es

$$\lim_{x \to 5} x+2 = 7$$

$$\lim_{x \to -\infty} -x+2 = +\infty$$

$$\lim_{x \to +\infty} \frac{1}{x} = 0$$

$f(x)$ verifica una proprietà $P$ definitivamente per $x \to x_0$

$$\Updownarrow$$

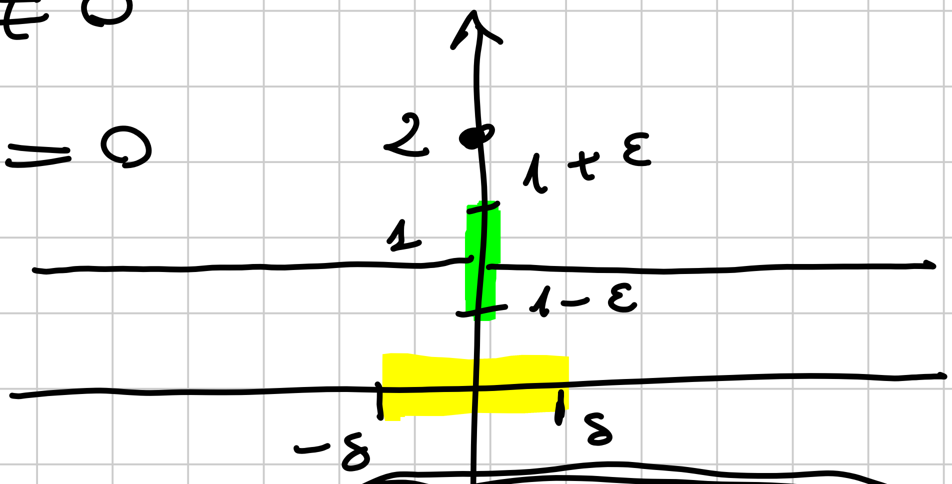
$\exists\, U$ intorno di $x_0$ t.c. $f(x)$ verifica $P$, $\forall x \in U$

$\lim_{x \to x_0} f(x) = l \iff \forall\, V$ intorno di $l$ $f(x) \in V$ definitivamente per $x \to x_0$.

es.

$$\forall x \in U \cap X, \quad x \neq x_0$$

$$f(x) = \begin{cases} 1 & x \neq 0 \\ 2 & x = 0 \end{cases}$$




$$\lim_{x \to 0} f(x) = 1$$

$$\Longleftrightarrow \underline{\forall \varepsilon > 0} \ \underline{\exists \delta > 0} \ \text{t.c.} \ |f(x) - 1| < \varepsilon$$

$$\text{se } |x| < \delta, \quad \underline{x \neq 0}$$

$$|1 - 1| = 0 < \varepsilon \qquad \forall x \neq 0$$

sempre vero $\forall \delta$

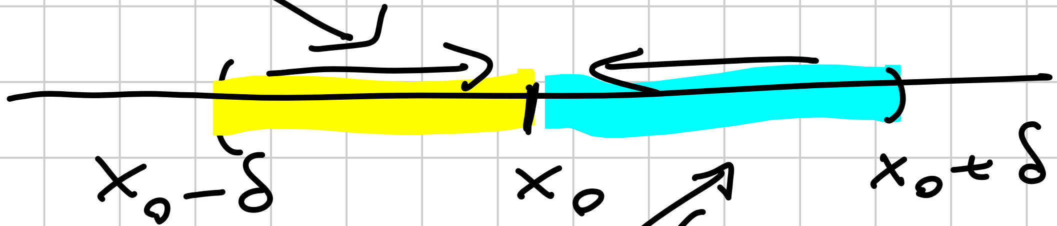
1) $\lim\limits_{x\to x_0} f(x) = l \in \mathbb{R} \Rightarrow f(x)$ è limitata definitivamente per $x \to x_0$

2) Permanenza del segno:

