Metodo di integrazione per parti

Metodo di integrazione per **sostituzione**

$$\int f(x)\,dx = \int f(\varphi(t))\,\varphi'(t)\,dt$$

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f(\varphi(t))\,\varphi'(t)\,dt \qquad a = \varphi(\alpha) \quad b = \varphi(\beta)$$

$$x = \varphi(t)$$

$$dx = \varphi'(t)\,dt$$

es. $\int \operatorname{tg} x \, dx = \int \frac{\operatorname{sen} x}{\cos x} \, dx$

$\cos x = t$

$-\operatorname{sen} x \, dx = dt$

$= \int \frac{-dt}{t} = - \log |t| = - \log |\cos x| + K$

$\log x = t$

es. $\int \frac{1}{x \log x} \, dx = \int \frac{dt}{t} =$

$\frac{1}{x} \, dx = dt$

$= \underline{\log |\log x| + K}$

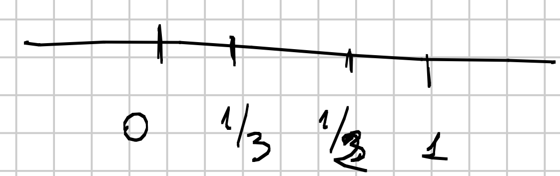
$$\int_2^3 \frac{1}{x\log x}\,dx = \log|\log 3| - \log|\log 2| =$$

$$= \log(\log 3) - \log(\log 2)$$

$$\int_{-5}^{-4} \frac{1}{x\log x}\,dx = \cancel{\log|\log(-}$$

non posso farlo

non è definita



$$\int_{1/3}^{1/2} \frac{1}{x \log x}\, dx = \log\left|\log\left(\frac{1}{2}\right)\right| - \log\left|\log\left(\frac{1}{3}\right)\right|$$

$$= \log\left(-\log 1/2\right) - \log\left(-\log 1/3\right) =$$

Integrali <u>impropri</u> (<u>generalizzati</u>)

$$\int_a^b f(x)\,dx$$

$f : [a,b] \to \mathbb{R}$

e limitato

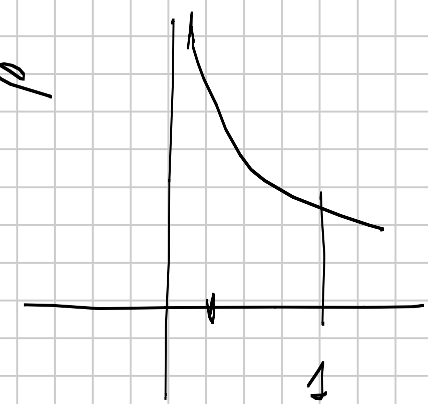
1) f limitato in $[a,b]$

2) $[a,b]$ limitato.

Cosa succede se f non è limitato in $[a,b]$

$f(x) = \frac{1}{x}$ $\quad [1,2]$ $\quad \int_1^2 \frac{1}{x}\,dx = \log 2 - \lg 1$

$f(x) = \frac{1}{x}$ $\quad [0,1]$ $\quad$ non è limitata



$\int_a^b \frac{1}{x}\,dx$ $\quad$ no!

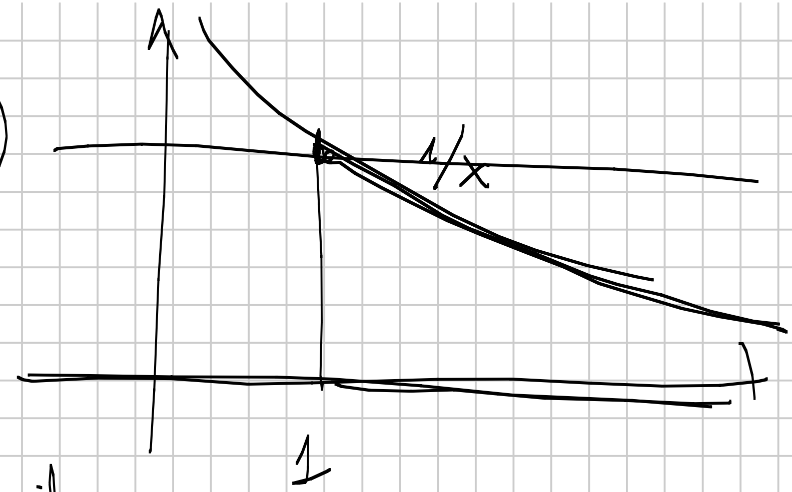
1 Vogliamo estendere la def. di $\int_a^b$ anche nel caso che f non sia limitata in (a,b).

