$f : \mathbb{R}^n \to \mathbb{R}$ $\quad n = 2 \quad$ $x = (x_1, \dots x_n)$

$f(x, y)$ $\quad (x_1, x_2)$

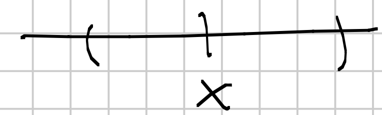
$f(x, y, z)$ $\quad n = 3 \quad (x_1, x_2, x_3)$

$x = (x_1, x_2)$

norma $\quad \| X \| = \sqrt{\sum_{i=1}^{n} x_i^2}$

$d(x, y) = \| X - y \| = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2}$

$$B_\varepsilon(x) = \{ y \in \mathbb{R}^n : d(x,y) < \varepsilon \}$$ Intorni

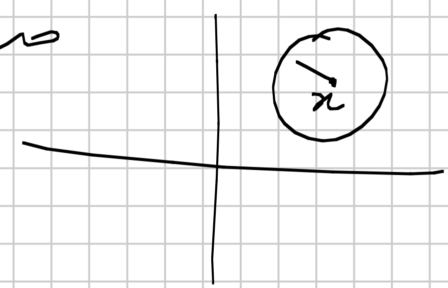


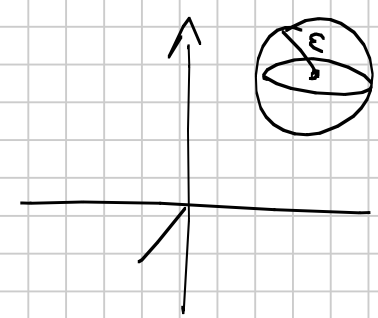
$$\lim_{x \to x_0} f(x) = l \iff \forall V$$ intorno di l $\exists U$ intorno di x_0: $f(x) \in V$ $\forall x \in U$.

$x \in \mathbb{R}^n$

x_0 di accumulazione

$x_0 \in \mathbb{R}^n$





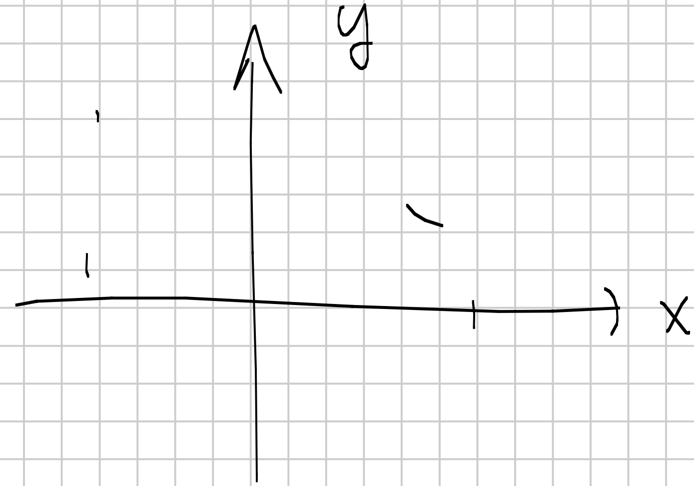
Limite all'infinito

$f : \mathbb{R} \to \mathbb{R}$ $\qquad\qquad$ $\mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$

$$\lim_{\substack{x \to +\infty \\ x \to -\infty}}$$

$f : \mathbb{R}^2 \to \mathbb{R}$

$f(x, y)$



(x, y) tende a infinito?

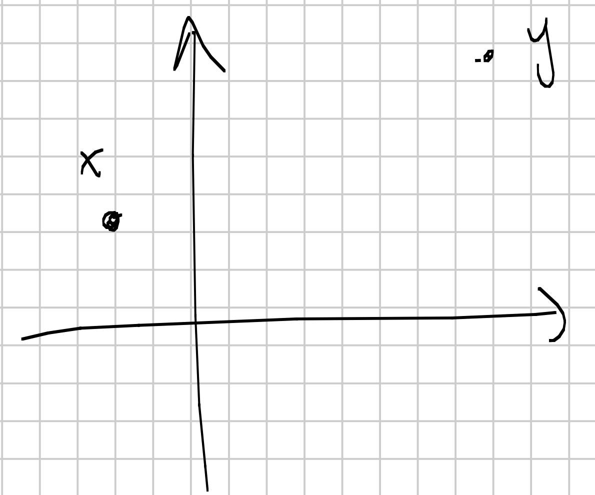
$\mathbb{R}$ insieme ordinato x_1, x_2

$x_1 < x_2$

$x_1 > x_2$

$x_1 = x_2$

$\mathbb{R}^2$ e in generale $\mathbb{R}^n$ <u>non</u> sono insiemi <u>ordinati</u>

