$\lim\limits_{x \to x_0} \dfrac{f}{g}$ se $\exists \lim\limits_{x \to x_0} \dfrac{f'}{g'} = l \Rightarrow \lim\limits_{x \to x_0} \dfrac{f}{g} = l$

pure $\nexists \lim \dfrac{f'}{g'}$ — ma $\exists \lim \dfrac{f}{g}$

Derivabilità

Derivabilità di f in un p.to. f continua in x_0

$\exists\ f'(x_0)$ $\lim\limits_{x \to x_0} \dfrac{f(x) - f(x_0) \to 0}{x - x_0 \to 0}$

n $\exists \lim\limits_{x \to x_0} \dfrac{f'(x)}{1} = \underbrace{\lim\limits_{x \to x_0} f'(x)}_{f(x_0)}$ allora

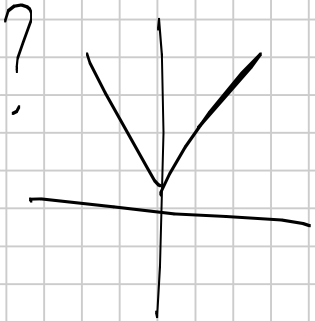
$\Downarrow = \lim\limits_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0}$.

$$f'_+(x_0) = (\text{se esiste}) = \lim_{x \to x_0^+} f'(x)$$

$$f'_-(x_0) = (\qquad\qquad) = \lim_{x \to x_0^-} f'(x)$$

<u>Es.</u> $f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x & x < 0 \end{cases}$ $\qquad f'(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$

f è derivabile in $x = 0$?



$$\lim_{x \to 0^+} f'(x) = 1 = f'_+(0)$$

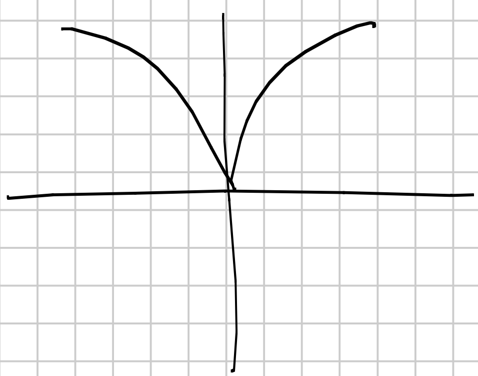
$$\lim_{x \to 0^-} f'(x) = -1 = f'_-(0)$$

Es. $f(x) = \sqrt[3]{x^2}$ è derivabile $x = 0$?

$f'(x) = \frac{2}{3} \frac{1}{x^{1/3}}$

$\lim_{x \to 0^+} f' = +\infty = f'_+(0) = +\infty$

$\lim_{x \to 0^-} f' = -\infty = f'_-(0) = -\infty$



Es.

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

dire se f è continua in $\mathbb{R}$
e se f è derivabile in $\mathbb{R}$.

$D = \mathbb{R}$

in $x = 0$ è continua ?

$$\lim_{x \to 0} f(x) = 0$$

$$\lim_{x \to 0} e^{-1/x^2} = 0$$

f è derivabile $\forall x \neq 0$

f è derivabile in $x = 0$?

$x \neq 0$ $\quad f'(x) = e^{-1/x^2} \dfrac{2}{x^3}$

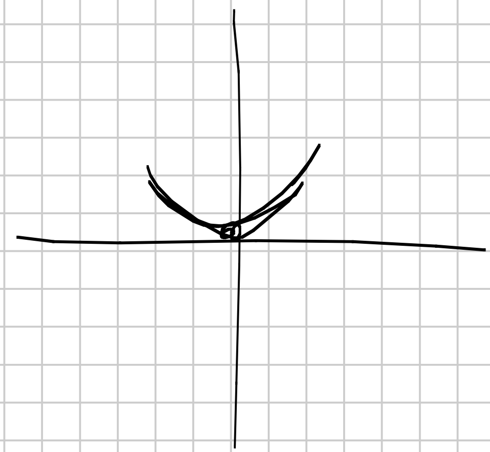
$$\lim_{x \to 0} \frac{e^{-1/x^2}}{x^3}$$

