Integrali in senso <u>improprio</u>

f definita in $(a, +\infty)$

$$\int_a^{+\infty} f(x)\,dx \doteq \lim_{b \to +\infty} \int_a^b f(x)\,dx$$

$+\infty$

$\exists$ finito

$$f(x) = \frac{1}{x^\alpha}$$

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integrabili in $(1, +\infty) \iff \alpha > 1$

Confronto

$f(x) \sim g(x)$, $x \to +\infty$ $\qquad \int_a^{+\infty} f$ e $\int_a^{+\infty} g$

$f(x) \sim \frac{1}{x^\alpha}$

Un'altra classe di funzioni

$g(x) = \frac{1}{x^\alpha (\log x)^\beta}$ in $[2, +\infty)$

Es. $\displaystyle\int_2^{+\infty} \frac{1}{x\log x}\,dx = \lim_{b\to+\infty} \int_2^{b} \frac{1}{x\log x}\,dx = \lim_{b\to+\infty} \left(\log(\log b) - \log(\log 2)\right) = +\infty$

oss. $\dfrac{1}{x\log x}$ non è asintotica a nessuna potenza di $\dfrac{1}{x^\alpha}$ per $x \to +\infty$

$$\frac{\cancel{x}}{\cancel{x}\log x} \to 0 \qquad \frac{x^\alpha}{x\log x} \begin{cases} \to 0 \\ \to +\infty \end{cases}$$

$$\int_2^{+\infty} \frac{1}{x^\alpha (\log x)^\beta}\,dx \quad \text{converge se} \begin{cases} \alpha > 1, & \forall \beta \in \mathbb{R} \\ \alpha = 1, & \beta > 1 \end{cases}$$

<u>Es.</u> $\displaystyle\int_2^{+\infty} \frac{1}{x(\log x)^5}\,dx$ converge $\qquad \displaystyle\int_2^{+\infty} \frac{1}{x^2(\log x)^{1/100}}$ converge

$$\int_2^{+\infty} \frac{1}{\sqrt{x}\,(\log x)^{1000}}$$ **diverge**

Esercizio

$$\int_2^{+\infty} \frac{(1+x^3)^{1-\alpha}}{x^{2-\alpha}\log^2 x}\,dx$$

$f(x)$

dire per quali α l'integrale converge e calcolarlo per $\alpha = 1$

$$f(x) \sim \frac{x^{3(1-\alpha)}}{x^{2-\alpha} \cdot \log^2 x} = \frac{1}{x^{2\alpha-1} \log^2 x} \qquad x \to +\infty$$

$$2 - \alpha - 3 + 3\alpha$$

$$2\alpha - 1 \geq 1 \qquad \alpha \geq 1$$

$$\log x = t$$

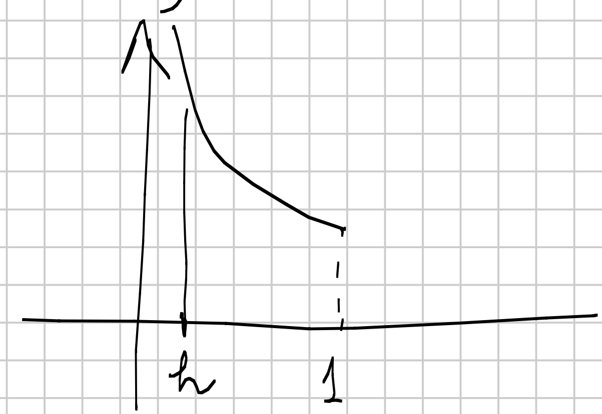
$$\alpha = 1 \qquad \int_2^{+\infty} \frac{1}{x \log^2 x}\, dx = \lim_{b \to +\infty} \int_2^b \frac{1}{x \log^2 x}\, dx$$

Altro tipo di <u>integrale</u> in senso <u>improprio</u>

f illimitata in (a, b) (limitato).