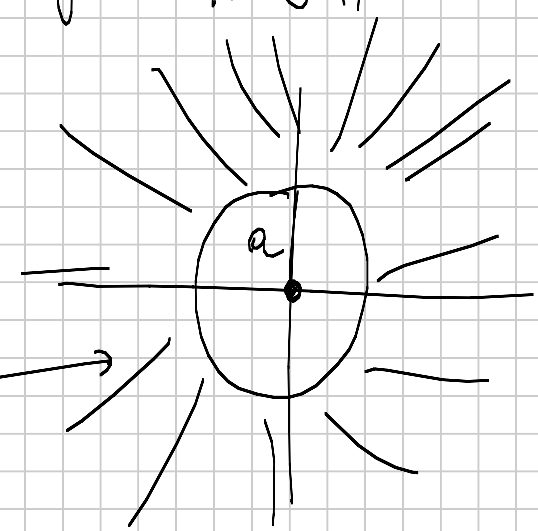


in $\mathbb{R}^n$ si introduce l'elemento ∞

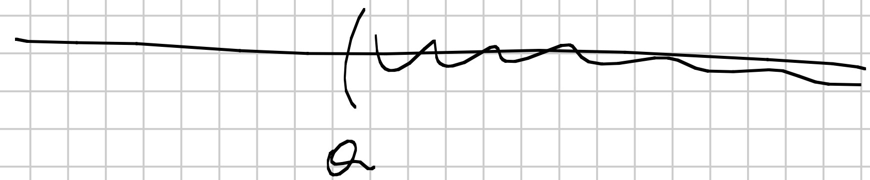
$\mathbb{R}^n \cup \{\infty\}$ ampliamento di $\mathbb{R}^n$

<u>Def.</u> Intorno di ∞ $= \{ y : \|y\| > a \}$

$\mathbb{R}^2$ $\{ y : \sqrt{y_1^2 + y_2^2} > a \}$



in $\mathbb{R}$ $\quad (a, +\infty]$



$x \in \mathbb{R}^n$

$$\lim_{x \to \infty} f(x) = \ell \iff \forall\, V \text{ intorno di } \ell\ \exists\, U \text{ intorno di } \infty \text{ t.c. } f(x) \in V\ \forall\, x \in U$$

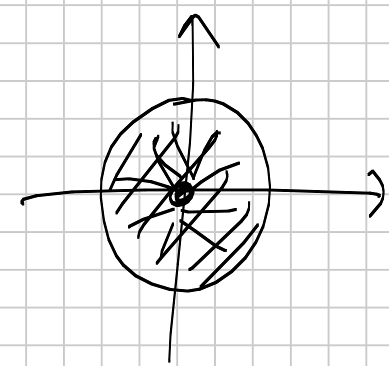
$f : \mathbb{R}^2 \to \mathbb{R}$

$f(x,y)$

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \ell$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \ell$$