$1/x = y$

$$\lim_{x\to 0} \frac{2e^{-1/x^2}}{x^3} = \lim_{y\to\pm\infty} 2e^{-y^2}\cdot y^3 =$$

$$\frac{1}{x} = y \qquad = \lim_{y\to\pm\infty} 2\frac{y^3}{e^{y^2}} = 0$$

$\exists$ finito $\lim_{x\to 0} f'(x) = f'(0)$

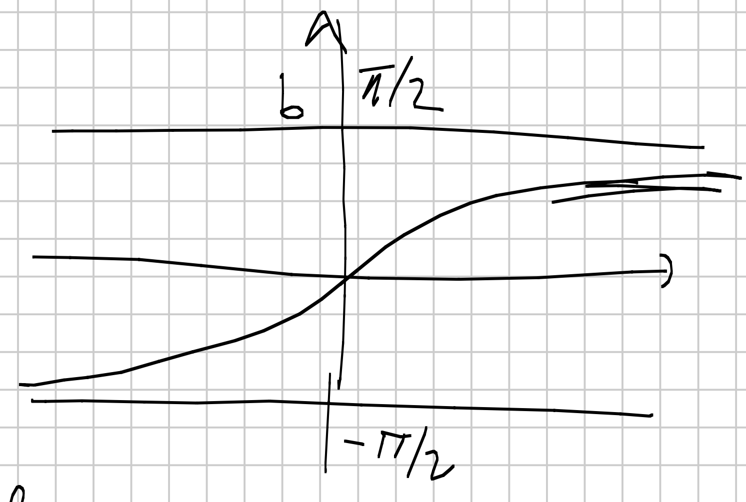


Asintoti di una funzione f

$y = b$ è asintoto orizzontale per f e per $x \to +\infty$

se $\lim_{x \to +\infty} f(x) = b$

$\lim_{x \to -\infty} f(x) = b$



es. $f(x) = \operatorname{arctg} x$

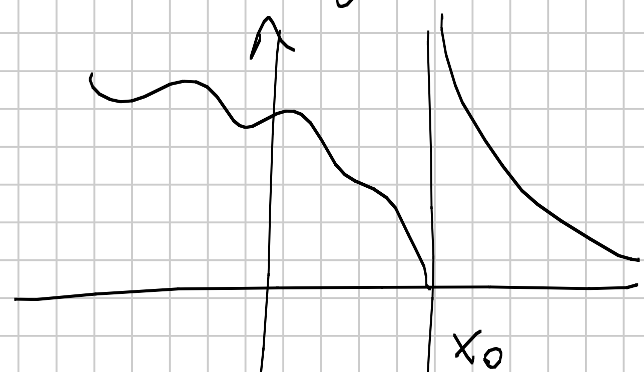
$\lim_{x \to +\infty} \operatorname{arctg} x = \frac{\pi}{2}$

$\lim_{x \to -\infty} \operatorname{arctg} x = -\pi/2$

Asintoto verticale

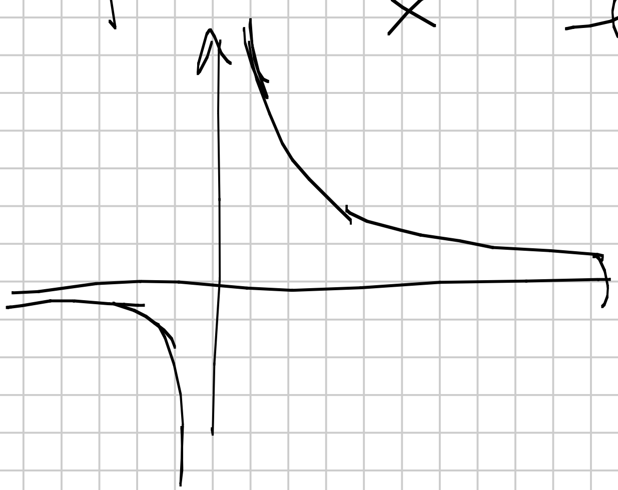
retta di equazione $x = x_0$

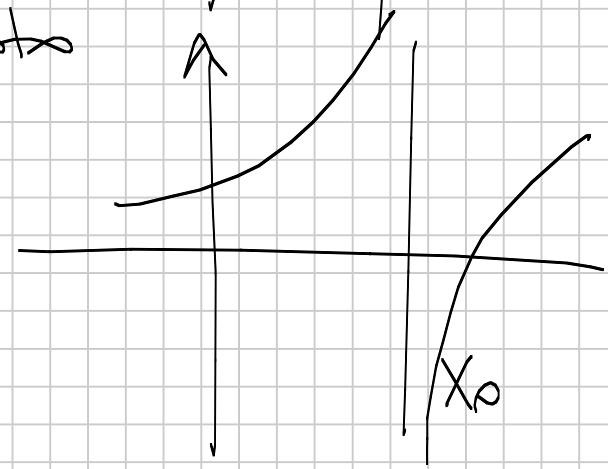
se $\lim_{x \to x_0^+} f$ o $\lim_{x \to x_0^-} f$ è $\pm\infty$