$\lim\limits_{x\to x_0} f(x) = l > 0 \Rightarrow f(x) > 0$ definitivamente per $x \to x_0$

$$\not\Leftarrow$$

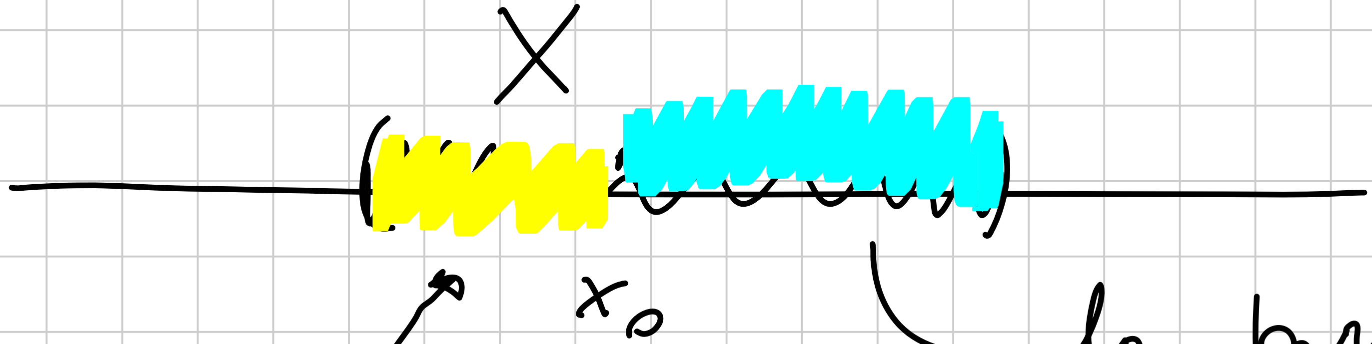
# Limite destro e sinistro




intorno sinistro di $x_0$ $\qquad (x_0-\delta, x_0)$

intorno destro di $x_0$ $\qquad (x_0, x_0+\delta)$

**Def.** $x_0$ punto di accumulazione **destro** (sinistro) per $X$ se è di accumulazione per $X \cap (x_0, +\infty)$




la parte di $X$ a destra di $x_0$.

$$\left( X \cap (-\infty, x_0) \right)$$

<u>Def.</u> $f : X \to \mathbb{R}$, $X \subseteq \mathbb{R}$. $x_0$ p.to di accumulazione destro (sinistro) per $X$.

$$\lim_{x \to x_0^+} f(x) = \ell \iff \forall \varepsilon > 0 \ \exists \delta > 0 : |f(x) - \ell| < \varepsilon \quad \forall x, \ x_0 < x < x_0 + \delta$$

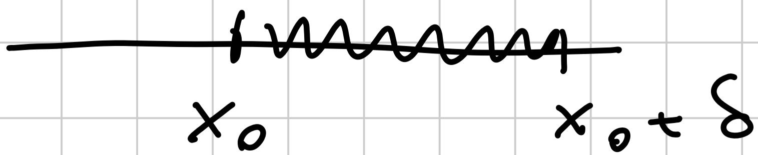
("limite destro")

$\ell \in \mathbb{R}$

$\big( x \in (x_0, x_0 + \delta) \big)$

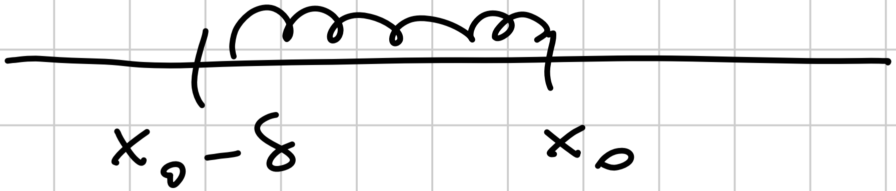
$\downarrow$

intorno destro



$$\lim_{x \to x_0^-} f(x) = \ell \iff \forall \varepsilon > 0 \ \exists \delta > 0 : |f(x) - \ell| < \varepsilon \quad \forall x : x_0 - \delta < x < x_0$$