<u>es.</u> $f(x,y) = \dfrac{x}{x^2+y^2}$ $\quad \mathbb{R}^2 \setminus \{0,0\}$

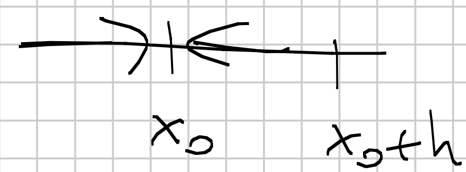


$$\lim_{(x,y)\to(0,0)} \frac{x}{x^2+y^2} = \frac{0}{0}$$

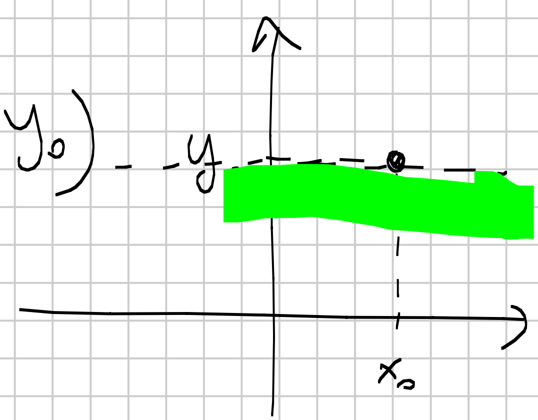
Derivate parziali

$f : \mathbb{R} \to \mathbb{R} \qquad f(x) \qquad x \in \mathbb{R}$

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$



$f : \mathbb{R}^2 \to \mathbb{R} \qquad f(x,y) \qquad (x_0, y_0)$



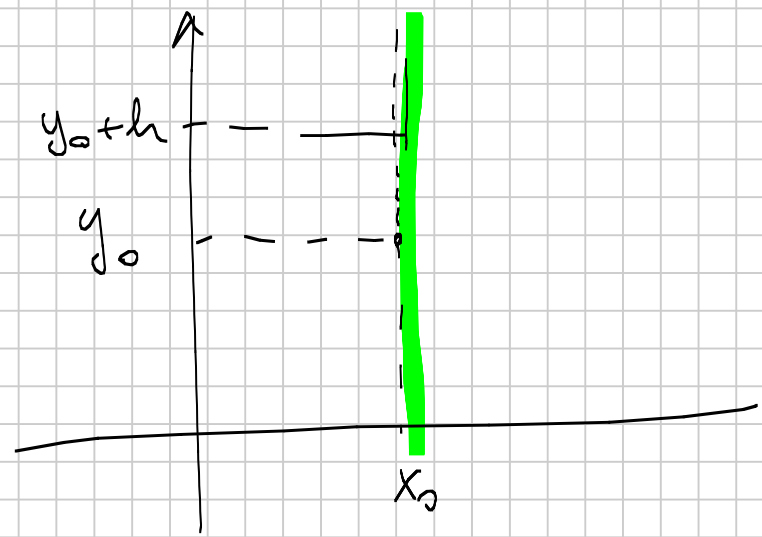
(x, y_0) varia solo la x
la y_0 è fissata

$$\frac{\partial f}{\partial x}(x_0,y_0) := \lim_{h\to 0} \frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}$$

Analogamente

(x_0, y)

$f(x_0, y)$



$$\frac{\partial f}{\partial y}(x_0,y_0) := \lim_{h\to 0} \frac{f(x_0,y_0+h)-f(x_0,y_0)}{h}$$

$$\frac{\partial f}{\partial x}(x_0, y_0) \quad , \quad \frac{\partial f}{\partial y}(x_0, y_0)$$

derivate parziali rispetto a x e a y di f in (x_0, y_0)

$$\Downarrow \qquad\qquad \Downarrow$$

$$f_x(x_0, y_0) \qquad f_y(x_0, y_0)$$

$$\left(f_x(x_0, y_0), f_y(x_0, y_0) \right) =$$

$\nabla f(x_0, y_0) =$ <u>gradiente</u> di f

$Df(x_0, y_0)$

$\text{grad} f(x_0, y_0)$

esempio $\quad f(x,y) = x^5 y^2 + 5 \qquad D = \mathbb{R}^2$

(x_0, y_0) $\qquad$ penso y come costante

$$\frac{\partial f}{\partial x} = f_x(x,y) = 5x^4 \cdot y^2$$

$$\frac{\partial f}{\partial y} = f_y(x,y) = x^5 \cdot 2y$$

$(x_0, y_0) = (1,2)$

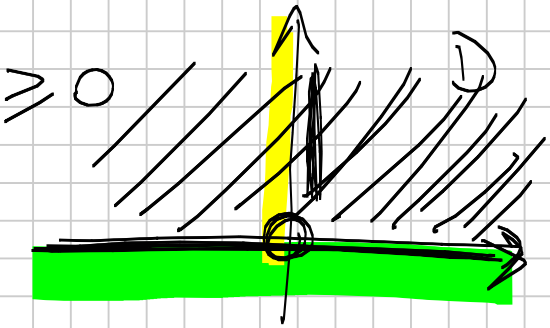
$$f_x(1,2) = 5 \cdot 1 \cdot 4 = 20$$

$$f_y(1,2) = 1 \cdot 2 \cdot 2 = 4$$

$$\nabla f(1,2) = (20, 4)$$

es. $f(x,y) = x\sqrt{y}$

$y \geq 0$



$f(x,0) = 0 \quad \forall x$

$f(0,y) = 0 \quad \forall y \geq 0$

$f_x(x,y) = \sqrt{y}$

$f_y(x,y) = x\dfrac{1}{2\sqrt{y}}$

$(x,y) = (1,2)$

domanda

$\exists\, f_y(0,0)$?