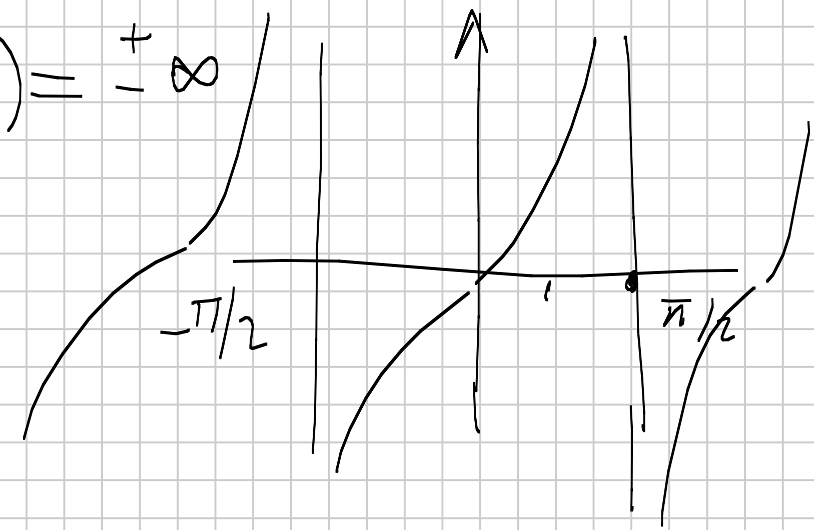
es. $f(x) = \frac{1}{x}$

$x = 0$ ha un asintoto verticale





es.: $f(x) = \operatorname{tg} x$ $\quad \lim_{x \to \frac{\pi}{2}^{\mp}} f(x) = \pm\infty$



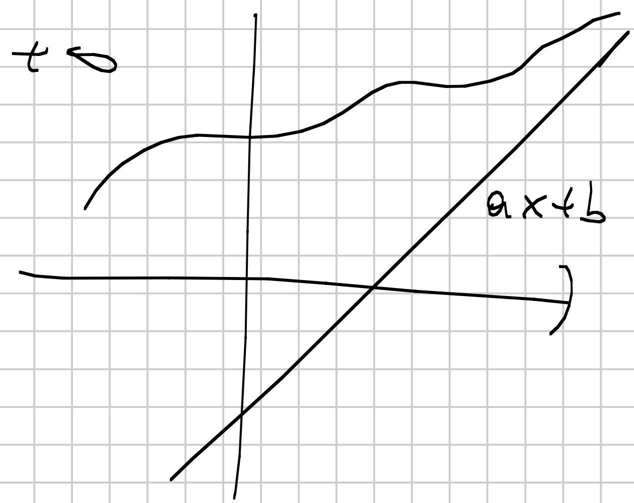
Asintoto obliquo $\quad y = ax + b$

f ha un asintoto obliquo per $x \to +\infty$

$$\lim_{x \to +\infty} \left(f(x) - (ax+b) \right) = 0$$

$$\Updownarrow$$

$$\begin{cases} \cdot \ \lim\limits_{x \to +\infty} \dfrac{f(x)}{x} = a \in \mathbb{R} \\ \cdot \ \lim\limits_{x \to +\infty} f(x) - ax = b \in \mathbb{R} \end{cases}$$



Es. $f(x) = \sqrt{x^2+1}$ $\qquad \lim_{x \to +\infty} \sqrt{x^2+1} = +\infty$

$$\lim_{x \to +\infty} \frac{f(x)}{x} = \lim_{x \to +\infty} \frac{\sqrt{x^2+1}}{x} = \lim_{x \to +\infty} \frac{\cancel{x}\sqrt{1+\frac{1}{x^2}}}{\cancel{x}} = 1$$