("limite sinistro")




$\ell \in \mathbb{R}$

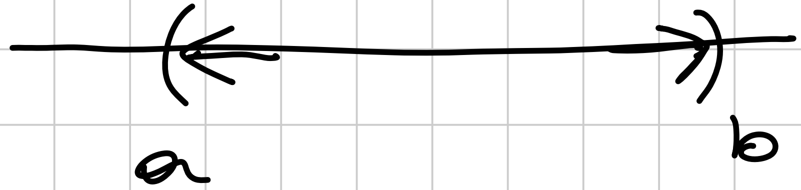
$\ell = \pm\infty$

Applicazione

$$\lim_{x \to a^+} f(x)$$

$$\lim_{x \to b^-} f(x)$$

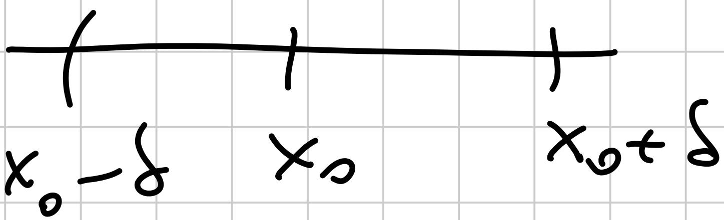
$$X = (a, b)$$



OSS.

$$\lim_{x \to x_0} f(x) = \ell \iff \lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = \ell$$

($x_0$ p.to di accumulazione per $X$)



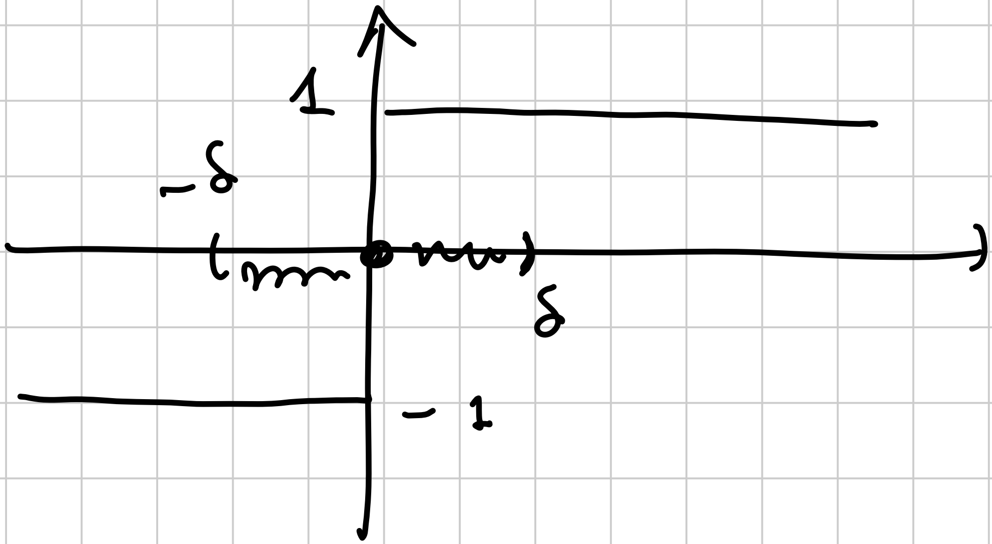

serve per dimostrare che una funzione non ha limite per $x \to x_0$

se $\lim_{x \to x_0^+} f(x) \neq \lim_{x \to x_0^-} f(x)$

$$\Rightarrow \nexists \lim_{x \to x_0} f(x)$$

(se esistesse il limite dx e sx dovrebbero essere uguali).

