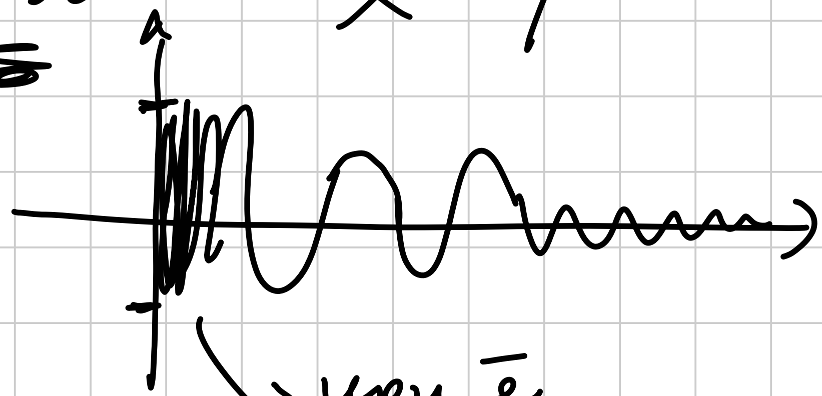
Teorema di esistenza del limite per funzioni <u>monotòne</u>

$$\lim_{x \to x_0^+} f(x)$$

$$\lim_{x \to x_0^-} f(x)$$

non esistono sempre!

<u>es.</u> $\lim_{x \to 0^+} \operatorname{sen} \frac{1}{x} \nexists$

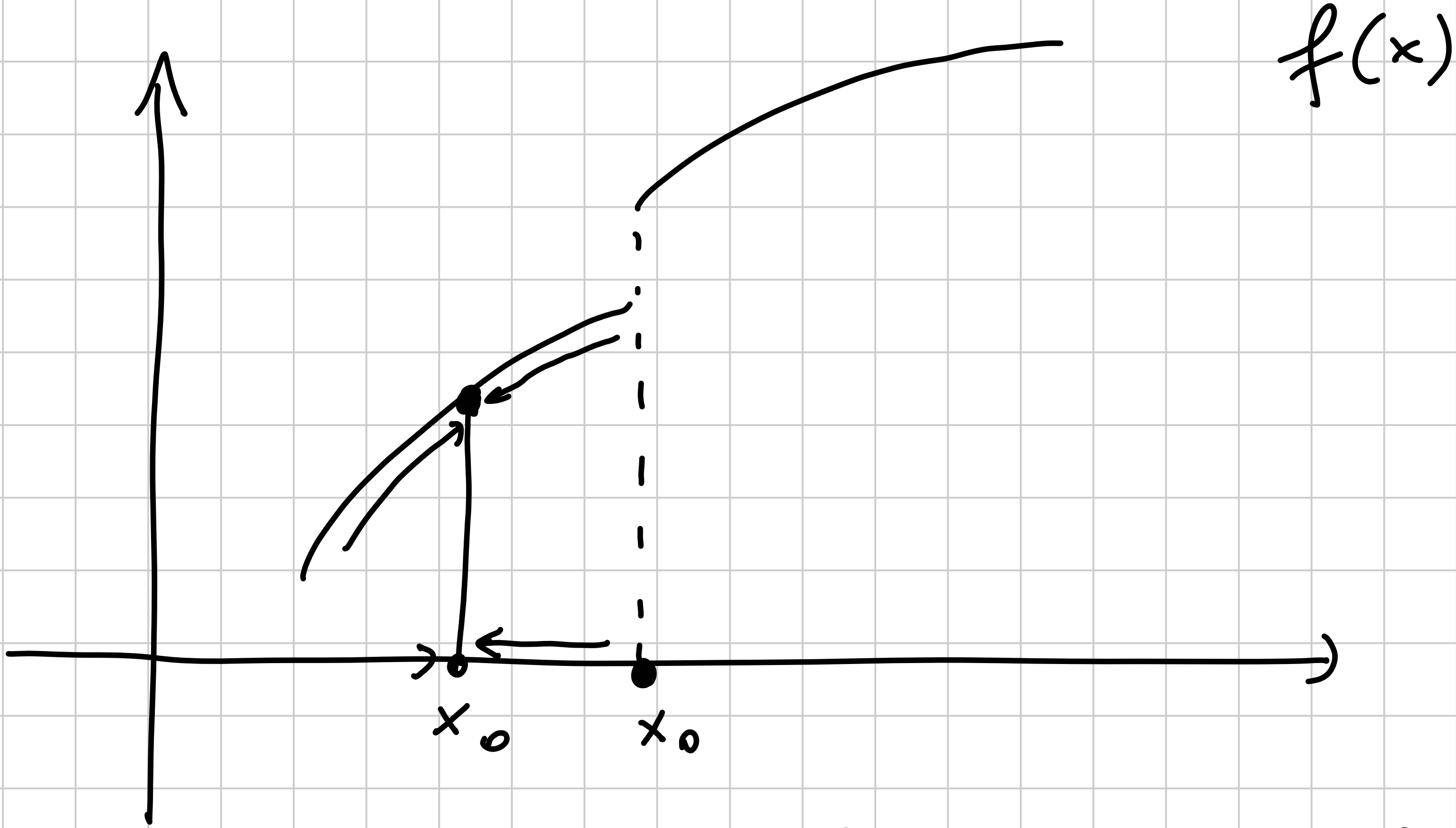


non è monotòne in un intorno destro di $x=0$.

