Teorema del limite di funzione composta.

$$f : X \subseteq \mathbb{R} \to \mathbb{R} \qquad g \circ f : X \to \mathbb{R}$$

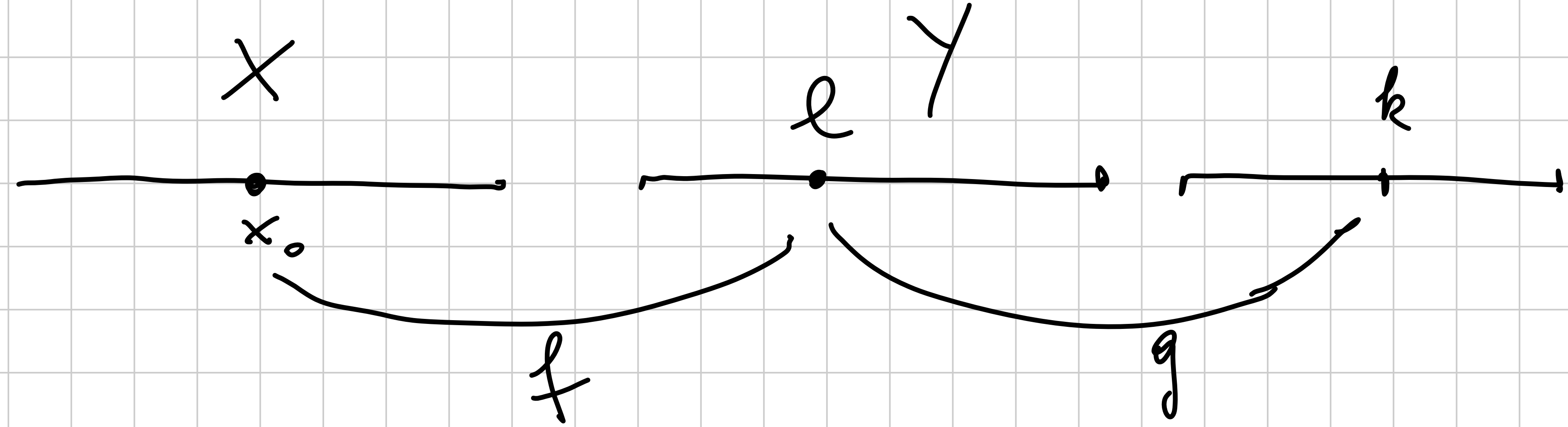
$$g : Y \subseteq \mathbb{R} \to \mathbb{R} \qquad f(X) \subseteq Y$$

x_0 punto di accumulazione per X, $x_0 \in \mathbb{R}^*$

$$\begin{cases} \lim\limits_{x \to x_0} f(x) = l \\ \lim\limits_{y \to l} g(y) = k \end{cases}$$

$f(x) \neq l$ in un intorno di x_0

$$\Longrightarrow \lim_{x \to x_0} g \circ f(x) = k$$



Es. $3^{1/x}$

$$\lim_{x \to +\infty} 3^{1/x} = 1$$

$$x \xrightarrow[f]{} \frac{1}{x} \xrightarrow[g]{} 3^{1/x}$$

$$g(y) = 3^{y}$$

$$\lim_{x \to +\infty} \frac{1}{x} = 0 = \ell$$

$$\lim_{y \to 0} 3^{y} = 3^{0} = 1$$

Dim.