**Es.** $f(x) = \operatorname{sgn} x = \begin{cases} 1 & x>0 \\ 0 & x=0 \\ -1 & x<0 \end{cases}$




$\nexists \lim\limits_{x\to 0} \operatorname{sgn} x$

$$\lim_{x\to 0^+} \operatorname{sgn} x = 1 \iff \forall \varepsilon>0 \ \exists \delta : \ |\operatorname{sgn} x - 1| < \varepsilon \quad \forall\ 0<x<\delta$$

$0 < \varepsilon$ vero $\forall \delta$.

$$\lim_{x\to 0^-} \operatorname{sgn} x = -1$$

(fate voi!)

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty \iff \forall m \quad \exists\, \delta > 0$$

A.c. $\frac{1}{x} > m$

$\forall \; x : \; 0 < x < \delta$

$x > 0$

$0 \qquad \delta$

$m \leq 0$ sempre vero!

$$\boxed{\frac{1}{x} > m}$$

$m > 0 \implies (x > 0)$

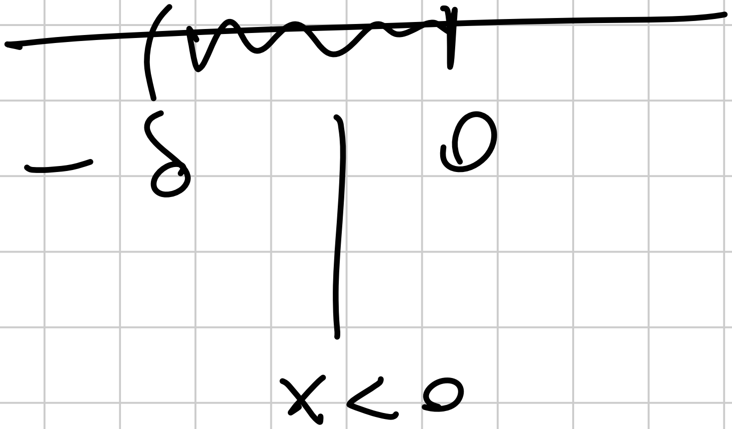
$$\boxed{x < \frac{1}{m}}$$

$$\delta = \frac{1}{m}$$

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

$$\iff \forall m \ \exists \delta \text{ t.c. } \boxed{\frac{1}{x} < m}, \quad \forall x : -\delta < x < 0$$

(Fate voi!)




$$\nexists \lim_{x \to 0} \frac{1}{x}$$

perché $\lim_{x \to 0^-} \neq \lim_{x \to 0^+}$.

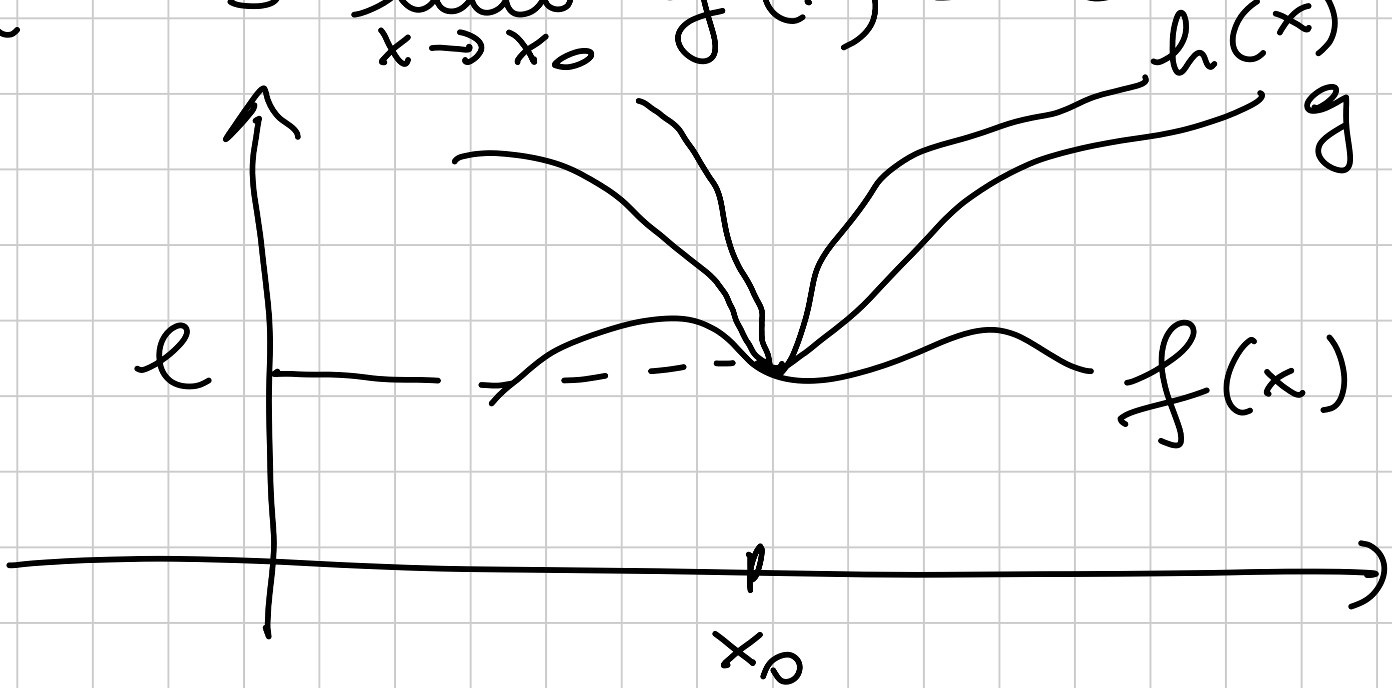
Teorema del confronto (dei "carabinieri")