$a = 1$

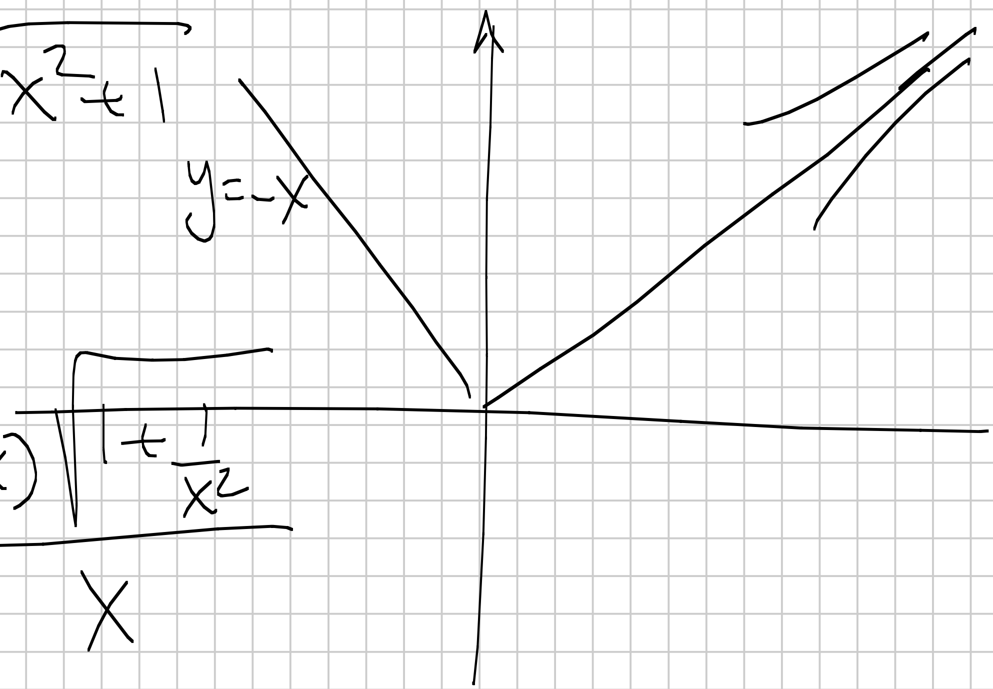
$$\lim_{x \to +\infty} f(x) - ax = \lim_{x \to +\infty} \sqrt{x^2+1} - x = \lim_{x \to +\infty} \frac{(\cancel{x^2}+1) - \cancel{x^2}}{\sqrt{x^2+1} + x} = 0$$

$ax + b = x$

$\sqrt{x^2+1}$ ha come asintoto, per $x \to +\infty$, la retta $y=x$

PC. asintoto di $\sqrt{x^2+1}$

$x \to -\infty$



$$\lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{x} = \lim_{x\to-\infty} \frac{(-x)\sqrt{1+\frac{1}{x^2}}}{x}$$

$$f(x) = \log|e^x - 4| - \operatorname{arctg}(e^x - 5) - \log 4$$

cerchiamo se ci sono asintoti.

$x \to +\infty$

$x \to -\infty$

$$\lim_{x \to +\infty} f(x) = +\infty$$

$$\lim_{x \to +\infty} \frac{f(x)}{x} = \lim_{x \to +\infty} \frac{\log(e^x - 4) - \operatorname{arctg}(e^x - 5) - \log 4}{x} =$$

$$= \lim_{x \to +\infty} \frac{\log(e^x - 4)}{x} = \lim_{x \to +\infty} \frac{\log\left(e^x\left(1 - \frac{4}{e^x}\right)\right)}{x} =$$

$$= \lim_{x \to +\infty} \frac{x + \log\left(1 - \frac{4}{e^x}\right)}{x} = 1 = a$$

$$b = \lim_{x \to +\infty} f(x) - x = \lim_{x \to +\infty} \log(e^x - 4) - \operatorname{arctg}(e^x - 5)$$

$$- \log 4 - x = \lim_{x \to +\infty} \log\left(e^x\left(1 - \frac{4}{e^x}\right)\right) - \operatorname{arctg}(e^x - 5)$$

$$- \log 4 - x = \lim_{x \to +\infty} \cancel{x} + \log\left(1 - \frac{4}{e^x}\right) - \operatorname{arctg}(e^x - 5)$$