2 $f(x) = \frac{1}{x}$

$$\int_1^{+\infty} \frac{1}{x}\,dx$$

$(1, +\infty)$

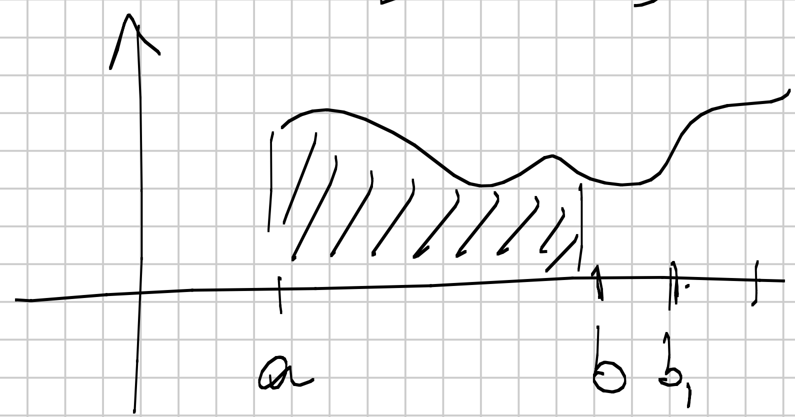
l'intervallo non è limitato



Integrale in senso improprio per f su intervalli illimitati

Caso $f \geq 0$, continua in $[a, +\infty)$

$$\int_a^{+\infty} f(x)\,dx := \lim_{b \to +\infty} \int_a^b f(x)\,dx$$



$f(x) \geq 0$

$$\int_a^x f(t)\,dt = F(x) \text{ è crescente}$$

$\exists$ sempre $\lim_{x \to +\infty} F(x)$

Quindi $\lim_{b \to +\infty} \int_a^b f(x)\,dx \ \exists$ sempre

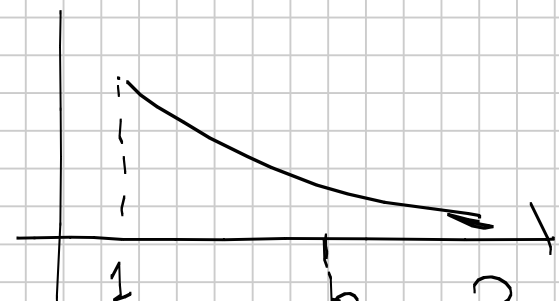
FINITO | INFINITO $(+\infty)$

$f(x)$ è integrabile in senso improprio in $[a, +\infty)$ (l'integrale converge)

$$\int_a^{+\infty} f(x)\,dx = l \in \mathbb{R}$$

$f(x)$ non è integrabile in $[a, +\infty)$ (l'integrale diverge).

Es. $f(x) = \frac{1}{x}$ $\quad (1, +\infty)$



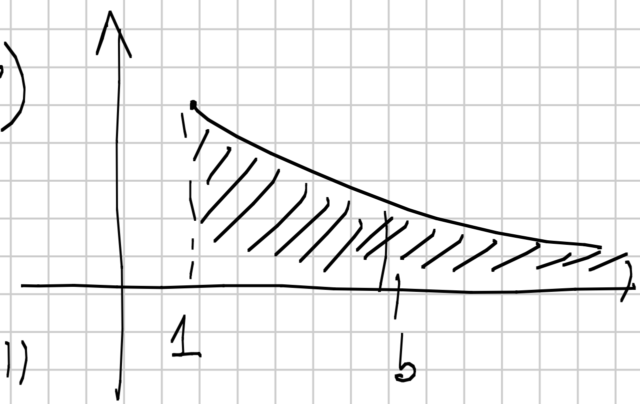
è integrabile in senso improprio in $(1, +\infty)$?

$$\int_1^{+\infty} \frac{1}{x}\,dx = \lim_{b\to+\infty} \int_1^b \frac{1}{x}\,dx = \lim_{b\to+\infty} \log|x| \Big/_1^b =$$

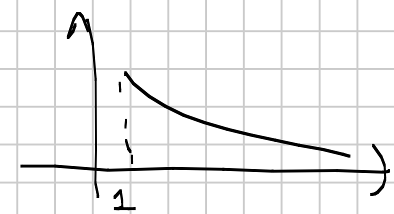
$$= \lim_{b\to+\infty} \log b = +\infty$$