$$\left|\operatorname{sen} \frac{1}{x}\right| \le \left|\frac{1}{x}\right|$$

$x \to +\infty$

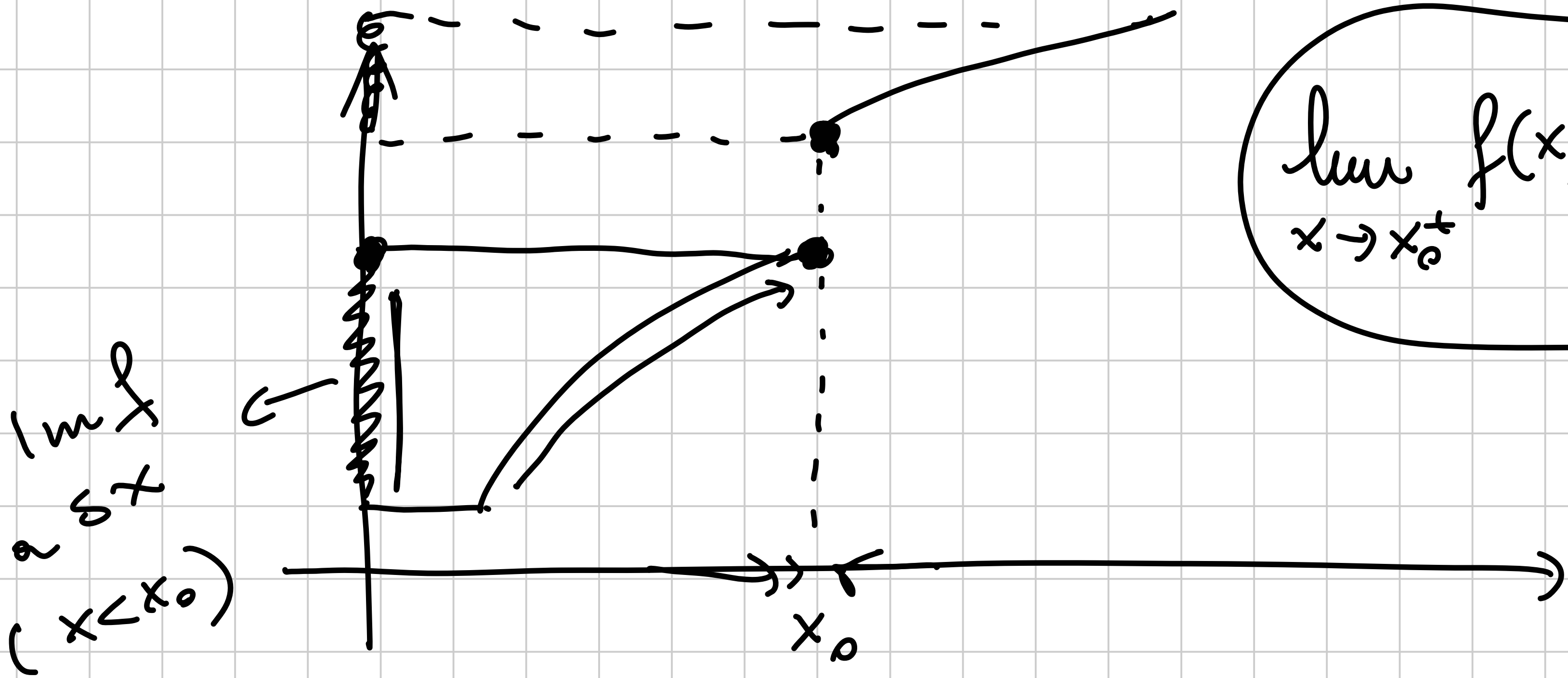
$\left|\operatorname{sen} \frac{1}{x}\right| \to 0$, $\left|\frac{1}{x}\right| \to 0$