$f, g, h : X \to \mathbb{R}$, $x_0 \in \mathbb{R}^*$ p.to di accum. per $X$

Supponiamo che $f(x) \leq g(x) \leq h(x)$
in un intorno di $x_0$

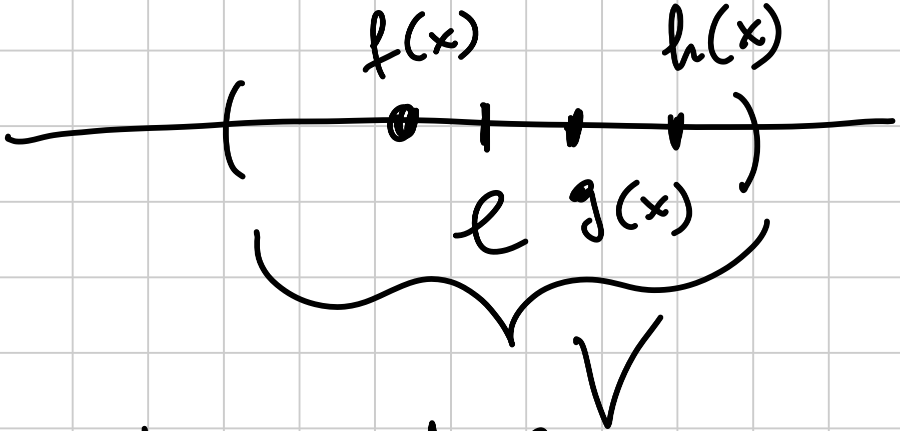
Se $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = \ell \in \mathbb{R}^*$

allora $\exists \lim_{x \to x_0} g(x) = \ell$



**Dim.** $\lim_{x\to x_0} f(x) = \ell \iff \forall V$ intorno di $\ell$ $\exists$ $U_1$ :

$\lim_{x\to x_0} h(x) = \ell \iff \forall V$ intorno di $\ell$ $\exists U_2$ : $x \in U_2$




$U_1 \cap U_2 = U$ intorno di $x_0$

$\forall V$ intorno di $\ell$ $\exists U$ intorno di $x_0$ t.c. $g(x) \in V$ se $x \in U$.