$1/x$ <u>NON</u> è integrabile in $(1, +\infty)$

$$\int_1^{+\infty} \frac{1}{x}\,dx = +\infty = \text{"Area della regione tratteggiata"}$$



es. $f(x) = \frac{1}{x^2}$ $(1, +\infty)$



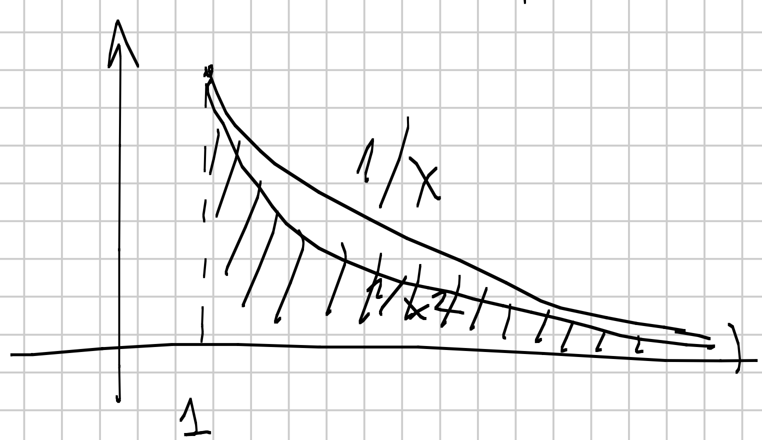
$f(x)$ è integrabile in senso improprio in $(1, +\infty)$?

$$\int_1^{+\infty} \frac{1}{x^2}\, dx \stackrel{.}{=} \lim_{b \to +\infty} \int_1^b \frac{1}{x^2}\, dx = \lim_{b \to +\infty} \left. -x^{-1} \right|_1^b =$$

$$= \lim_{b \to +\infty} \left(-\frac{1}{b} + 1 \right) = 1$$

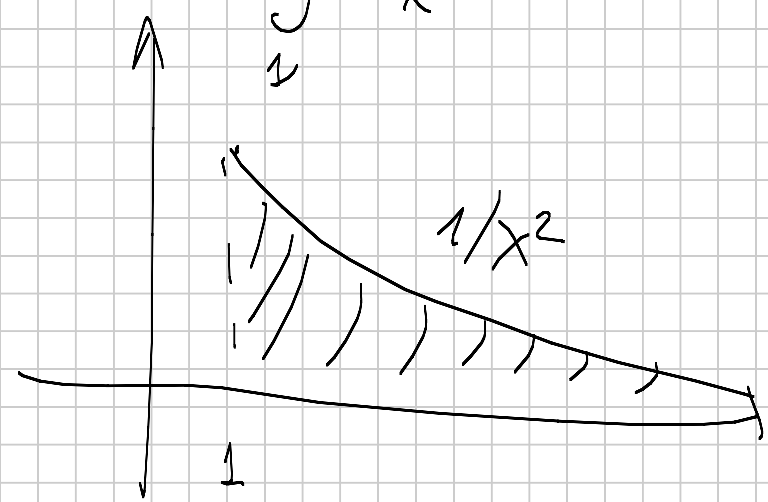
$\frac{1}{x^2}$ è integrabile in s.i. in $(1, +\infty)$.