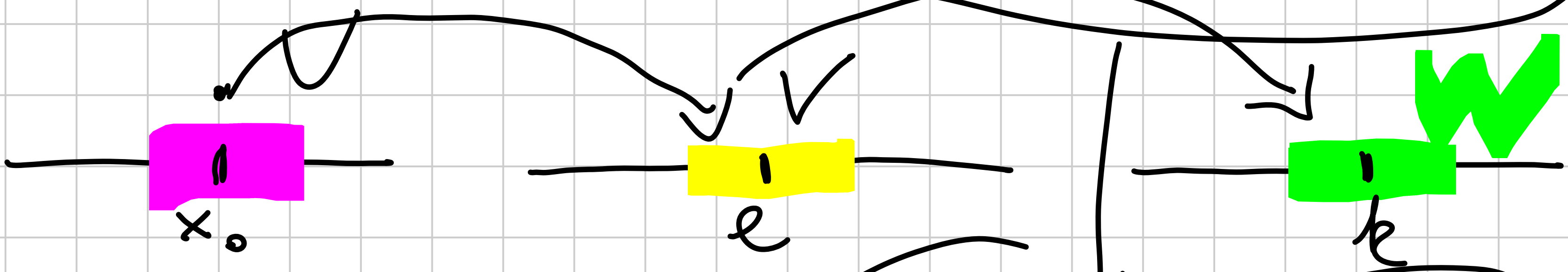
Hp

$$\lim_{x \to x_0} f(x) = \ell \quad \cdot$$

$$\lim_{y \to \ell} g(y) = k \quad \cdot\cdot$$

$\Longrightarrow$ Ts.

$$\lim_{x \to x_0} g(f(x)) = k$$

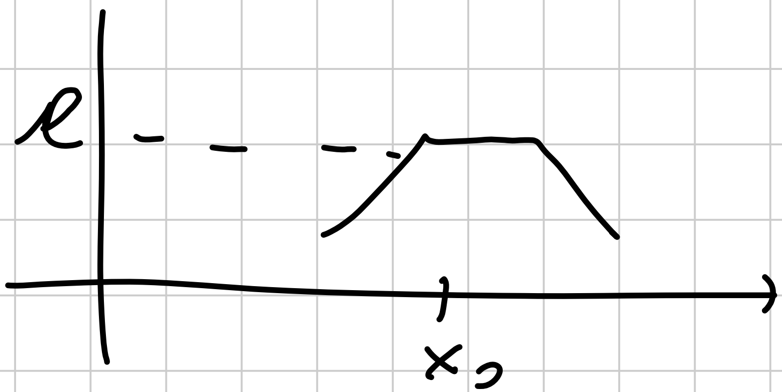


$\cdot\cdot$ $\forall\, W$ intorno di k $\exists\, V$ intorno di ℓ t.c. $g(y) \in W$ se $y \in V$

$\cdot$ $\forall\, I$ di ℓ (prendo V) $\exists\, U$ di x_0 t.c. $f(x) \in V$ se $x \in U$ $\Rightarrow g(f(x)) \in W$

Ts. $\forall\, W$ intorno di k $\exists\, U$ di x_0 t.c. $g(f(x)) \in W$, se $x \in U$

Oss. Il teorema non è vero se $f(x) = \ell$ in un intorno di x_0.



f, g

$$g \circ f = \begin{cases} 27 & x \le 1 \\ x-1 & x > 1 \end{cases}$$

errore nel libro!

Formule di cambiamento di variabili

$$\lim_{x \to +\infty} 3^{1/x} = 1$$

$1/x \to 0 = l \quad x \to +\infty$

$3^y \to 1 = k \quad y \to 0$

Cambio di variabili

$$\frac{1}{x} = y$$

$$\lim_{x \to +\infty} 3^{1/x} = \lim_{y \to 0} 3^y = 1$$

Es. $\lim_{x \to +\infty} \operatorname{arctg} \underbrace{\left(\sqrt{x+1} - \sqrt{x} \right)}_{f(x)} = 0$

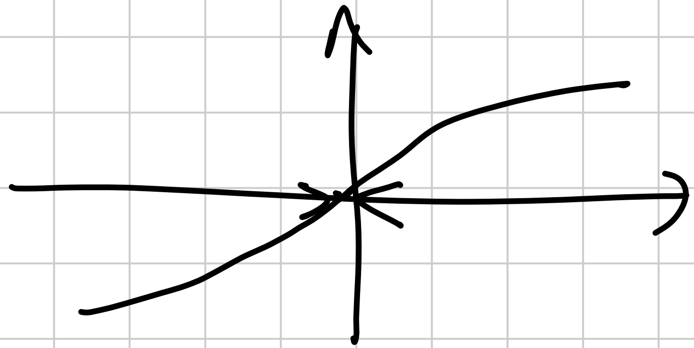
$$\lim_{x \to +\infty} \left(\underbrace{\sqrt{x+1}}_{+\infty} - \underbrace{\sqrt{x}}_{+\infty} \right) = 0$$

$$\infty - \infty$$

$$\left(\sqrt{x+1} - \sqrt{x} \right) \frac{\left(\sqrt{x+1} + \sqrt{x} \right)}{\sqrt{x+1} + \sqrt{x}} = \frac{(\cancel{x}+1) - \cancel{x}}{\sqrt{x+1} + \sqrt{x}} =$$