$$\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = f(x_0)$$

se $\exists \lim_{x \to x_0} f(x) = f(x_0)$

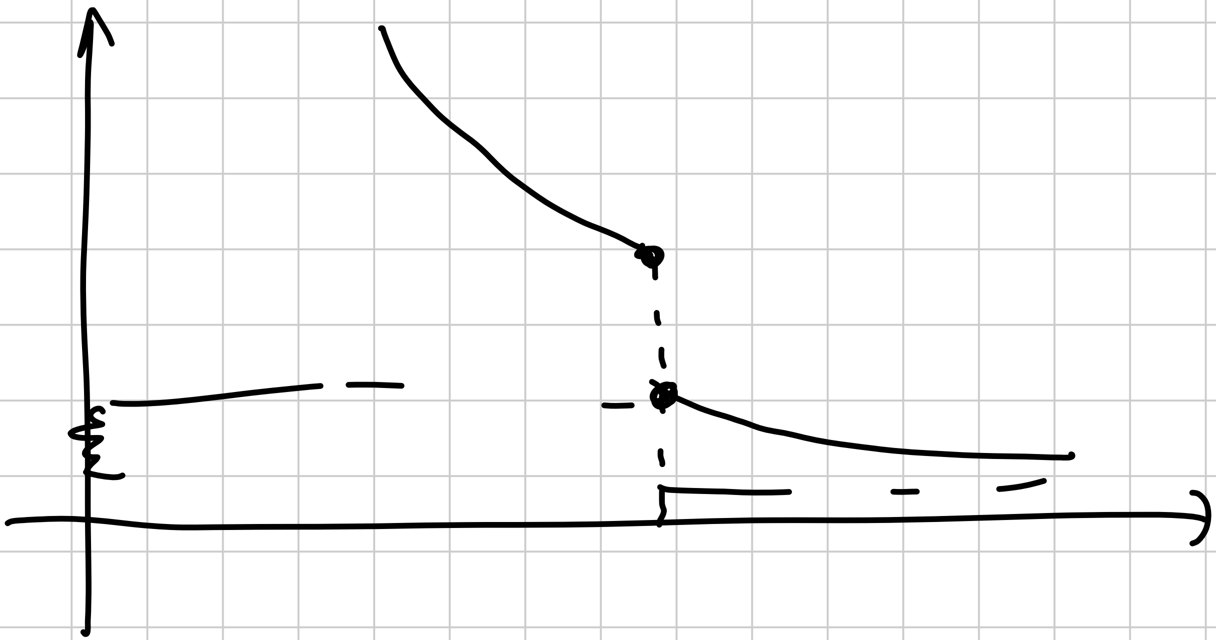
si dice che f è <u>continua</u> in x_0.