$f(x) = \frac{1}{x}$ $\qquad \int_1^{+\infty} \frac{1}{x}\,dx = +\infty$



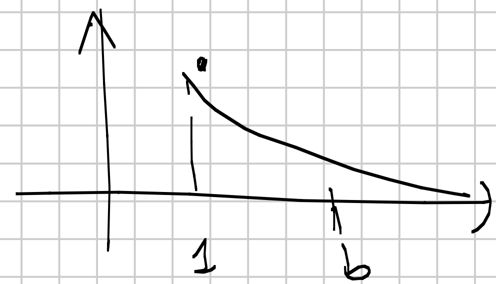
$f(x) = \frac{1}{x^2}$

$$\int_1^{+\infty} \frac{1}{x^2}\,dx = 1$$



In generale

$f(x) = \frac{1}{x^\alpha}$ in $(1, +\infty)$ $\quad \alpha > 0$

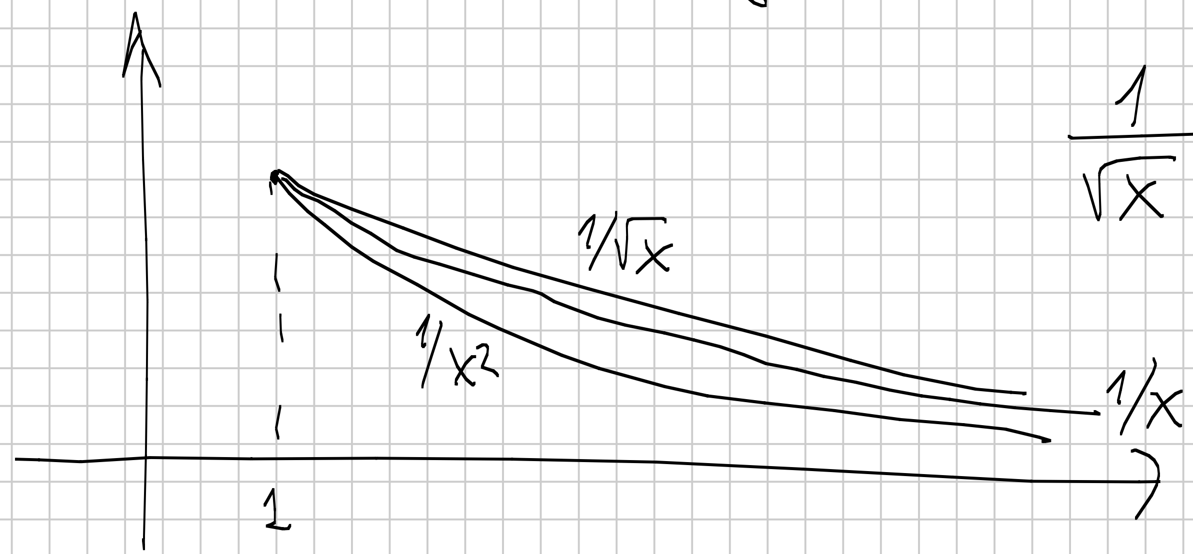


$$\int_1^{+\infty} \frac{1}{x^\alpha}\,dx = \lim_{b\to+\infty} \int_1^b \frac{1}{x^\alpha}\,dx = \lim_{b\to+\infty} \left(\frac{x^{-\alpha+1}}{-\alpha+1} \right)\Bigg|_1^b$$

$$= \lim_{b\to+\infty} \left(\frac{b^{1-\alpha}}{1-\alpha} - \frac{1}{1-\alpha} \right) \begin{cases} +\infty & (1-\alpha) > 0 \quad \alpha < 1 \\ \frac{1}{\alpha-1} & 1-\alpha < 0 \quad \alpha > 1 \end{cases}$$

Riassumendo

$$\int_1^{+\infty} \frac{1}{x^\alpha}\,dx \quad \begin{cases} \text{diverge se } \alpha \leq 1 \\ \text{converge se } \alpha > 1 \end{cases}$$



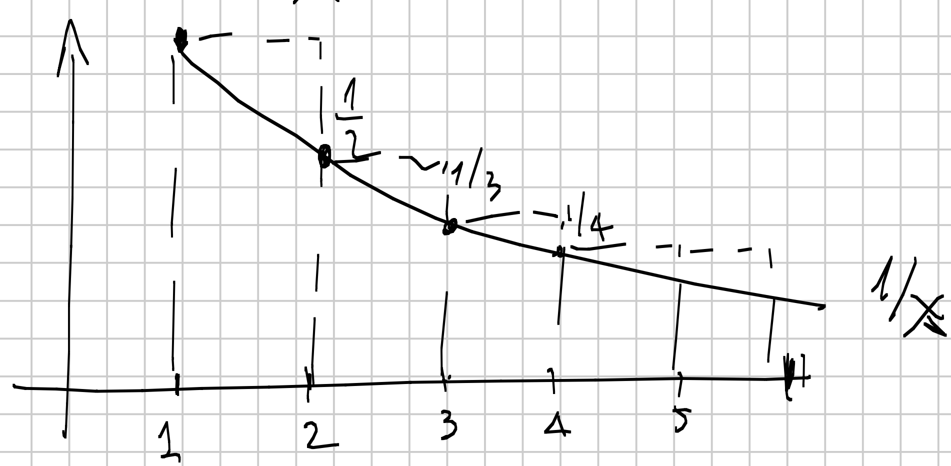
$$\frac{1}{\sqrt{x}} \qquad \alpha = 1/2$$