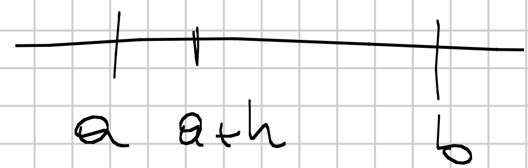
<u>Es.</u> $f(x) = \frac{1}{x}$ in $(0, 1)$

$$\int_0^1 \frac{1}{x}\, dx = \;?$$



$\frac{1}{x}$ è limitata in $[h, 1]$

$$\int_0^1 \frac{1}{x}\,dx =: \lim_{h \to 0^+} \int_h^1 \frac{1}{x}\,dx$$



In generale f definita in (a, b), $f \geq 0$

$$\int_a^b f(x)\,dx := \lim_{h \to 0^+} \int_{a+h}^b f(x)\,dx$$

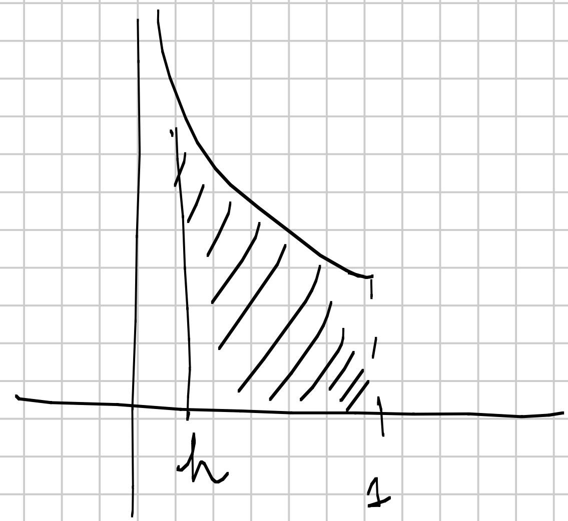
Finito $\Rightarrow f$ è integrabile in s.i. in (a, b)

$+\infty$ $\Rightarrow$ f <u>non</u> è integrabile in (a,b), (l'$\int$ diverge)

<u>Es.</u> $f(x) = 1/x$

$$\int_h^1 \frac{1}{x}\,dx = \log|1| - \log|h|$$

$$= \underbrace{-\log h}_{>0}$$



$$\int_0^1 \frac{1}{x}\,dx =: \lim_{h\to 0^+} \int_h^1 \frac{1}{x}\,dx = \lim_{h\to 0^+} \left(-\log h\right) = +\infty$$

$\frac{1}{x}$ <u>non</u> è integrabile in $(0,1)$

<u>Caso generale</u> $\frac{1}{x^\alpha}$ in $(0,1)$

$$\int_0^1 \frac{1}{x^\alpha}\,dx = \lim_{h\to 0^+} \int_h^1 \frac{1}{x^\alpha}\,dx = \lim_{h\to 0^+} \left(\frac{x^{-\alpha+1}}{-\alpha+1} \right)\Big|_h^1$$

$$= \lim_{h\to 0^+} \left(\frac{1}{1-\alpha} - \frac{h^{1-\alpha}}{1-\alpha} \right) \text{ è finito} \iff 1-\alpha>0$$