$$\lim_{x \to x_0^+} f(x) = \inf_{x > x_0} f$$

$$\lim_{x \to x_0^-} f(x) = \sup_{x < x_0} f$$

Im f a sinistra di x_0



$$\lim_{x \to x_0^-} f(x) = \inf_{x < x_0} f$$

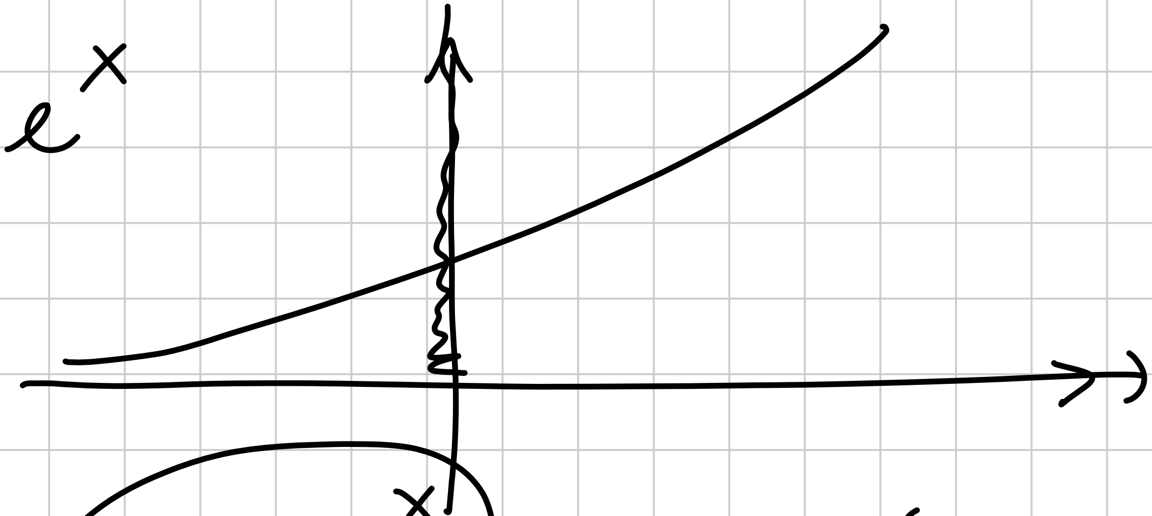
$$\neq$$

$$\lim_{x \to x_0^+} f(x) = \sup_{x > x_0} f$$

$$\nexists \lim_{x \to x_0} f(x)$$

es. $f(x) = e^x$

è crescente
su $\mathbb{R}$



$$\lim_{x \to +\infty} e^x = \sup_{\mathbb{R}} e^x = \sup(0,+\infty) = +\infty$$

$\downarrow$

$$\lim_{x \to x_0^-} e^x$$

$x_0 = +\infty$

$$\lim_{x \to -\infty} e^x =$$

$$\lim_{x \to x_0^+} e^x = \inf(0,+\infty) = 0$$

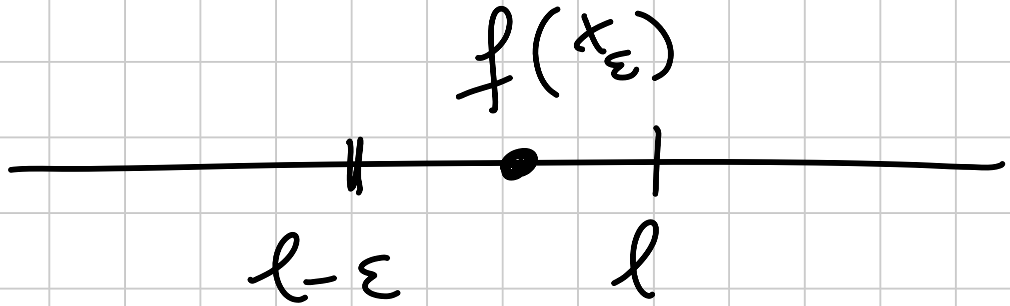
$x_0 = -\infty$

$$\sup_X f = l = \sup \underbrace{f(X)}_{\text{insieme di } \mathbb{R}}$$

1) $l \geq f(x), \ \forall x \in X$

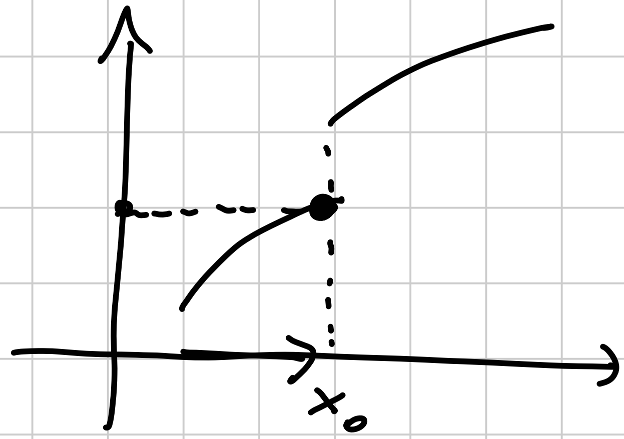
2) $\forall \varepsilon > 0 \ \exists x_\varepsilon \in X$ t.c. $f(x_\varepsilon) > l - \varepsilon$

caratterizzazione del $\sup_X f$



Teorema $f : X \subseteq \mathbb{R} \to \mathbb{R}$

monotona **crescente**



1) x_0 p.to di accumulazione sinistro per f. Allora

$$\lim_{x \to x_0^-} f(x) = \sup_{x < x_0} f$$

2) x_0 p.to di accumulazione destro per f.

Allora $$\lim_{x \to x_0^+} f(x) = \inf_{x > x_0} f$$

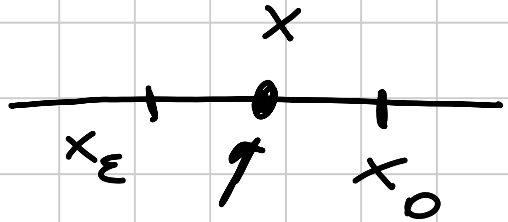
(se f è decrescente, vale l'inverso).

Dim 1)

Dim. f crescente in X

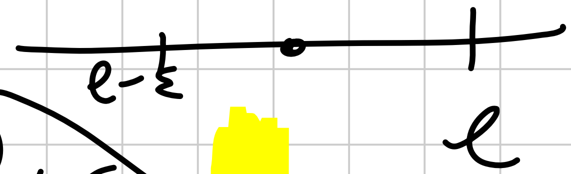
Voglio dim. che $\lim_{x \to x_0^-} f(x) = \sup_{x<x_0} f = l$

$l = \sup_{x<x_0} f$ $\qquad \boxed{l \in \mathbb{R}}$



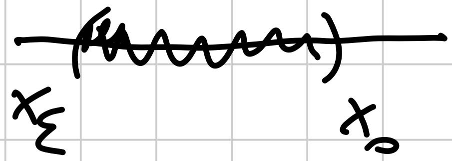
$$\begin{cases} 1)\ f(x) \le l, \quad \forall x < x_0 \\ 2)\ \forall \varepsilon > 0 \ \exists\, x_\varepsilon < x_0 : \ l - \varepsilon < f(x_\varepsilon) \end{cases}$$