$$- \log 4 - \cancel{x} = -\frac{\pi}{2} - \log 4 = b$$

$$y = x - \left(\frac{\pi}{2} + \log 4\right)$$

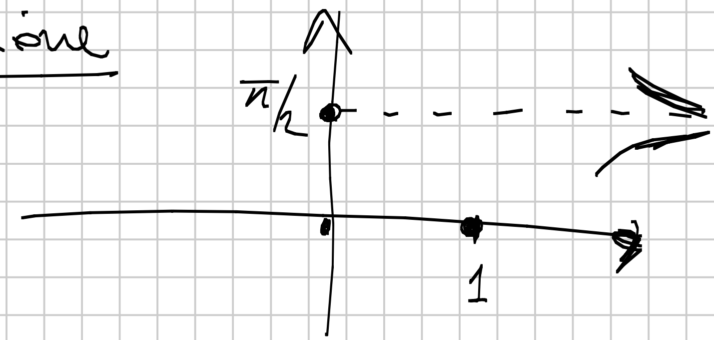
asintoto obliquo

$x \to +\infty$

Per caso asintoto di f per $x \to -\infty$

sugg. $\log|e^x - 4| = \log(4 - e^x)$
$x \to -\infty$

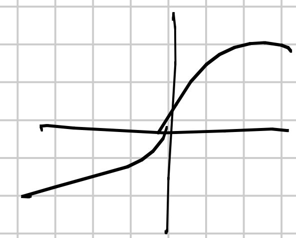
Studio di un grafico di funzione



$$f(x) = |\arctan(\log x)|$$

$$D = \{x > 0\} \qquad f \geq 0$$

$$f = 0 \iff \log x = 0 \iff x = 1$$



$$f(x) = \begin{cases} \arctan(\log x), & x \geq 1 \\ -\arctan(\log x), & x < 1 \end{cases}$$

$$\lim_{x \to 1} f(x) = 0$$

$(0, +\infty)$

$\lim_{x \to 0^+} f(x) = \frac{\pi}{2}$ se si pone $f(0) = \frac{\pi}{2}$

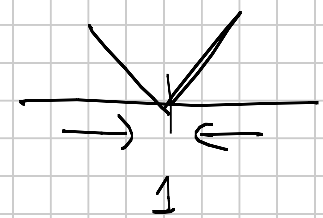
la $f(x)$ si può estendere per continuità

anche in $x = 0$

$\lim_{x \to +\infty} f(x) = \frac{\pi}{2}$ asintoto orizzontale $x \to +\infty$

$$f(x) = \begin{cases} \operatorname{arctg}(\log x), & x \geq 1 \\ -\operatorname{arctg}(\log x), & x < 1 \end{cases}$$

$$f'(x) = \begin{cases} \dfrac{1}{1+\log^2 x} \cdot \dfrac{1}{x} > 0 & x > 1 \\ -\dfrac{1}{1+\log^2 x} \cdot \dfrac{1}{x} < 0 & x < 1 \end{cases}$$



$$\lim_{x\to 1^-} f'(x) = -1 \neq 1 = \lim_{x\to 1^+} f'(x)$$

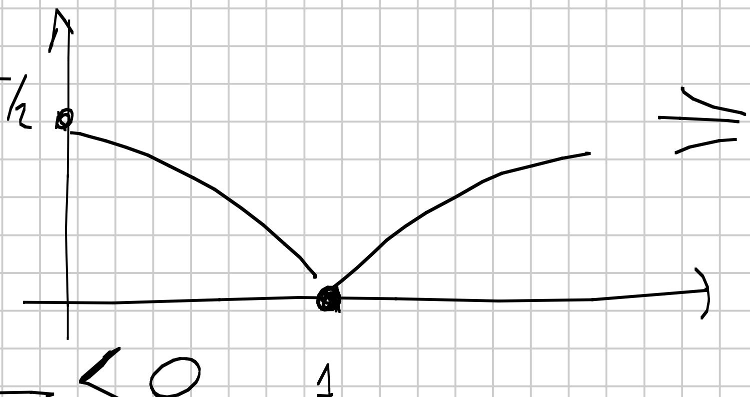
$$f'_-(1) \neq f'_+(1)$$ f <u>non</u> è derivabile in $x=1$

(p.to angoloso)