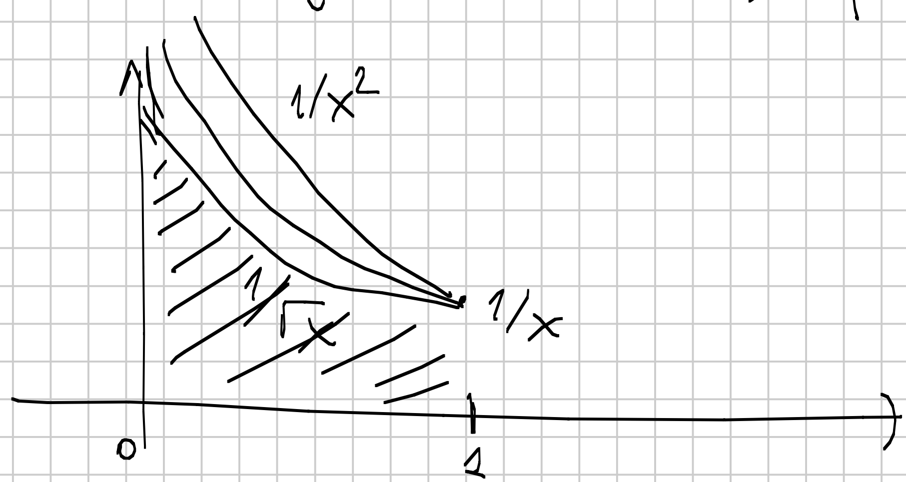
$$\iff \alpha<1$$

Riassumendo

$$\boxed{\int_0^1 \frac{1}{x^\alpha}\,dx \text{ converge} \iff \alpha < 1}$$

$\int_0^1 \frac{1}{\sqrt{x}}\,dx$ converge $\quad \int_0^1 \frac{1}{x^2}\,dx$

$\int_1^{+\infty} \frac{1}{x^2}\,dx$ converge $\Leftrightarrow \alpha > 1$



$\alpha = 1$ $\qquad \dfrac{1}{x}$ non è integrabile nè tra $(0,1)$

nè tra $(1,+\infty)$

Anche in questo caso valgono i principi di confronto e confronto asintotico

Quindi $\quad f(x) \longrightarrow +\infty \quad x \to 0^+$

$$\int_0^1 f(x)\,dx$$

$$f(x) \sim \frac{1}{x^\alpha} \qquad x \to +\infty$$

<u>Es.</u>

$$\int_0^1 \frac{1}{\operatorname{sen} x}\, dx$$

$$\frac{1}{\operatorname{sen} x} \to +\infty \qquad x \to 0^+$$

$$\operatorname{sen} x \sim x \qquad x \to 0^+$$

$$\frac{1}{\operatorname{sen} x} \sim \frac{1}{x} \qquad x \to 0^+$$

<u>non</u> è integrabile tra $(0,1)$

$\Downarrow$

No!

<u>Es.</u> $\displaystyle\int_0^1 \frac{1}{(e^x-1)^\alpha}\,dx \quad , \quad \alpha > 0$

$e^x = 1 + x + o(x) \qquad x \to 0$

$e^x - 1 = x + o(x) \qquad \dfrac{1}{(e^x-1)^\alpha} \sim \dfrac{1}{x^\alpha}$

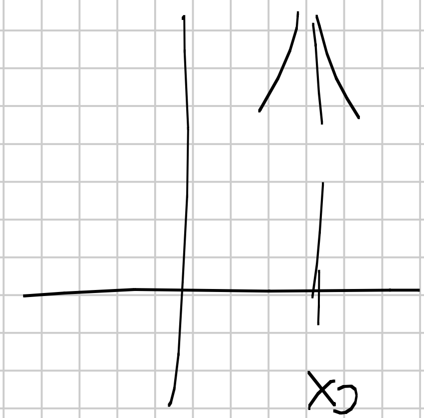
è integrabile in $(0,1)$ $\forall \alpha < 1$

è integrabile in $(0,1)$

$\forall\, \alpha < 1$

Se non siamo in $x=0$?

$$\frac{1}{(x-x_0)^\alpha} \longrightarrow +\infty \qquad x \to x_0$$



$$\int_{x_0}^{b} \frac{1}{(x-x_0)^\alpha}\,dx$$

$$x - x_0 = y$$

$$dx = dy$$

$$ = \int_0^{b-x_0} \frac{1}{y^\alpha}\,dy \text{ converge } \iff \underline{\underline{\alpha < 1}}$$

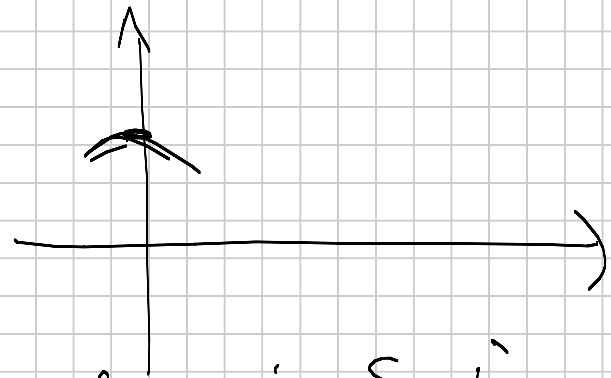
<u>OSS.</u> Deve essere $\lim\limits_{x \to 0} f(x) = +\infty$

ma se questo non avviene, meglio!