Prendo $x \in (x_\varepsilon, x_0)$



$$l - \varepsilon \underset{2)}{<} f(x_\varepsilon) \underset{\substack{f \text{ crescente} \\ x_\varepsilon < x}}{\le} f(x) \underset{1)}{\le} l < l + \varepsilon$$

$$\ell - \varepsilon < f(x) < \ell + \varepsilon$$



$$-\varepsilon < f(x) - \ell < \varepsilon$$

$$|f(x) - \ell| < \varepsilon, \quad \forall x \in \underbrace{(x_\varepsilon, x_0)}_{\text{intorno sinistro di } x_0}$$

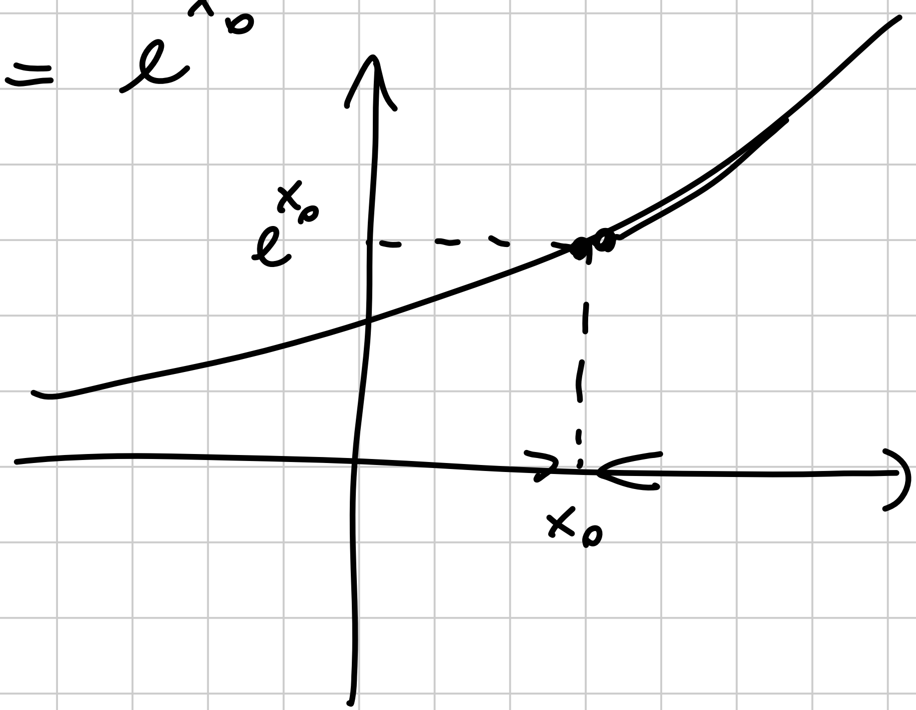
$$\Downarrow$$

$$\ell = \lim_{x \to x_0^-} f(x)$$

quindi $\displaystyle\lim_{x \to x_0^-} f(x) = \sup_{x < x_0} f$. //

es. $f(x) = e^x$ $\qquad \lim_{x \to +\infty} e^x = +\infty$

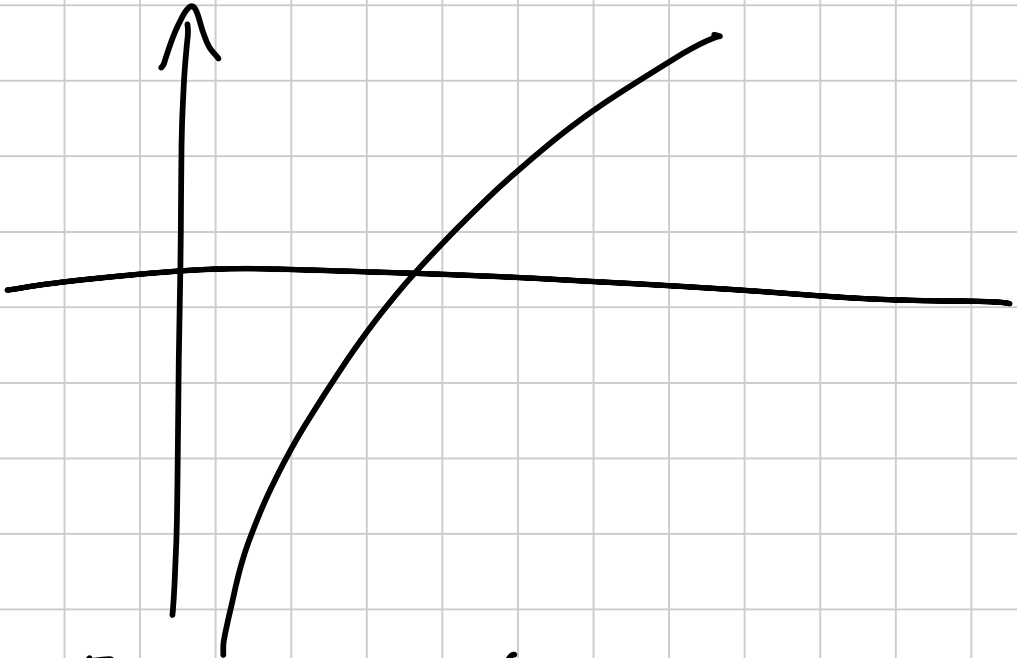
$$\lim_{x \to x_0^-} e^x = \lim_{x \to x_0^+} e^x = e^{x_0}$$



