'g$$

$$\int fg' = \int (fg)' - \int f'g$$

$$\int fg' = fg - \int f'g$$

<u>OSS.</u>

$$\int f'(x)dx = f$$

$$\boxed{\int f(x)\, g'(x)\, dx = f(x)\, g(x) - \int f'(x)\, g(x)\, dx}$$

formula di integrazione per PARTI

vale anche

$$\int_a^b f(x)\, g'(x)\, dx = f(x) g(x) \Big|_a^b - \int_a^b f'(x)\, g(x)\, dx$$

$f(x) = x$
$g'(x) = \operatorname{sen} x$
$g(x) = -\cos x$

es. $\int x \operatorname{sen} x \, dx = x \cdot (-\cos x) -$

$$- \int 1 \cdot (-\cos x)\, dx =$$

$$= -x\cos x + \int \cos x\,dx = -x\cos x + \operatorname{sen} x$$

$$\int x \operatorname{sen} x = \operatorname{sen} x \, \frac{x^2}{2} - \int \cos x \cdot \frac{x^2}{2} dx$$

$g'(x) = x$

$f(x) = \operatorname{sen} x$

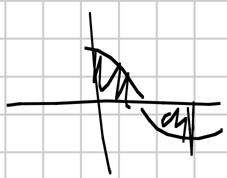
$g(x) = \frac{x^2}{2}$

es.

$$\int_0^{\pi} \underbrace{x}_{f}\,\underbrace{\operatorname{sen} x}_{g'}\,dx = x(-\cos x)\Big|_0^{\pi} - \int_0^{\pi} (-\cos x)\,dx$$

$$= -\pi(-1) + \int_0^{\pi} \cos x\,dx =$$

$$= \pi + 0 = \pi$$



es.

$$\int \underbrace{x^2}_{f} \underbrace{\operatorname{sen} x}_{g'} \, dx = x^2(-\cos x) - \int 2x(-\cos x)\,dx$$

$$= -x^2 \cos x + \int 2x \cos x \, dx = -x^2\cos x + 2x \operatorname{sen} x + 2\cos x + K$$

$$\int \underbrace{x}_{f} \underbrace{\cos x}_{g'} = x \operatorname{sen} x - \int 1 \cdot \operatorname{sen} x \, dx =$$

$$= x \operatorname{sen} x + \cos x$$

Es.

$$\int \underbrace{1}_{g'} \cdot \underbrace{\log x}_{f} \, dx = (\log x)\, x - \int \frac{1}{\cancel{x}} \cancel{x} \, dx$$

$$= x \log x - x + K$$

Es.

$$\int \underbrace{e^x}_{f} \underbrace{\operatorname{sen} x}_{g'} \, dx = e^x(-\cos x) - \int e^x(-\cos x)$$

$$= -e^x \cos x + \int \underbrace{e^x}_{f} \underbrace{\cos x}_{g'} = -e^x \cos x + e^x \operatorname{sen} x$$

$$- \int e^x \operatorname{sen} x \quad \Rightarrow \quad 2 \int e^x \operatorname{sen} x = -e^x \cos x + e^x \operatorname{sen} x$$

$$\Rightarrow \int e^x \operatorname{sen} x = \frac{-e^x \cos x + e^x \operatorname{sen} x}{2}$$

<u>PC.</u> $\int \operatorname{sen}^2 x \, dx$

$$\cos^2 x = 1 - \operatorname{sen}^2 \cancel{x}$$

$$\left(x - \operatorname{sen} x \cos x \right)$$

Integrazione per <u>sostituzione</u>

F primitiva di f $\qquad F'(x) = f(x)$

$x = \varphi(t) \qquad f(x) = f(\varphi(t))$

$F(x) = F(\varphi(t))$

$$[F(\varphi(t))]' = F'(\varphi(t)) \cdot \varphi'(t) = f(\varphi(t))\varphi'(t)$$

$$(F(x))' \longrightarrow f(x)$$

$$\int f(x)\,dx = \int f(\varphi(t))\,\varphi'(t)\,dt$$

Formula di integrazione per sostituzione

$x = \varphi(t)$

cambio di variabile

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f(\varphi(t))\,\varphi'(t)\,dt$$

$x \in [a, b]$

$x = \varphi(t)$

$t \in [\alpha, \beta]$

Formalmente

$$x = \varphi(t)$$

$$1\,dx = \varphi'(t)\,dt \quad \rightarrow \quad dx = \varphi'(t)\,dt$$

$$\frac{1}{2x+1} \qquad \int \frac{1}{t}\,dt$$

$$\log|t|$$

<u>es.</u> $\displaystyle\int \frac{1}{2x+1}\,dx$

$$2x+1 = t$$

$$2x = t-1$$

$$x = \frac{1}{2}(t-1) = \varphi(t)$$

$$dx = \frac{1}{2}\,dt$$

$$= \int \frac{1}{t}\,\frac{1}{2}\,dt =$$

$$= \frac{1}{2}\int \frac{1}{t}\,dt = \frac{1}{2}\log|t| = \frac{1}{2}\log|2x+1| + K$$