f è strett. crescente per $x>1$

f è " decrescente per $x<1$

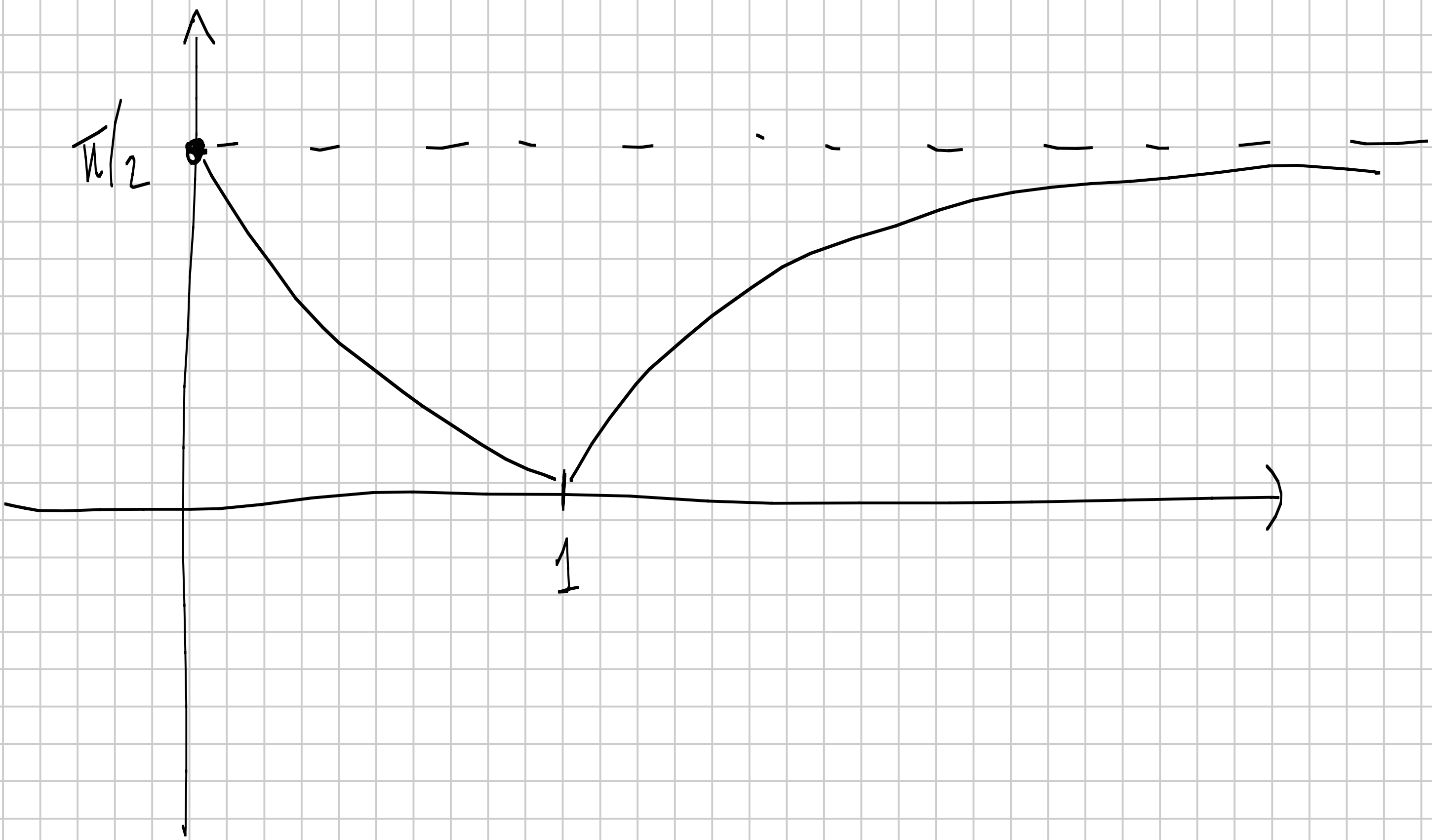
$x=1$ p.to di minimo globale (assoluto)



$x>1$ $\quad f''(x) = -\dfrac{(1+\log x)^2}{x^2(\log^2 x+1)^2} < 0 \quad f$ strett. concava $x>1$

controllare!

$x<1$ $\quad f''(x) = \dfrac{(1+\log x)^2}{x^2(\log^2 x+1)^2} > 0 \quad$ strett. convessa $x<1$


π/2
1

ES. dire se esistono soluzioni dell'equazione

$$(x-1)e^{x} - (x+1)e^{-x} = 0$$

e se sì, quante sono.

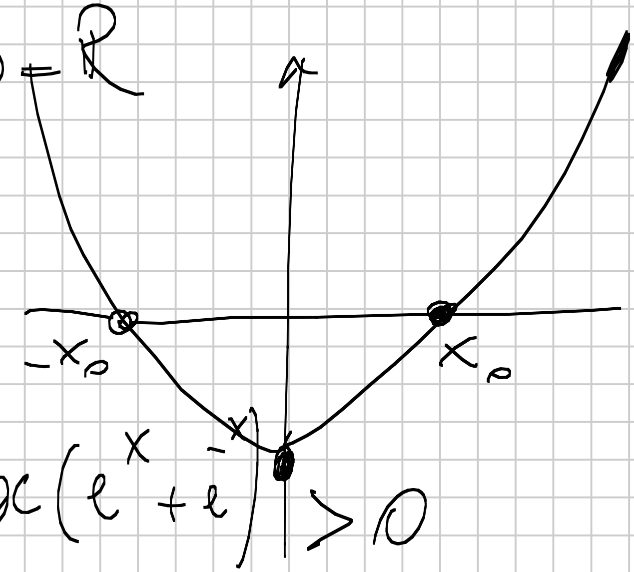
$$f(x) = (x-1)e^{x} - (x+1)e^{-x} \qquad D = \mathbb{R}$$