<u>es.</u> $f(x) = \dfrac{\operatorname{sen} x}{x}$ $\underline{\underline{\text{non}}}$ è definita in $x=0$

$$\lim_{x\to 0} \frac{\operatorname{sen} x}{x} = 1$$

) è limitata



$$\int_0^1 \frac{\operatorname{sen} x}{x}\, dx$$

non è integrale in S. i.

$(0,1)$

$$\int_0^1 \frac{1}{x^\alpha}\, dx \quad \text{converge} \iff \alpha < 1$$

$$\int_0^{1/2} \frac{1}{x^\alpha \, |\log x|^\beta} \, dx \quad \text{converge se} \quad \begin{cases} \alpha < 1, & \forall \beta \in \mathbb{R} \\ \alpha = 1, & \forall \beta > 1 \end{cases}$$

Es.

$$\int_0^1 \frac{\operatorname{sen}(x^\alpha)}{x^5 (x+1) \cos x} \, dx \qquad \alpha > 0$$

$f(x)$

$x \to 0$

$x^\alpha = y$

$\operatorname{sen} y \sim y$

$$f(x) \sim \frac{x^\alpha}{x^5 \cdot 1} = \frac{1}{x^{5-\alpha}}$$

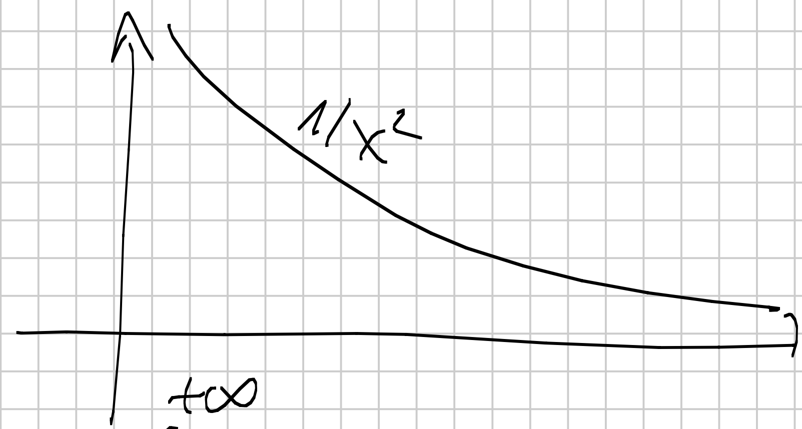
è integrabile in $(0,1)$ se $5 - \alpha < 1$

in $(0,1)$

$\alpha > 4$

Es.

$$\int_0^{+\infty} \frac{1}{x^2}\,dx =$$



$$= \underbrace{\int_0^1 \frac{1}{x^2}\,dx}_{\text{diverge}} + \underbrace{\int_1^{+\infty} \frac{1}{x^2}\,dx}_{\text{converge}} \Longrightarrow \text{diverge}$$

In generale $\displaystyle\int_0^{+\infty} \frac{1}{x^\alpha}\,dx$ divenge $\forall \alpha$

<u>Es.</u> $\displaystyle\int_3^{\frac{7}{2}} \frac{\operatorname{sen}\left((x-3)^\alpha\right)(x-1)}{(x-3)^2\, e^x \left(\log|x-3|\right)^2}\,dx$ $\quad \alpha>0$