non si può usare questa espressione

si usa la definizione di f_y

$$\frac{\partial f}{\partial y}(0,0) = \lim \frac{f(0, 0+h) - f(0,0)}{h}$$

$$\frac{0}{h}$$

perché la f vale zero sull'asse y

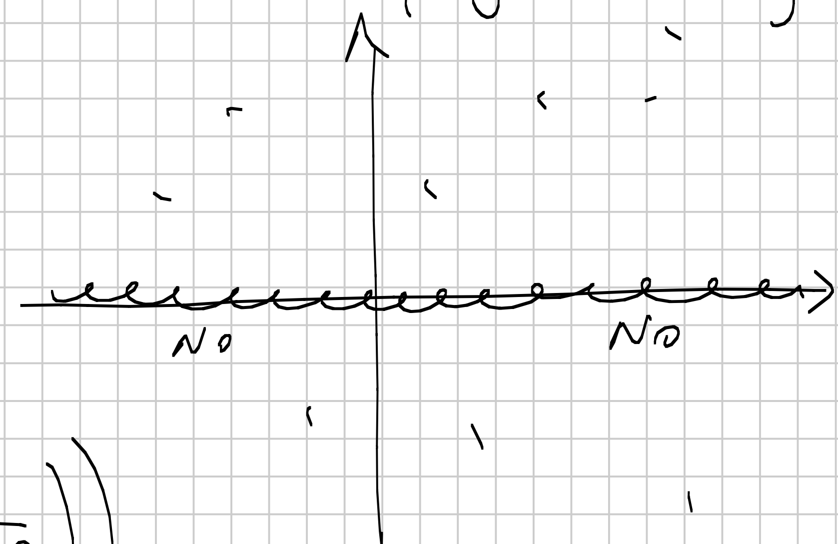
$$= 0$$

$\exists$

$$\frac{\partial f}{\partial y}(0,0) = 0$$

es. $f(x, y) = e^{x/y}$

$D = \left\{ (x,y) \in \mathbb{R}^2 : y \neq 0 \right\}$



$f_x(x,y) = e^{\frac{x}{y}} \cdot \frac{1}{y}$

$f_y(x,y) = e^{\frac{x}{y}} \left(x \left(-\frac{1}{y^2} \right) \right)$.

$\exists\ f_x$ e f_y in ogni punto del dominio.

<u>**Def.**</u> f è <u>DERIVABILE</u> in (x_0, y_0) del dominio se esistono le sue derivate parziali in (x_0, y_0).

$$f(x,y) = x\sqrt{y}$$

$f_x =$

$f_y =$

$f_y(0,0) = 0$

$f_x(0,0) = 0$

$\downarrow$

f è derivabile in $(0,0)$

Es. di funzione non derivabile in un p.to

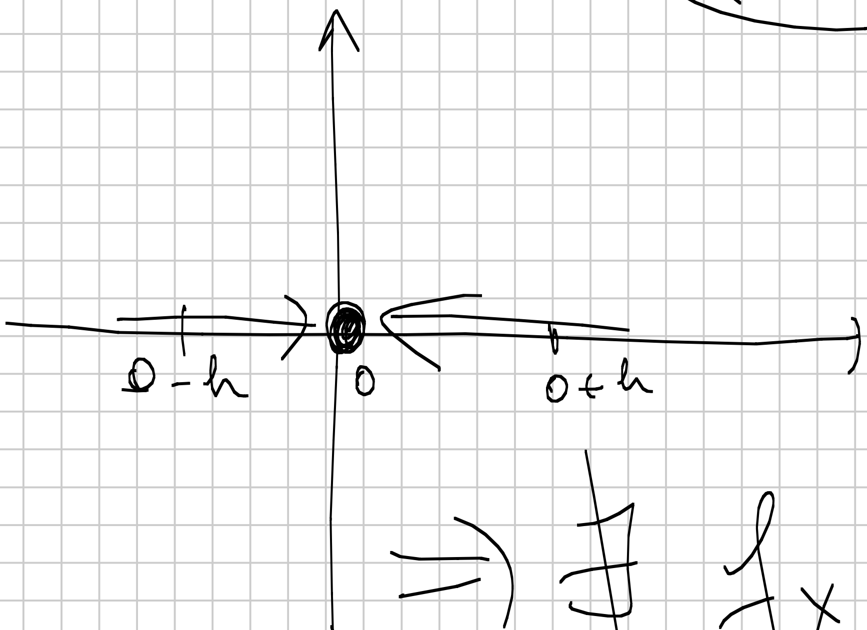
$$f(x,y) = \sqrt{\underbrace{x^2+y^2}_{\geq 0}} \qquad D = \mathbb{R}^2$$