$$f(-x) = f(x) \quad (\text{fare}) \quad \underline{\text{Pari}}$$

$x > 0 \qquad \lim_{x \to +\infty} f(x) = +\infty$

$$f(0) = -1-1 = -2$$

$$f'(x) \overset{?}{=} x(e^{x} + e^{-x}) > 0$$

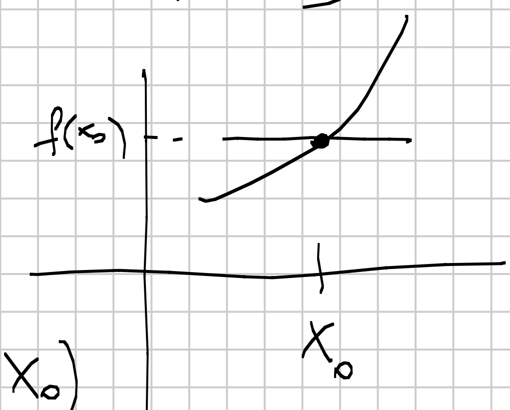


Polinomio di Taylor

Scopo: approssimare una funzione con una più semplice, per es. polinomi.

1) f continua in x_0 $\quad \lim_{x \to x_0} f(x) = f(x_0)$

$$f(x) = f(x_0) + o(1), \quad x \to x_0$$



2) f derivabile

$$f(x) = \underbrace{f(x_0) + f'(x_0)(x-x_0)}_{T_1(x)} + o(x-x_0) \quad x \to x_0$$

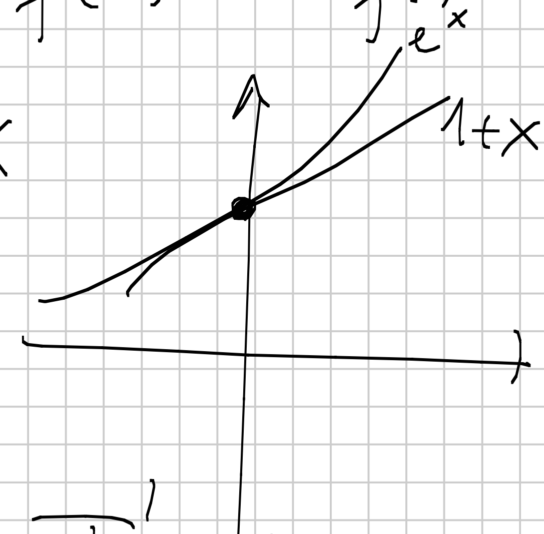
$$f(x) = T_1(x) + o(x - x_0)$$

polinomio di grado 1

es.

$f(x) = e^x \qquad x_0 = 0 \qquad f(0) = 1 \qquad f'(x) = e^x \qquad f'(0) = 1$

$$T_1(x) = 1 + 1 \cdot (x) = 1 + x$$



$$e^x = 1 + x + o(x) \qquad x \to 0$$

$$f(x_0) = T_1(x_0) \qquad f'(x_0) = T_2'(x_0)$$

Cerco un polinomio di grado 2 $T_2(x)$

t.c.

$$f(x_0) = T_2(x_0)$$

$$f'(x_0) = T_2'(x_0)$$

$$f''(x_0) = T_2''(x_0)$$

$$T_2(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2$$

polinomio di grado 2

$$T_2(x_0) = \boxed{a_0 = f(x_0)}$$

$$T_2'(x) = a_1 + 2a_2(x-x_0) \qquad T_2'(x_0) = \boxed{a_1 = f'(x_0)}$$

$$T_2''(x) = 2a_2 \qquad T_2''(x_0) = 2a_2 = f''(x_0)$$

$$\boxed{a_2 = \frac{f''(x_0)}{2}}$$

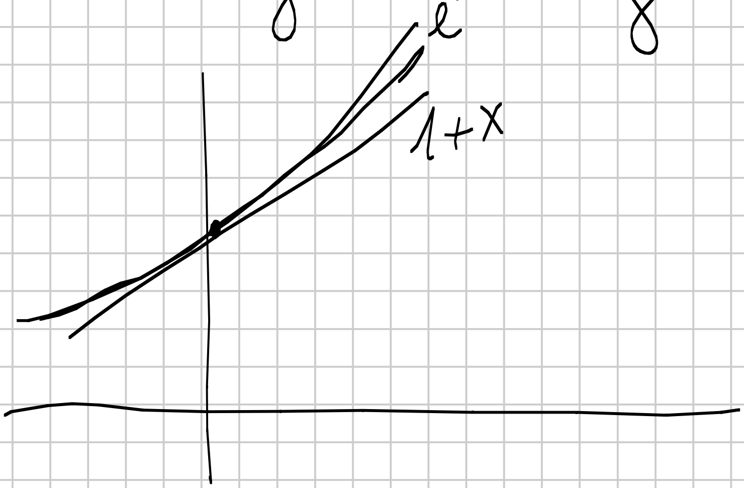
$$T_2(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2$$