$\to$ non integrabile in $(1, +\infty)$

$$\sum_{n=1}^{+\infty} \frac{1}{n^\alpha} \begin{cases} \text{converge} & \text{se } \alpha > 1 \\ \text{diverge} & \text{se } \alpha \leq 1 \end{cases}$$

collegamento tra serie e integrali generalizzati

$$f(x) = \frac{1}{x} \quad \text{in } (1, +\infty)$$



$$S(D) = \sum_{n=1}^{\infty} \frac{1}{n}$$

$\sum a_k$ converge $\Rightarrow \lim_{k \to \infty} a_k = 0$

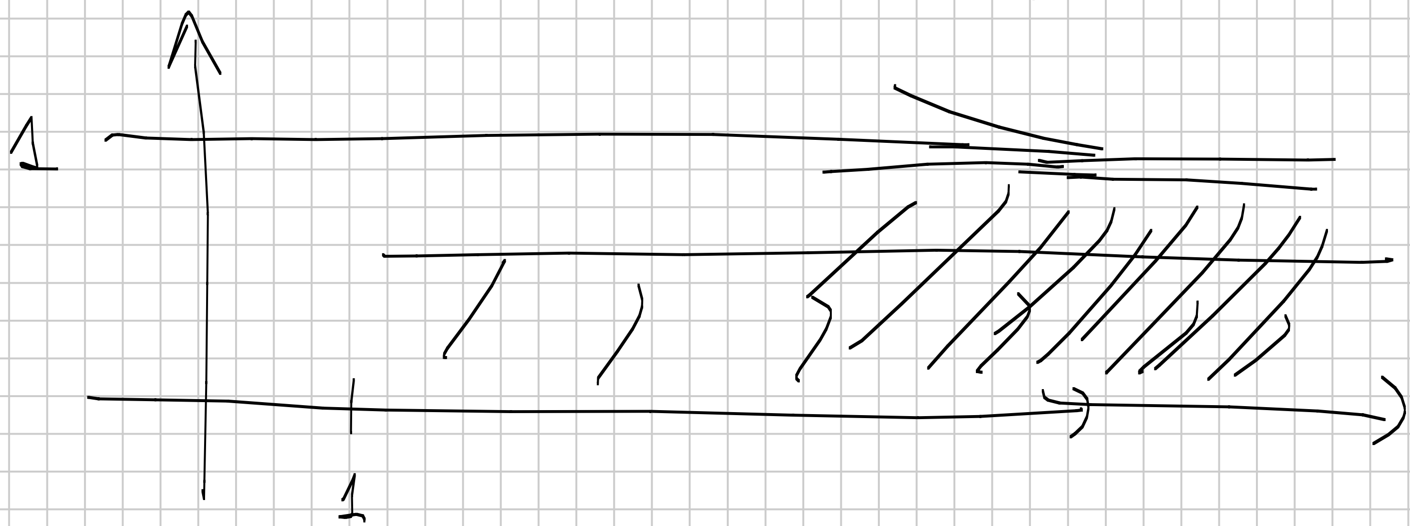
$f \geq 0$

$\int_a^{+\infty} f(x)\,dx$ converge e se $\exists \lim_{x \to +\infty} f(x) \Rightarrow \lim_{x \to +\infty} f(x) = 0$

Se il $\lim_{x \to +\infty} f(x)$ $\exists$ ed è $\neq 0 \Rightarrow \int_a^{+\infty} f(x)\,dx$ <u>diverge</u> !

<u>Es.</u> $f(x) = \frac{x}{x+1}$ è integrabile in s.i. in $(1, +\infty)$?

$$\int_1^{+\infty} \frac{x}{x+1}\,dx = +\infty \qquad \lim_{x\to+\infty} \frac{x}{x+1} = 1 \neq 0$$