$$f_x(x,y) = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}}$$

$$f_y(x,y) = \frac{\cancel{2}y}{\cancel{2}\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}}$$

non sono definite in $(0,0)$

Calcoliamo $f_x(0,0)$ non posso usare



l'espressione sopra

$$f(x,y) = \sqrt{x^2+y^2}$$

$$f_{/\text{retta}}(x,0) = \sqrt{x^2} = |x|$$

non è derivabile in $x=0$

$$\Rightarrow \nexists \; f_x(0,0)$$

$f(x,y) = \sqrt{x^2+y^2}$ <u>non</u> è derivabile rispetto a x in $(0,0)$.

$f : \mathbb{R} \to \mathbb{R}$

f derivabile in $x_0 \implies f$ continua in x_0

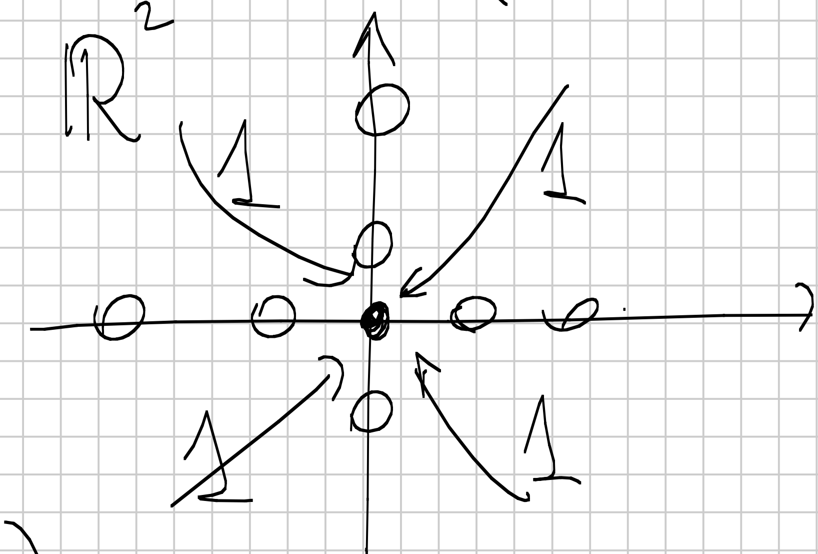
$f : \mathbb{R}^2 \to \mathbb{R}$

f derivabile in $x_0 \not\Longrightarrow f$ continua in x_0

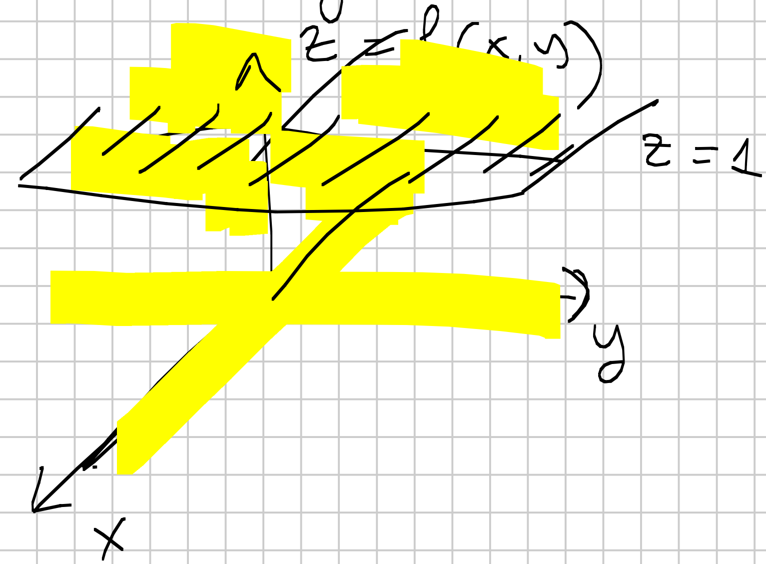
<u>Es.</u>

$$f(x,y) = \begin{cases} 0 & x \cdot y = 0 \\ 1 & x \cdot y \neq 0 \end{cases}$$

$D = \mathbb{R}^2$



$(0,0)$



$f(x,y)$ <u>non</u> è continua in $(0,0)$

$$\nexists \lim_{(x,y)\to(0,0)} f(x,y)$$

<u>ma</u> $f_x(0,0) = 0 \qquad 0 = f_y(0,0)$

quindi f è <u>derivabile</u> in $(0,0)$ ma <u>non</u> è <u>continua</u> in $(0,0)$.