$x=3$

$3 \qquad \frac{7}{2}$

$$x - 3 = y \qquad x = y + 3 \qquad dx = dy$$

$$I = \int_0^{1/2} \frac{\operatorname{sen}(y^\alpha)\,(y+2)}{y^2\, e^{y+3} \left(\log|y|\right)^2}\, dy \qquad y \to 0$$

$$f(y) \sim \frac{\cancel{y^\alpha}\; 2}{\cancel{y^2}\; e^3 \log^2|y|} = \frac{2}{e^3} \frac{1}{y^{2-\alpha} \log^2|y|}$$

$$\frac{1}{x^{\alpha}|\log x|^{\beta}}$$

$(0, 1/2)$

✓ è integrabile in $(0, 1/2)$

$\Updownarrow$

$2 - \alpha \leq 1$

$\alpha \geq 1$

Esercizio

$$\int_2^{+\infty} \frac{6x^{2\alpha}}{(x-1)^2\,(x^2+x^\alpha+1)}\,dx$$

D. $\neq 0$ in $[2, +\infty)$

$$f(x) \sim \frac{6x^{2\alpha}}{x^2\,(x^2+x^\alpha)} = \frac{6}{x^{2-2\alpha}\cdot(x^2+x^\alpha)}$$

$\alpha \leq 2$ $\quad f(x) \sim \dfrac{6}{x^{2-2\alpha}\cdot x^2} = \dfrac{6}{x^{4-2\alpha}}$

$4-2\alpha > 1$

$\alpha > 2$ $\qquad f(x) \sim \dfrac{6}{x^{2-2\alpha} \cdot x^{\alpha}} = \dfrac{1}{x^{2-\alpha}}$

è integrabile se $\quad 2-\alpha > 1 \qquad \alpha < 1$

non ci sono valori $\alpha > 2$ per i quali f è integrabile

$\alpha < 3/2$

$\forall\, \alpha < 3/2$

$3/2 \qquad 2$

Stesso esercizio tra 1 e 2

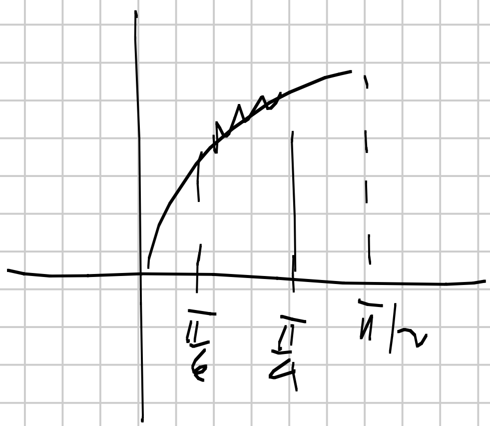
$$\int_1^2 \frac{6x^{2\alpha}}{(x-1)^2\,(x^2+x^\alpha+1)}\,dx \rightsquigarrow \int_1^2 \frac{1}{(x-1)^2}\,dx$$

Es.

$$\int_{\pi/6}^{\pi/4} \frac{1}{\operatorname{tg} x \log(\operatorname{sen} x)}\,dx = \int_{\pi/6}^{\pi/4} \frac{\cos x}{\operatorname{sen} x \log(\operatorname{sen} x)}\,dx$$

$\operatorname{sen} x = t$

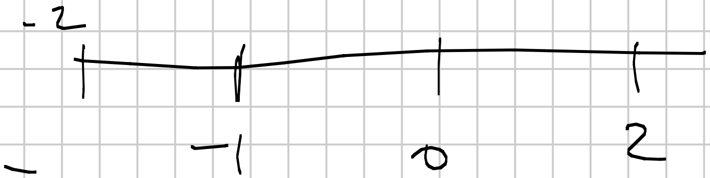
$\cos x\,dx = dt$



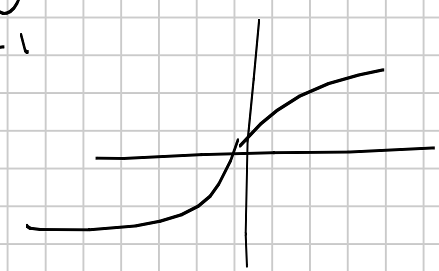
$$= \int_{\operatorname{sen}\frac{\pi}{6}}^{\frac{\sqrt{2}}{2}} \frac{1}{t \log t}\,dt = \log(\log t)\Big|_{\operatorname{sen}\frac{\pi}{6}}^{\frac{\sqrt{2}}{2}}.$$