polinomio di Taylor
in $x = x_0$ di $f(x)$.
di ordine 2

es. $f(x) = e^x$ $\quad x_0 = 0$ $\quad f''(x) = e^x$

$T_1(x) = 1 + x$ $\quad T_2(x) = 1 + x + \frac{1}{2}x^2$

polinomio di Taylor di grado 2 di e^x, centrato in $x=0$



In generale il polinomio di Taylor $T_K(x)$ di grado K della funzione $f(x)$ ha coefficienti

$$a_K = \frac{f^{(K)}(x_0)}{K!}$$

<u>Def</u>. f n volte derivabile in x_0. Chiamo <u>polinomio di Taylor</u> $T_n(x)$ di ordine n, di f di centro x_0

$$T_n(x) = \sum_{K=0}^{n} \frac{f^{(K)}(x_0)}{K!} (x - x_0)^K$$

$$T_n(x) = \underbrace{f(x_0)}_{k=0} + \underbrace{f'(x_0)(x-x_0)}_{k=1} + \frac{f''(x_0)}{2}(x-x_0)^2 + $$

$$+ \frac{f'''(x_0)}{3!}(x-x_0)^3 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

<u>oss.</u> Se $x_0 = 0$ il polinomio di McLaurin

<u>es.</u> $f(x) = e^x$ $\quad x = 0 \quad f'(x) = e^x \quad f^{(k)}(x) = e^x$

$f^{(k)}(0) = 1$

$$\boxed{T_n(x) = 1 + 1 \cdot x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \dots + \frac{1}{n!}x^n}$$

dimostreremo / polinomio di McLaurin di e^x di grado n

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + o(x^n) \quad x \to 0$$

<u>es.</u> $f(x) = \operatorname{sen} x \qquad x = 0$

$$T_5(x) = \underbrace{f(0)}_{0} + \underbrace{f'(0)}_{1}x + \cancel{\frac{f''(0)}{2}x^2} + \frac{f'''(0)}{3!}x^3 + \cancel{\frac{f^{(IV)}(0)}{4!}x^4} + \frac{f^{(V)}(0)}{5!}x^5$$

$f'(x) = \cos x \qquad f'(0) = 1$

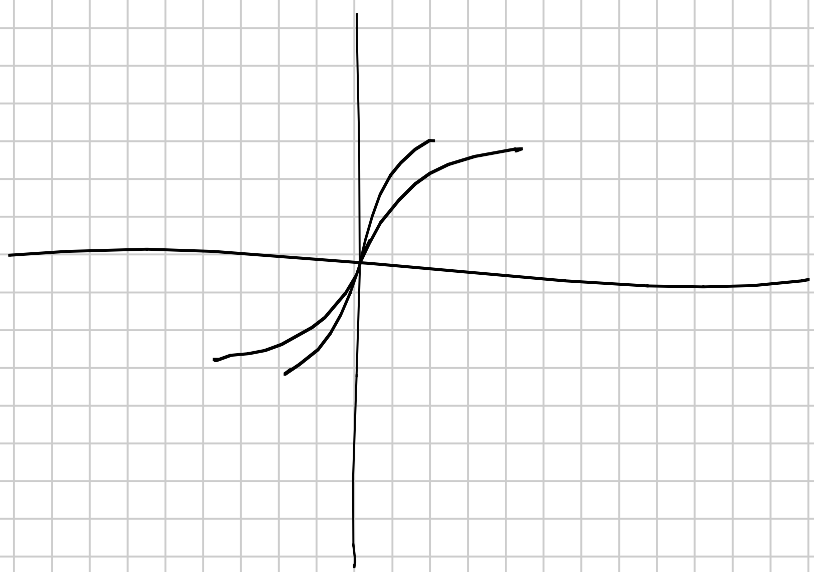
$f''(x) = -\operatorname{sen} x \qquad f''(0) = 0$

$f'''(x) = -\cos x \qquad f'''(0) = -1$

$f^{IV}(x) = \operatorname{sen} x \qquad f^{IV}(0) = 0$

$f^{V}(x) = \cos x \qquad f^{V}(0) = 1$

$$T_5(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9$$



$f(x) = \log(1+x)$

Polinomio di McLaurin di ordine n

$$T_n(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$= \sum_{k=0}^{n} \frac{(-1)^{k-1}}{k} x^k$$

controllare