$$\exists \lim_{x \to x_0} e^x = e^{x_0}$$

e^x è continua in $x_0 \in \mathbb{R}$

es. $f(x) = \log_a x$



$a > 1$

$$\lim_{x \to +\infty} \log_a x = +\infty$$

$$\lim_{x \to 0^+} \log_a x = -\infty$$

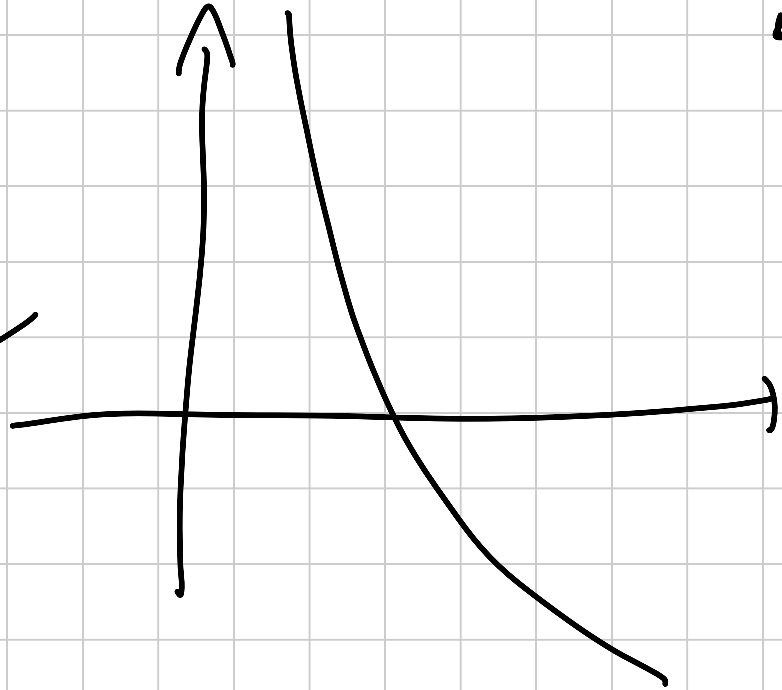
$\operatorname{Im} f = (-\infty, +\infty)$

$a<1$

$$\lim_{x \to 0^+} \log_a x = +\infty$$

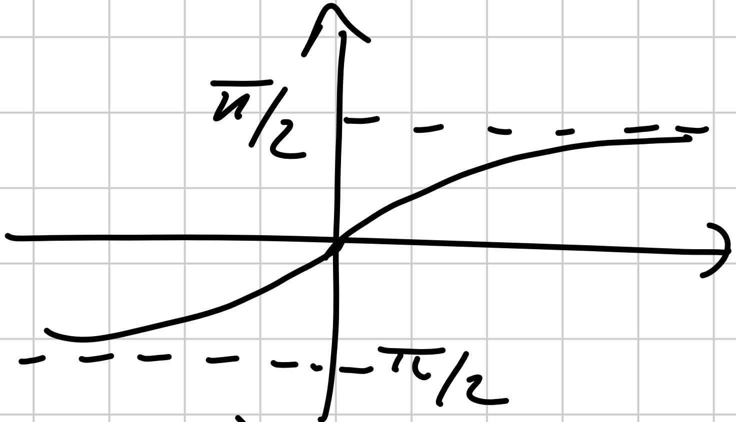
$$\lim_{x \to +\infty} \log_a x = -\infty$$