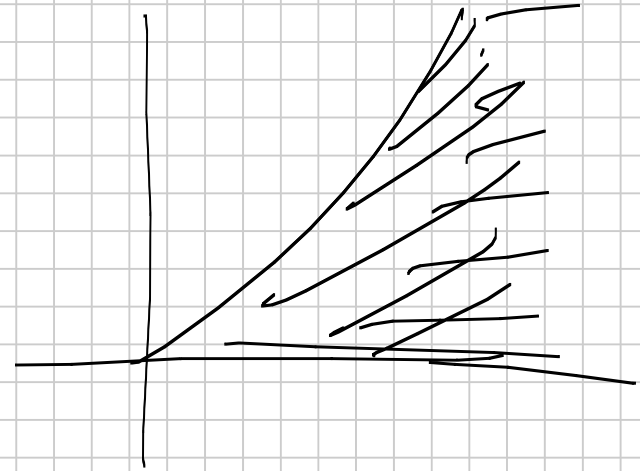


es

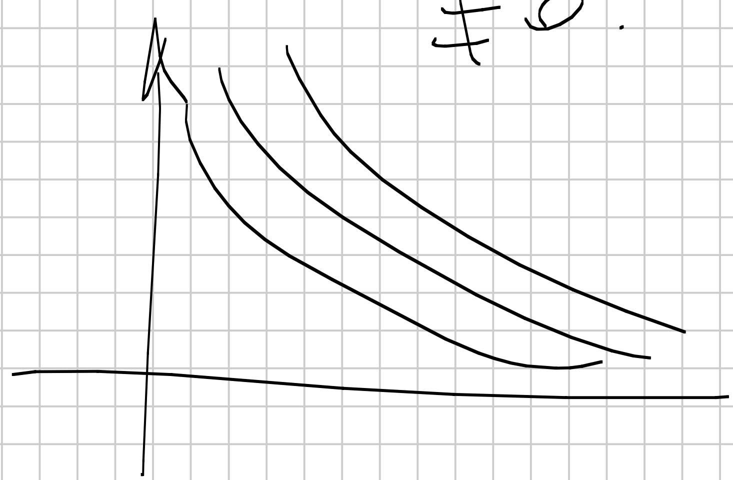
$$\int_1^{+\infty} x^2 \, dx$$

$$\lim_{x \to +\infty} x^2 = +\infty$$

$\neq 0$.



$\frac{1}{x^\alpha}$



Criteri di convergenza

non sempre però si riesce a calcolare una funzione di f

Criterio del confronto

$f, g : [a, +\infty) \to \mathbb{R}$ $\quad 0 \le f \le g$

se $\int_a^{+\infty} g(x)\,dx$ converge $\Rightarrow \int_a^{+\infty} f(x)\,dx$

se $\int_a^{+\infty} f(x)\,dx$ diverge $\Rightarrow \int_a^{+\infty} g(x)\,dx$ diverge es. $\quad f(x) = \left(\frac{\operatorname{sen} x}{x}\right)^2$

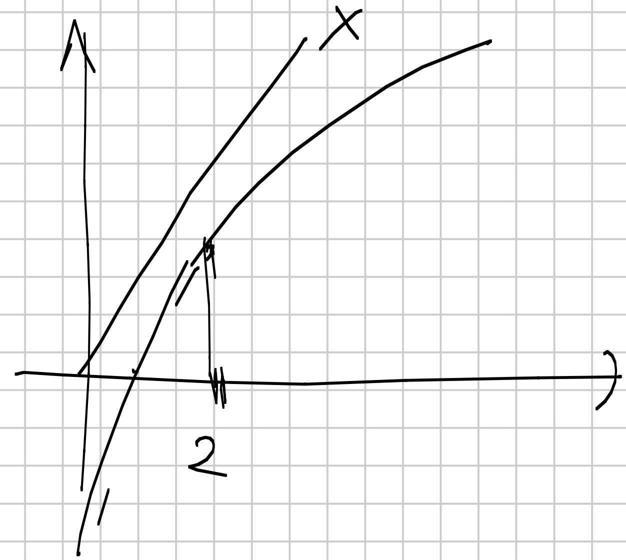
$$\int_1^{+\infty} \left(\frac{\operatorname{sen} x}{x}\right)^2 dx \quad \text{converge?}$$

$$\underbrace{\frac{(\operatorname{sen} x)^2}{x^2}}_{f} \leq \underbrace{\frac{1}{x^2}}_{g}$$

poiché $\int_1^{+\infty} \frac{1}{x^2}\, dx$ converge $\Rightarrow \int_1^{+\infty} \left(\frac{\operatorname{sen} x}{x}\right)^2 dx$ converge

Es.

$$\int_2^{+\infty} \frac{1}{\log x}\, dx \quad \text{converge ?}$$



$$x > \log x \quad \forall x > 0$$

$$\frac{1}{\log x} > \frac{1}{x} > 0$$

$\frac{1}{x}$ non è integrabile in $(2, +\infty)$

$\Rightarrow \frac{1}{\log x}$ <u>non</u> è integrabile in $(2, +\infty)$.

Principio del confronto asintotico

$f, g : [a, +\infty) \to \mathbb{R}$, $f > 0$, $g > 0$.

a) $\lim_{x \to +\infty} \frac{f(x)}{g(x)} = L > 0$, $L \in \mathbb{R}$

f è integrabile in s.i. $\iff$ g è integrabile in s.i.

b) $\lim_{x \to +\infty} \frac{f(x)}{g(x)} = 0$ se g è integrabile in s.i

$\left(f = o(g) \right)_{x \to +\infty}$ $\Longrightarrow$ f è integrabile in s.i.