stesse definizioni $\quad f : \mathbb{R}^3 \to \mathbb{R}$

$f(x, y, z)$ $\quad f_x =$

f_y

f_z

es: $f(x, y, z) = \operatorname{sen}(xz) + \log(y+x)$

$f_x = \cos(xz) \cdot z + \frac{1}{y+x} \cdot 1$

$$\begin{cases} f_y = 0 + \dfrac{1}{y+x} \cdot 1 \\ f_z = \cos(xz) \cdot x + 0 \end{cases}$$

$$(f_x, f_y, f_z) = \operatorname{grad} f = Df$$

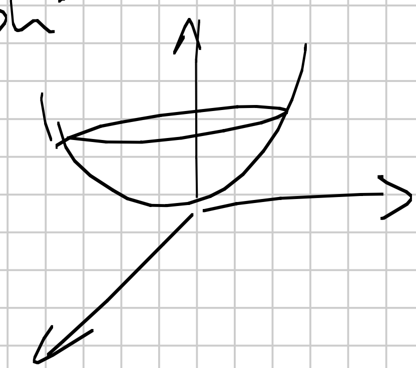
$f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) \quad x \in \mathbb{R}$

$f'(x) = 0$ p.ti critici (stazionari)

$f: \mathbb{R}^2 \to \mathbb{R}$ punti critici

Def. $(x_0, y_0) \in D$ è un punto critico per f se $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$

<u>Es.</u> Determinare i p.ti critici di



$$f(x,y) = x^2 + y^2$$

$$D = \mathbb{R}^2$$

$$f_x(x,y) = 2x = 0 \qquad x = 0$$

e

$$f_y(x,y) = 2y = 0 \qquad y = 0$$

$(0,0)$ l'unico p.to <u>critico</u>.

$f(x,y) = e^{\frac{x}{y}}$ $\qquad y \neq 0$

$f_x = e^{\frac{x}{y}} \cdot \frac{1}{y} \overset{?}{=} 0 \qquad$ mai!

$f_y =$

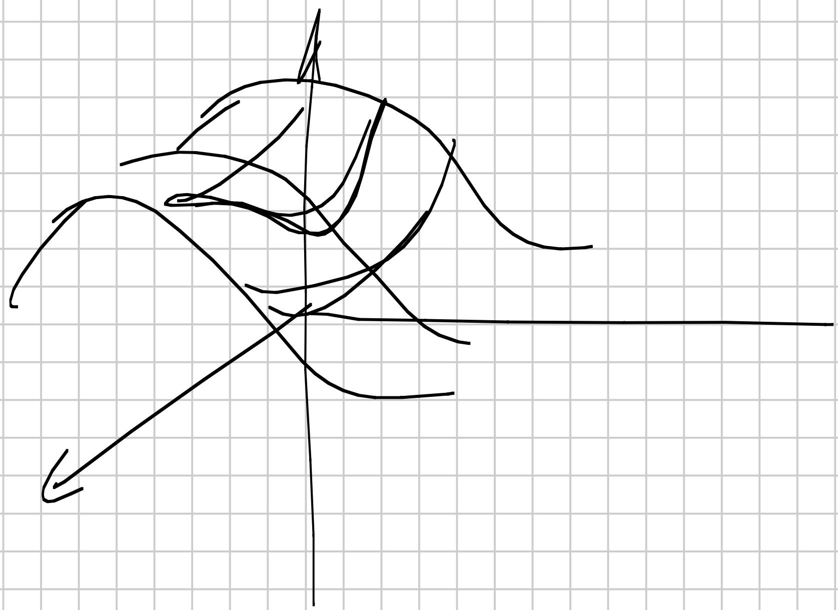
$f(x,y) = x^2 - y^2$

paraboloide iperbolico

$f_x = 2x = 0$

$f_y = -2y = 0$

$(0,0)$ } stazionario



guardate in rete!