$a<1$



es. $f(x) = \operatorname{arctg} x$

$f : \mathbb{R} \longrightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

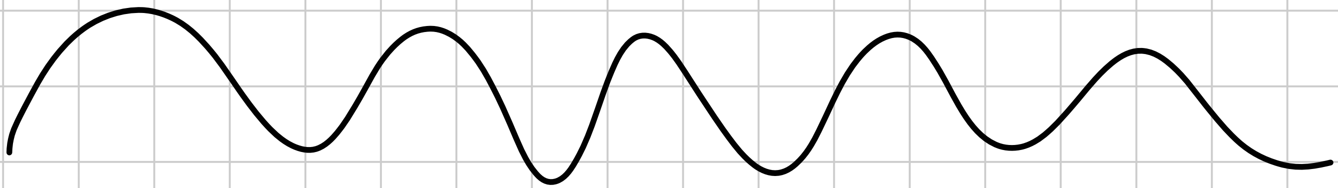


$\lim_{x \to +\infty} \operatorname{arctg} x = \sup\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) = \frac{\pi}{2}$

$\lim_{x \to -\infty} \operatorname{arctg} x = -\pi/2$

$\lim_{x \to x_0^+} \operatorname{arctg} x = \lim_{x \to x_0^-} \operatorname{arctg} x = \operatorname{arctg} x_0$

è continua $\quad \forall x_0 \in \mathbb{R}.$



Operazioni sui limiti

Teorema $\lim_{x \to x_0} f(x) = l_1$, $\lim_{x \to x_0} g(x) = l_2$, $l_1, l_2 \in \mathbb{R}$, $x_0 \in \mathbb{R}^*$

1) $f+g \xrightarrow[x \to x_0]{} l_1 + l_2$

2) $c f \xrightarrow[x \to x_0]{} c\, l_1$, $c \in \mathbb{R}$

3) $f \cdot g \longrightarrow l_1 \cdot l_2$

4) $\frac{1}{g} \longrightarrow \frac{1}{l_2}$ (se $l_2 \neq 0$)

3) + 4) $\Rightarrow \frac{f}{g} \longrightarrow \frac{l_1}{l_2}$ (se $l_2 \neq 0$).

Dim. di 1)

$$\text{Hp} \begin{cases} f \underset{x \to x_0}{\longrightarrow} l_1 \quad (\text{[illegible]} \quad \forall x \in U_1) \\ g \underset{x \to x_0}{\longrightarrow} l_2 \quad (\text{[illegible]} \quad \forall x \in U_2) \end{cases} \Longrightarrow \text{Ts.} \quad (f+g) \underset{x \to x_0}{\longrightarrow} (l_1 + l_2)$$

$$|(f+g) - (l_1 + l_2)| = |(f - l_1) + (g - l_2)| \leq$$

$$\leq \text{[illegible]} + \text{[illegible]} < \underbrace{\text{[illegible]} + \text{[illegible]}}_{= 2\varepsilon}$$

$$\boxed{\begin{array}{c} (\, |a+b| \leq |a| + |b| \,) \\ \text{disug. triangolare} \end{array}}$$

$$x \in U_1 \cap U_2$$

esempi

$$\lim_{x \to -1} x = -1$$

$$x^2 = x \cdot x$$

$$\lim_{x \to -1} x^2 = 1 \qquad (-1)(-1)$$

$$\lim_{x \to -1} 85x = 85 \cdot (-1) = -85$$

$$f \to l_1 \qquad cf \to c\, l_1$$

$$\lim cf = c \lim f$$

es. $\lim_{x \to -1} \frac{x^2 - 3x + 1}{2x^3 + 7}$ = applico il teo. sulle operazioni dei limiti

$\to 1$

$x \to -1 \qquad x^2 \to 1$

$x \to -1 \qquad -3x \to 3$

$x^2 - 3x + 1 \to 1 + 3 + 1 = 5$

$f \to 5$

$g \to 5$

$x \to -1 \qquad 2x^3 \to -2$

$2x^3 + 7 \to 5$

Estensioni del teorema sulle operazioni dei limiti

$f \to l_1 \in \mathbb{R}$
$g \to l_2$

1) $f \to 0 \quad x \to x_0$ (prodotto di una infinitesima per limitata)
g limitata in un intorno di x_0. Allora

$$f \cdot g \to 0 \quad x \to x_0$$

Dim. $0 \le |f \cdot g| \le |f| \cdot M$

$|g| \le M$

$|f \cdot g| \to 0$, $|f| \cdot M \to 0$

es:

$$\lim_{x \to 0} \underbrace{x}_{f} \cdot \underbrace{\operatorname{sen} \frac{1}{x}}_{g} = 0$$

$$\nexists \lim_{x \to 0} \operatorname{sen} \frac{1}{x}$$

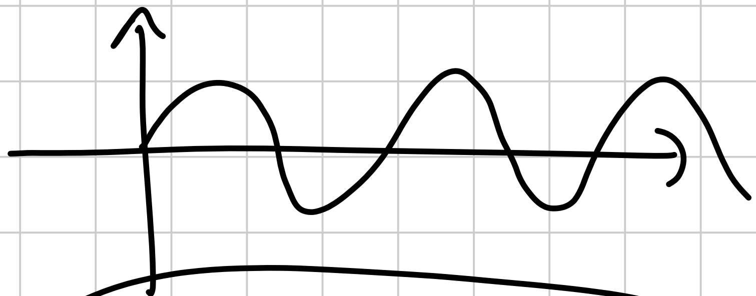
2) $\displaystyle f \xrightarrow[x \to x_0]{} +\infty$, $g \geq M$