$(\lim_{x\to x_0} g(x) = \ell)$

$x \in U_1 \Rightarrow f(x) \in V$

$x \in U_2 \Rightarrow h(x) \in V$

Es.

$$\lim_{x \to +\infty} \underbrace{\frac{\operatorname{sen} x}{x}}_{g(x)}$$

$$-1 \leq \operatorname{sen} x \leq 1$$

$$\underbrace{-\frac{1}{x}}_{f(x)} \leq \frac{\operatorname{sen} x}{x} \leq \underbrace{\frac{1}{x}}_{h(x)} \qquad x \to +\infty$$

$x \to +\infty$: $f(x) \to 0$

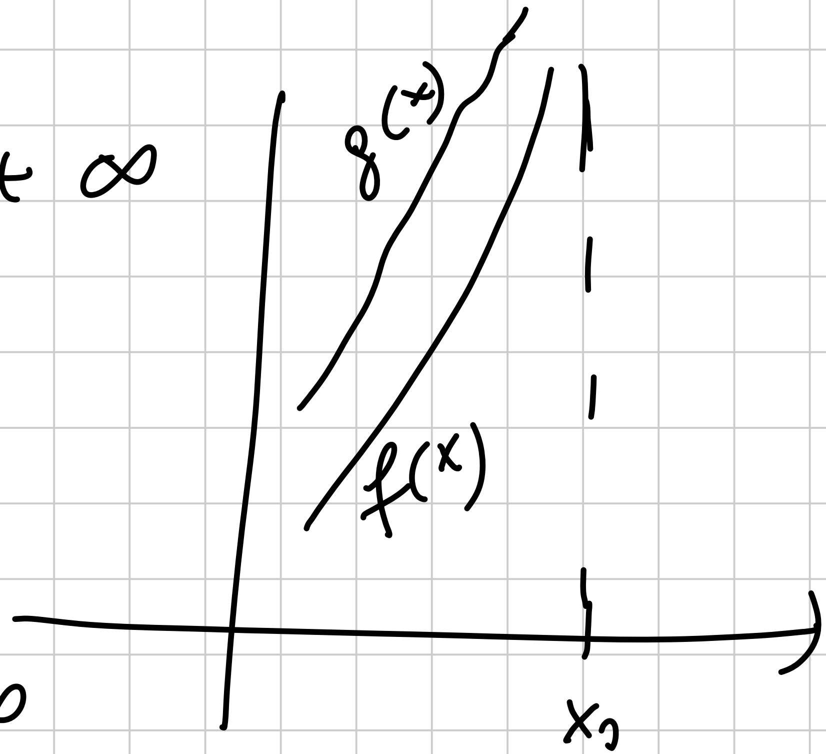
$x \to +\infty$: $h(x) \to 0$

$\dfrac{\operatorname{sen} x}{x} \to 0$   $x \to +\infty$   Dal teo. del confronto!

<u>oss.</u> Se $l = +\infty$ basta una funzione (per il confronto)

Se $f(x) \leq g(x)$ e $f \to +\infty$ (in un intorno di $x_0$)

allora $g \to +\infty$




Se $l = -\infty$

Se $g(x) \leq h(x)$ e $h \to -\infty$ $\Rightarrow g \to -\infty$

ES.

$$\lim_{x \to +\infty} \frac{x^2 + \operatorname{sen} x + 1}{x}$$

$g(x)$

$$\underbrace{\frac{x^2 + \operatorname{sen} x + 1}{x}}_{+\infty} > \frac{x^2 - \cancel{1} + \cancel{1}}{x} = x \xrightarrow{x \to +\infty} +\infty$$

$f(x)$

$+\infty$ se $x \to +\infty$

**oss.** $\displaystyle \lim_{x\to x_0} g(x) = 0 \iff \lim_{x\to x_0} |g(x)| = 0$

se $g(x) \xrightarrow[x\to x_0]{} 0$ si dice $g(x)$ è infinitesimo per $x \to x_0$

Il teo. del confronto spesso si applica così:

$$0 \le |g| \le h$$

$$0 \to 0, \qquad |g| \to 0, \qquad h \to 0 \quad x \to x_0$$

se $|g| \to 0 \implies g \to 0$.