$$= \frac{1}{\underbrace{\left(\sqrt{x+1} + \sqrt{x} \right)}_{\to +\infty}} \to 0$$

$$\lim_{y \to 0} \operatorname{arctg} y = 0$$



$$\lim_{x \to x_0} f(x)^{g(x)} \quad , \quad f(x) > 0$$

in un intorno di x_0

$$f(x)^{g(x)} = e^{\log\left(f(x)^{g(x)}\right)} = e^{\underbrace{g(x)}\,\underbrace{\log f(x)}}$$

$$y = e^{\log y}$$

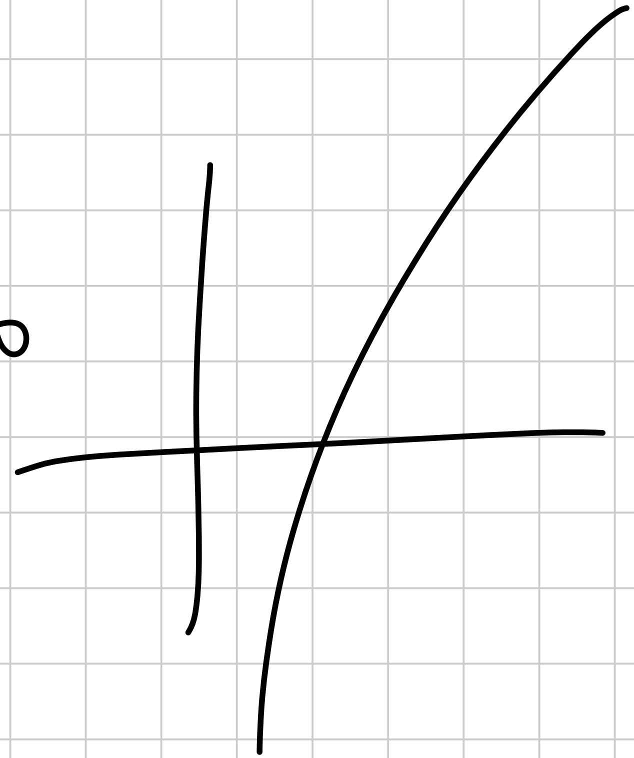
$$\lim_{x \to x_0} f(x)^{g(x)} = \lim_{x \to x_0} e^{g(x) \log f(x)}$$

$$\lim_{x \to +\infty} x^{x} = \lim_{x \to +\infty} e^{\log x^{x}} =$$

$$= \lim_{x \to +\infty} e^{x \log x} = +\infty$$

$$x \log x \to +\infty$$

$$\lim_{x \to +\infty} \log x = +\infty$$



$$f(x)^{g(x)}$$

$$f(x)^{g(x)}$$

$f \to 0$
$g \to 0$

0^0

$0 \cdot (-\infty)$

$$f(x)^{g(x)} = e^{g(x) \log f(x)}$$

~~forma~~
indeterminata

$f \to 1$
$g \to \infty$

1^∞

$0 \cdot (\infty)$

$f \to \infty$
$g \to 0$

∞^0

$0 \cdot \infty$

Limiti notevoli

$$\lim_{x \to 0} \frac{\operatorname{sen} x}{x} = \frac{0}{0} = 1$$

dimostreremo

OSS. non con l'Hôpital

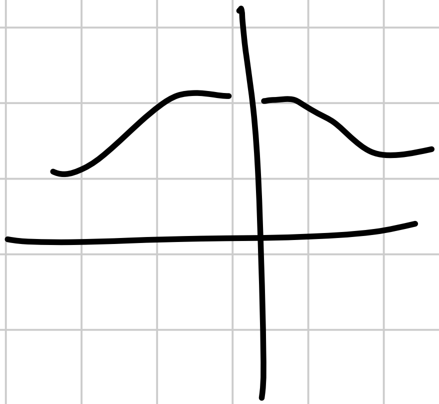
(per favore ... Tanto)