Es. $f(x) = \operatorname{sen}\left(\frac{1}{x^2}\right)$ è integrabile in s.i. in $[1, +\infty)$?

$f \geq 0$

$x \to +\infty \quad \frac{1}{x^2} \to 0 \quad \operatorname{sen}\left(\frac{1}{x^2}\right) \sim \frac{1}{x^2}$

$\int_1^{+\infty} \operatorname{sen}\frac{1}{x^2}\,dx$ ha lo stesso comportamento di $\int_1^{+\infty} \frac{1}{x^2}\,dx$

$\operatorname{sen}\frac{1}{x^2} = \frac{1}{x^2} + o\left(\frac{1}{x^2}\right)$

f, g

Confronto asintotico con $\dfrac{1}{x^\alpha}$

$f \geq 0$ in $[1, +\infty)$

a) $\displaystyle\lim_{x \to +\infty} \frac{f(x)}{\frac{1}{x^\alpha}} = L \geq 0$, per qualche $\alpha > 1$

allora f è integrabile in s.i. in $[1, +\infty)$

b) $\displaystyle\lim_{x \to +\infty} \frac{f(x)}{\frac{1}{x^\alpha}} = L > 0$ (anche $+\infty$) per qualche

$\alpha \leq 1 \implies f$ <u>non</u> è int. in s.i.

es.

$$\int_1^{+\infty} \left(e^{1/x} - 1\right) dx$$

$f(x)$

diverge !

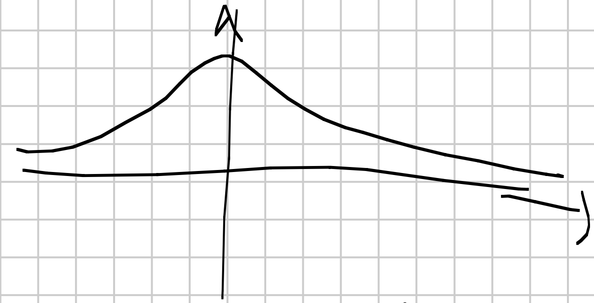
$$\cancel{e^{1/x} > 1 \quad \frac{1}{x} > 0} \quad \text{sì}!$$

$$\underbrace{e^{\frac{1}{x}} - 1}_{f} = \underbrace{\frac{1}{x}}_{\frac{1}{x}} + o\left(\frac{1}{x}\right) \quad x \to +\infty$$

$$\int_1^{+\infty} \frac{1}{x} dx \text{ diverge}$$

es.

$$\int_1^{+\infty} e^{-x^2}\,dx \quad \text{converge!}$$



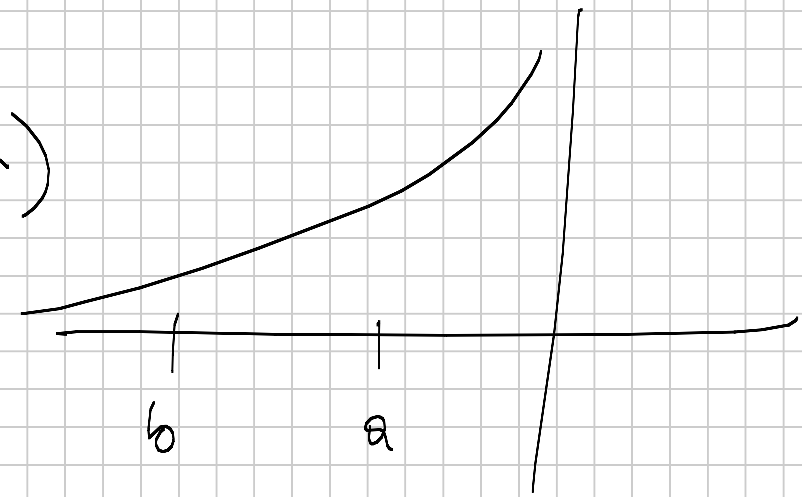
(caso 2) con $L = 0$)

$$e^{-x^2} = o\left(\frac{1}{x^2}\right) \quad x \to +\infty$$

$$\int_1^{+\infty} \frac{1}{x^2}\,dx \text{ converge} \implies \int_1^{+\infty} e^{-x^2}\,dx \text{ converge}$$

$(a, +\infty)$

Analog. in $(-\infty, a)$



$$\lim_{b \to -\infty} \int_b^a f(x)\,dx$$

f illimitata in $[a,b]$.

prossima volta.