$$\lim_{x \to 0} \operatorname{sen} x = 0$$

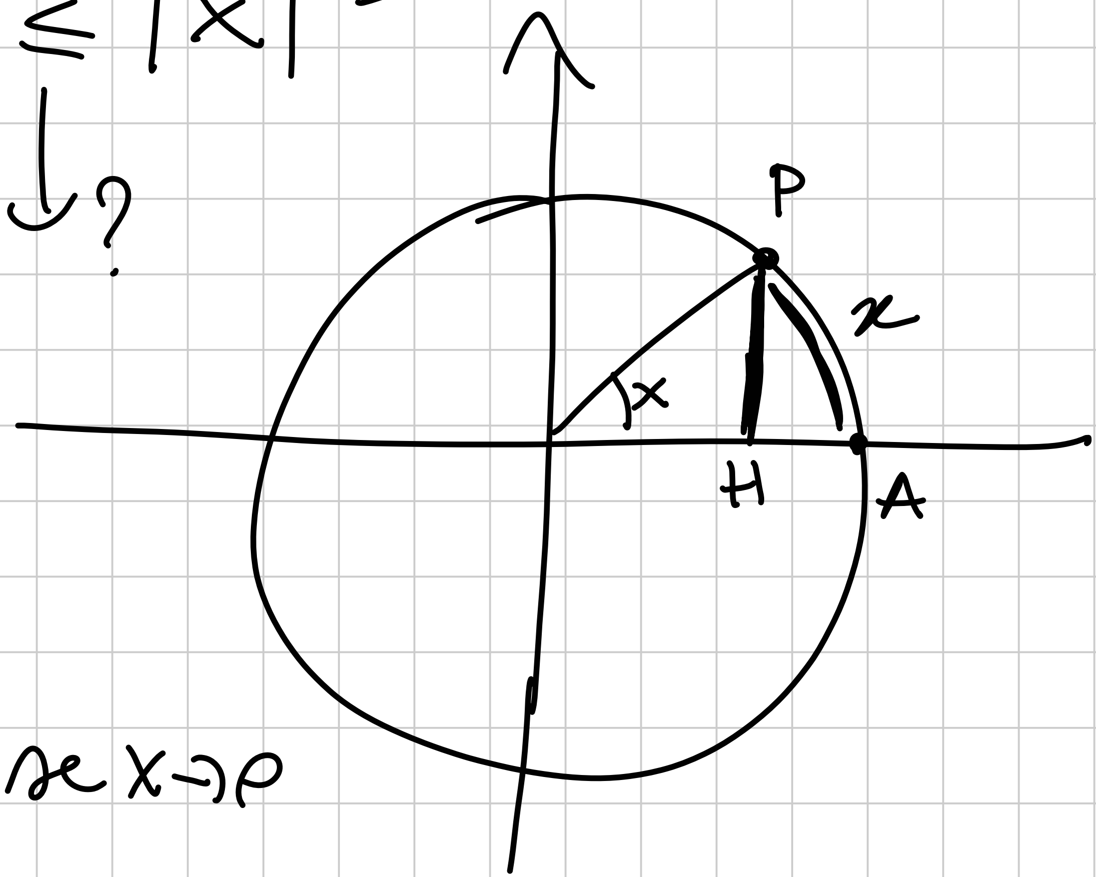
1) Verifico con la def. di limite

2) $$0 \le \underbrace{|\operatorname{sen} x|}_{\overline{PH}} \le |x| \xrightarrow{x \to 0} 0$$

↓ ?

$|\operatorname{sen} x| = \overline{PH}$

0 se $x \to 0$



$$\lim_{x \to +\infty} \frac{\operatorname{sen} x}{x}$$

$$0 \leq \left| \frac{\operatorname{sen} x}{x} \right| \leq \frac{1}{x} \to 0$$

$x \to +\infty$