$$\lim_{x \to 0} \frac{\operatorname{sen} x}{x} = 1$$

$$f(x) = \frac{\operatorname{sen} x}{x}$$

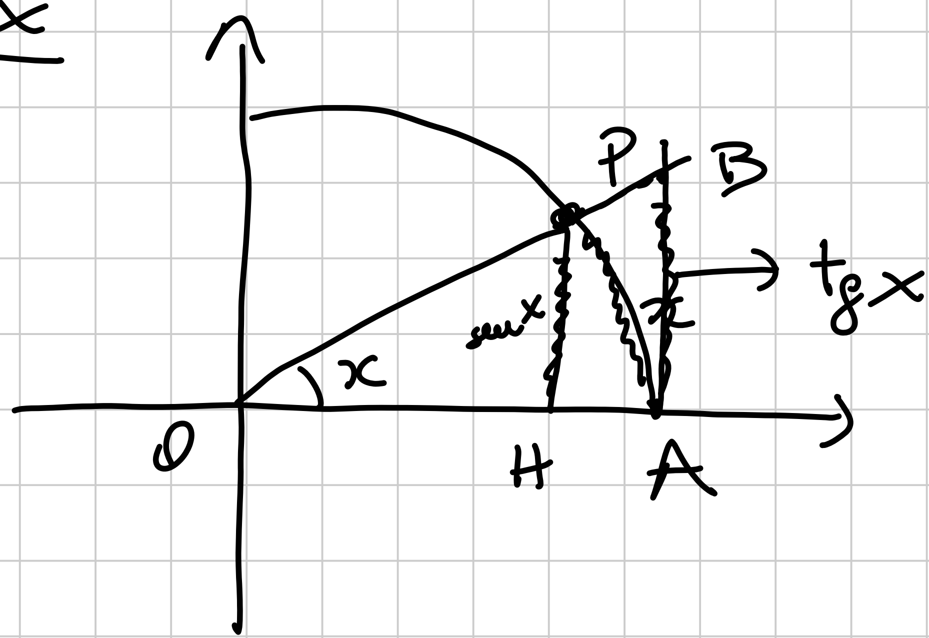
è pari

$$f(-x) = \frac{\operatorname{sen}(-x)}{-x} = \frac{\operatorname{sen} x}{x} = f(x)$$



$$\lim_{x \to 0^+} \frac{\operatorname{sen} x}{x} = \lim_{x \to 0^-} \frac{\operatorname{sen} x}{x}$$

Calcolo $\lim_{x \to 0^+} \frac{\operatorname{sen} x}{x}$



$PH = \operatorname{sen} x$

$BA = \operatorname{tg} x$

$$\frac{PH}{HO} = \frac{BA}{AO}$$

$$\operatorname{tg} x = \frac{\operatorname{sen} x}{\cos x} = \frac{BA}{1}$$

$x > 0$

nel primo quadrante

$$\operatorname{sen} x \leq x \leq \operatorname{tg} x \qquad \forall x \in \left[0, \frac{\pi}{2}\right).$$

$\operatorname{sen} x > 0$

$$1 \leq \frac{x}{\operatorname{sen} x} \leq \frac{1}{\cos x}$$

$$1 \le \frac{x}{\operatorname{sen} x} \le \frac{1}{\cos x} \qquad x \in \left(0, \frac{\pi}{2}\right).$$

$$\lim_{x \to 0^+}$$

$$\cos x \le \frac{\operatorname{sen} x}{x} \le 1$$

$x \to 0^+$: $\cos x \to 1$, $\quad \frac{\operatorname{sen} x}{x} \xrightarrow{x \to 0^+} 1$, $\quad 1 \to 1$

$$\lim_{x \to 0^+} \frac{\operatorname{sen} x}{x} = 1 = \lim_{x \to 0^-} \frac{\operatorname{sen} x}{x}$$

$$\Rightarrow \lim_{x \to 0} \frac{\operatorname{sen} x}{x} = 1$$

$$\lim_{x \to 0} \frac{\operatorname{sen} x}{x} = 1 \qquad \frac{0}{0}$$

$\operatorname{sen} x \to 0$ $\quad x \to 0$

$x \to 0$

sen x e x sono funzioni infinitesime per $x \to 0$

si dice che sen x e x sono infinitesime dello **stesso** ordine

$x \to 0$

Applicazioni del limite notevole.