allora $\displaystyle f + g \xrightarrow[x \to x_0]{} +\infty$

NO

es. $\displaystyle \lim_{x \to +\infty} (x \pm \operatorname{sen} x) = +\infty$

$\displaystyle \lim_{x \to +\infty} \operatorname{sen} x \not\exists$



no! $\Rightarrow$ $f \to +\infty$, $g \to -\infty$

$f + g \to +\infty - \infty$

3) $f \underset{x \to x_0}{\longrightarrow} +\infty$, $\quad g \geq \underline{M} > 0$

allora $\quad f \cdot g \underset{x \to x_0}{\longrightarrow} +\infty$

limitata con $M > 0$

<u>es.</u> $\lim_{x \to +\infty} \underbrace{x}_{f} \cdot \underbrace{(2 + \operatorname{sen} x)}_{g} \to +\infty$

$\lim_{x \to +\infty} (2 + \operatorname{sen} x) \nexists$

$2 + \operatorname{sen} x \geq \underset{M}{1} \quad \forall x \in \mathbb{R}$

$\underline{\text{es.}}$ $$\lim_{x \to +\infty} \underbrace{x}_{f} \cdot \underbrace{\text{sen}\, x}_{g} \quad \nexists$$

$$x(\text{sen}\, x + 2)$$

non si può applico 3)

4) $f \xrightarrow[x \to x_0]{} +\infty$, $g \leq -M < 0$

$$f \cdot g \xrightarrow[x \to x_0]{} -\infty$$

4)

$\underline{\text{es.}}$ $$\lim_{x \to +\infty} \underset{\substack{\downarrow \\ +\infty}}{x} (-5 + \text{sen}\, x) = -\infty$$

$$x^2 = x \cdot x$$

2) , 3) 4) hanno le analoghe

quando $f \to -\infty$ $x \to x_0$

es. $\lim_{x \to +\infty} \underset{\to +\infty}{x} - \underbrace{5 \operatorname{sen} x}_{\text{limitata}} + \underset{\to 0}{\frac{1}{x}} + 3 =$

$= +\infty$

5) $f \xrightarrow[x \to x_0]{} +\infty \Rightarrow \frac{1}{f} \xrightarrow[x \to x_0]{} 0$

6) $f \xrightarrow[x \to x_0]{} 0$ e $f > 0$ in un intorno di x_0 $\Rightarrow \frac{1}{f} \xrightarrow[x \to x_0]{} +\infty$

se $f < 0$ in un intorno di x_0 $\Rightarrow \frac{1}{f} \xrightarrow[x \to x_0]{} -\infty$

es. $f(x) = x^2$

$\lim_{x \to 0} x^2 = 0 \Rightarrow \lim_{x \to 0} \frac{1}{x^2} = +\infty$

Riassunto $f \to l_1$
$g \to l_2$ $\in \mathbb{R}$

Casi che non rientrano nei precedenti:

es. $\lim_{x \to +\infty} \underbrace{x^3} - x$

$\boxed{+\infty - \infty}$

$\boxed{-\infty + \infty}$

1

$x^3 \to +\infty$

$-x \to -\infty$

$f \to \pm\infty$
$g \to 0$

$f \cdot g \to ?$

$\infty \cdot 0$

casi che non rientrano nei precedenti

$f \to 0$
$g \to 0$

$\dfrac{f}{g}$ $\quad \dfrac{0}{0}$

$f \to \pm\infty$
$g \to \pm\infty$

$\dfrac{f}{g}$ $\quad \dfrac{\infty}{\infty}$

Limiti di polinomi

$$\lim_{x \to +\infty} x^3 - x = \lim_{x \to +\infty} \underbrace{x^3}_{f} \underbrace{\left(1 - \frac{1}{x^2}\right)}_{g} =$$

$$= +\infty$$

In generale polinomio

$$\lim_{x \to +\infty} a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n =$$

$$= \lim_{x \to +\infty} \underbrace{x^n}_{f} \underbrace{\left(a_n + \overbrace{\frac{a_{n-1}}{x}}^{0} + \dots + \overbrace{\frac{a_0}{x^n}}^{0} \right)}_{g \to a_n} =$$

$$= a_n \left(\lim_{x \to +\infty} x^n \right) \begin{cases} +\infty & \text{se } a_n > 0 \\ -\infty & \text{se } a_n < 0 \end{cases}$$