Derivate successive

$f_x(x,y)$

$f_y(x,y)$

$f: \mathbb{R}^2 \to \mathbb{R}$

$f(x,y)$

$\frac{\partial}{\partial x} f_x(x,y) = f_{xx}$

$\frac{\partial}{\partial y} f_x(x,y) = f_{xy}$

$\frac{\partial}{\partial x} f_y(x,y) = f_{yx}$

$\frac{\partial}{\partial y} f_y(x,y) = f_{yy}$

4 derivate seconde di f

es. $f(x,y) = x^5 y^2 + 5$

$f_x(x,y) = 5x^4 y^2$

$f_y(x,y) = 2x^5 y$

$f_{xx} = 5 \cdot 4 x^3 y^2 = 20x^3 y^2$

$f_{xy} = 5x^4 \cdot 2y = 10x^4 y$

$f_{yx} = 2 \cdot 5x^4 y = 10x^4 y$

$f_{yy} = 2x^5$

es. $f(x,y) = e^{xy^2}$ | P. critici

$f_x = e^{xy^2}\, y^2$

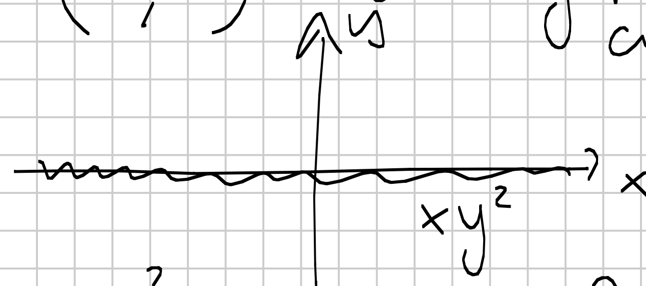
$f_y = e^{xy^2}\, 2xy$

$f_x = 0$
$f_y = 0$

$\begin{cases} y^2 = 0 \\ 2xy = 0 \end{cases}$

$y = 0$

$(x, 0)$ sono p.ti critici



$f_{xx} = y^2 e^{xy^2} \cdot y^2 = y^4 e^{xy^2}$

$f_{xy} = (f_x)_y = e^{xy^2} \cdot 2xy\, y^2 + e^{xy^2} \cdot 2y$

$= e^{xy^2}(2xy^3 + 2y)$

$$f_{yx} = (f_y)_x = e^{xy^2} y^2\, 2xy + e^{xy^2}\, 2y =$$

$$= e^{xy^2}(2xy^3 + 2y)$$

$$f_y = \underbrace{e^{xy^2} \cdot 2xy}$$

$$f_{yy} = e^{xy^2} \cdot 2xy \cdot 2xy +$$

$$+ e^{xy^2} \cdot 2x = e^{xy^2}(4x^2y^2 + 2x)$$

$f : \mathbb{R}^2 \to \mathbb{R}$ $\qquad f_{xx}, f_{xy}, f_{yx}, f_{yy}$

matrice 2×2

$$\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} =: D^2 f$$

Matrice hessiana

Es. precedente Calcolare la matrice hessiana nel p.to $(1,0)$

$$D^2 f(1,0) = \begin{pmatrix} f_{xx}(1,0) & f_{xy}(1,0) \\ f_{yx}(1,0) & f_{yy}(1,0) \end{pmatrix}$$

$x=1$
$y=0$

$f_{xx}(1,0) = 0$ $\quad$ $f_{yy}(1,0) = 2$

$f_{xy}(1,0) = 0$ $\quad$ $f_{yx}(1,0) = 0$

$$\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$$

Def. Se $\exists$ tutte le derivate seconde in (x_0, y_0) dico che f è *derivabile due volte* in (x_0, y_0).

<u>Teorema di Schwarz</u>

$f : X \to \mathbb{R} \qquad X \subseteq \mathbb{R}^2 \quad , \; f(x,y)$

$f(x,y)$ <u>derivabile</u> due volte in X

se $f_{xy}(x,y)$ e $f_{yx}(x,y)$ sono continue in (x_0, y_0) allora $f_{xy}(x_0,y_0) = f_{yx}(x_0,y_0)$.

$$\begin{pmatrix} f_{xx} & f_{xy} \\ & f_{yy} \end{pmatrix}$$

Def. di f differenziabile $\Longrightarrow$

f derivabile