Esercizi di ricerca di primitive

funzioni razionali

~~polinomio~~

polinomio

1) $$\int \frac{1}{ax+b}\,dx =$$

$$ax+b = y$$

$$a\,dx = dy$$

$$= \int \frac{1}{y}\,\frac{1}{a}\,dy = \frac{1}{a}\log|ax+b| + K$$

2) $\dfrac{\text{polinomio di } 1^\circ \text{ grado}}{\text{polinomio di } 2^\circ \text{ grado}}$) $\dfrac{1}{\text{polinomio di } 2^\circ \text{ grado.}}$

$$\int \frac{1}{1+x^2}\,dx = \operatorname{arctg} x + K$$

$$\int \frac{1}{a^2+x^2}\,dx = \int \frac{1}{a^2\left(1+\left(\frac{x}{a}\right)^2\right)}\,dx =$$

$$= \frac{1}{a^2} \int \frac{1}{1 + \left(\frac{x}{a}\right)^2} dx =$$

$$\frac{x}{a} = y$$

$$x = a y$$

$$dx = a\, dy$$

$$= \frac{1}{a^{\cancel{2}}} \int \frac{1}{1 + y^2} \cancel{a}\, dy$$

$$= \frac{1}{a} \operatorname{arctg}\left(\frac{x}{a}\right) + K$$

es.

$$\int \frac{1 \cdot \operatorname{arctg} x}{(1+x^2)(1+\operatorname{arctg}^2 x)}\, dx =$$

$\operatorname{arctg} x = t$

$\frac{1}{1+x^2}\, dx = dt$

$$= \int \frac{t}{1+t^2}\, dt =$$

$1+t^2 = y$

$2t\,dt = dy$

$$= \int \frac{1}{2} \frac{dy}{y} = \frac{1}{2} \log(1+\operatorname{arctg}^2 x)$$

Dire se

$$\int_1^{+\infty} \frac{\arctan x}{(1+x^2)(1+\arctan^2 x)}\,dx$$

dire se converge e in caso affermativo calcolarlo.

$$f(x) \qquad \frac{\frac{\pi}{2}}{(1+x^2)\left(1+\left(\frac{\pi}{2}\right)^2\right)}$$

$1+x^2 \sim x^2 \quad x \to +\infty$

$$f(x) \sim \frac{1}{x^2} \cdot L$$

Sì! converge

$$\int_1^{+\infty} \frac{\operatorname{arctg} x}{(1+x^2)\operatorname{arctg} x}\, dx = \lim_{b\to+\infty} \int_1^{b} \frac{\operatorname{arctg} x}{(1+x^2)\operatorname{arctg} x}\, dx =$$

$$= \lim_{b\to+\infty} \frac{1}{2} \log\left(1+\operatorname{arctg}^2 x\right)\Big/_1^b =$$

$$= \lim_{b\to+\infty} \frac{1}{2}\left(\log\left(1+\operatorname{arctg}^2 b\right) - \log\left(1+\operatorname{arctg} 1\right)\right)$$

$$= \frac{1}{2}\left(\log\left(1+\left(\frac{\pi}{2}\right)^2\right) - \log\left(1+\operatorname{arctg} 1\right)\right)$$

es. $$\int \frac{\sqrt{x}}{2+\sqrt{x}}\,dx = \int \frac{t}{2+t}\cdot 2t\,dt$$

$$\sqrt{x} = t \qquad x = t^2 \qquad dx = 2t\,dt$$

$$\frac{t+2-2}{2+t} = 1 - \frac{2}{2+t}$$

$$\int 2t\left(1 - \frac{2}{2+t}\right) dt = \underbrace{\int 2t\,dt}_{\text{o.k.}} - \int \frac{4t}{2+t} dt$$

$$4 \int \frac{t+2-2}{2+t} dt = 4\int\left(1 - \frac{2}{t+2}\right) dt$$

$$= 4\left(t - 2\log|t+2|\right)$$

es.

$$\int \operatorname{sen}^5 x \cos x \, dx \qquad \operatorname{sen} x = t$$

$$\int \operatorname{sen}^3 x \, dx = \int \operatorname{sen} x \operatorname{sen}^2 x \, dx = \int \operatorname{sen} x (1 - \cos^2 x) dx$$

$$\cos x = t$$