E5.

$$\int_{-2}^{2} |t+1| \operatorname{arctg} |t| \, dt =$$



$$= \int_{-2}^{-1} -(t+1) \operatorname{arctg}(-t) \, dt + \int_{-1}^{0} (t+1) \operatorname{arctg}(-t) \, dt$$



$$+ \int_{0}^{2} (t+1) \operatorname{arctg} t =$$

$$= \int_{-2}^{-1} (t+1) \operatorname{arctg} t \, dt - \int_{-1}^{0} (t+1) \operatorname{arctg}(t) \, dt + \int_{0}^{2} (t+1) \operatorname{arctg} t \, dt$$

$$\int (t+1) \operatorname{arctg} t = \underline{\underline{\text{fine}}} \quad (\text{per parti} \ldots)$$

$$\int \frac{e^{x}}{e^{2x}+2e^{x}+5}\,dx =$$

$$e^{x} = y$$

$$e^{x}\,dx = dy$$

$$= \int \frac{1}{y^{2}+2y+5}\,dy$$

$$y^{2}+2y+5 =$$

$$\Delta = 1-5<0$$

$$= y^{2}+2y+1+4 = (y+1)^{2}+4$$

$$\int \frac{1}{y^2+2y+5}\,dy = \int \frac{1}{4\left(\left(\frac{y+1}{2}\right)^2+1\right)}\,dy = \qquad \frac{y+1}{2} = t \qquad \frac{1}{2}\,dy = dt$$

$$= \int \frac{1}{4\left(t^2+1\right)}\,2\,dt =$$

$$= \frac{1}{2}\arctg\left(\frac{y+1}{2}\right) + K$$

<u>es.</u> $\displaystyle\int \frac{1}{x^2-5x+6}\,dx$

$$x^2 - 5x + 6 = (x-3)\cdot(x-2)$$

$$\int \frac{1}{x^2 - 5x + 6} = \int \frac{1}{(x-3)(x-2)}$$

$$\frac{1}{(x-3)(x-2)} = \frac{A}{(x-3)} + \frac{B}{(x-2)} = \frac{A(x-2) + B(x-3)}{(x-3)(x-2)}$$

$$= \frac{Ax - 2A + Bx - 3B}{(\quad)(\quad)} = \frac{x(A+B) - 2A - 3B}{(\quad)\cdot(\quad)}$$

$$1 = x(A+B) - 2A - 3B \quad \forall x$$

$$\begin{cases} A + B = 0 \\ -2A - 3B = 1 \end{cases} \qquad B = -A \qquad -2A + 3A = 1 \qquad \begin{array}{l} A = 1 \\ B = -1 \end{array}$$

$$\int \frac{1}{x^2 - 5x + 6} = \int \frac{1}{(x-3)} - \int \frac{1}{(x-2)}$$

&

$$\int \frac{x}{x^2 - 5x + 6}\,dx = \int \frac{x}{(x-3)(x-2)}$$

$$\frac{x}{(x-3)(x-2)} = \frac{A}{(x-3)} + \frac{B}{(x-2)} = \frac{x(A+B)-2A-3B}{(\ )(\ )}$$

$$\begin{cases} A + B = 1 \\ -2A - 3B = 0 \end{cases} \Rightarrow \begin{array}{l} A = 3 \\ B = -2 \end{array}$$

$$\int \sqrt{1-x^2}\, dx$$

$$\int \sqrt{x^2-1}\, dx$$

$$\int \sqrt{1+x^2}\, dx$$

$$\cosh^2 t - \operatorname{senh}^2 t = 1$$

$$\cosh^2 t - 1 = \operatorname{senh}^2 t$$

$$1 + \operatorname{senh}^2 t = \cosh^2 t$$

$$x = \operatorname{senh} t$$

$$dx = \cosh t$$

$$\int \sqrt{1+\sinh^2 t}\,\cosh t \, dt$$

$$= \int \cosh^2 t \, dt$$

per parti