$$\lim_{x \to 0} \frac{1-\cos x}{x^2} = \frac{0}{0} = \frac{1}{2}$$

$$\frac{(1-\cos x)}{x^2} \cdot \frac{(1+\cos x)}{(1+\cos x)} = \frac{1-\cos^2 x}{x^2(1+\cos x)} =$$

$$= \frac{\operatorname{sen}^2 x}{x^2(1+\cos x)} = \left(\frac{\operatorname{sen} x}{x}\right)^2 \cdot \frac{1}{1+\cos x}$$

$$\left(\frac{\operatorname{sen} x}{x}\right)^2 \to 1 \qquad \frac{1}{1+\cos x} \to 1/2$$

$1-\cos x$ e x^2 sono infinitesimi dello stesso ordine

$$\frac{1}{2}$$

- $$\lim_{x \to 0} \frac{\operatorname{sen}(e^x - 1)}{(e^x - 1)} = \lim_{y \to 0} \frac{\operatorname{sen} y}{y} = 1$$

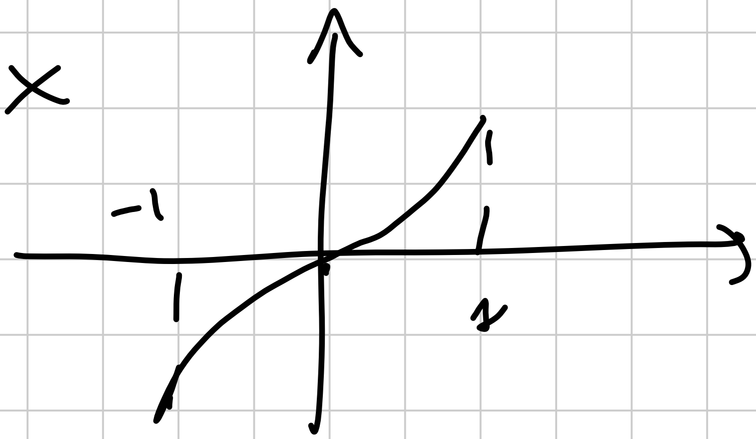
$$y = e^x - 1$$

$$x \to 0 \qquad y \to 0$$

- $$\lim_{x \to +\infty} \underbrace{x}_{\infty} \underbrace{\operatorname{sen} \frac{1}{x}}_{0} = \lim_{y \to 0} \frac{1}{y} \operatorname{sen} y = 1$$

$$\frac{1}{x} = y \qquad x = \frac{1}{y}$$

arcsen x



$$\lim_{x \to 0} \arcsin x = \arcsin 0 = 0$$

$$\lim_{x \to 0} \frac{\arcsin x}{x} = \lim_{y \to 0} \frac{y}{\operatorname{sen} y} = 1$$

$\arcsin x = y$

$x = \operatorname{sen} y$

$$\frac{\operatorname{sen} y}{y} \to 1$$

$y \to 0$

$$\frac{1-\cos x}{x^2} \to \frac{1}{2} \qquad x \to 0$$

$$\frac{\arcsin x}{x} \to 1 \qquad x \to 0$$

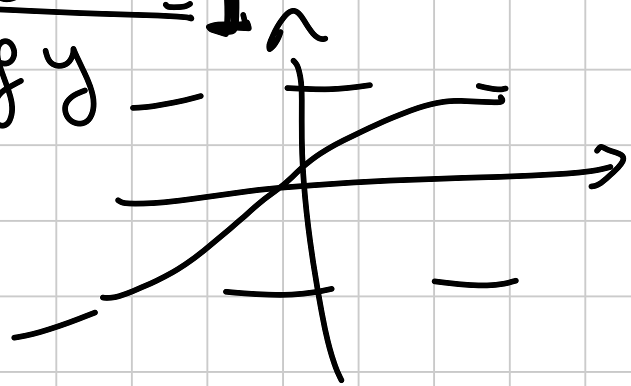
$$\lim_{x\to 0} \frac{\operatorname{tg} x}{x} = \lim_{x\to 0} \frac{\sin x}{\cos x} \cdot \frac{1}{x} = 1$$

$\to 1 \qquad \to 1$

$$\lim_{x\to 0} \frac{\operatorname{arctg} x}{x} = \lim_{y\to 0} \frac{y}{\operatorname{tg} y} = 1$$

$\operatorname{arctg} x = y$

$x = \operatorname{tg} y$



$$\lim_{x\to 0} \frac{(\operatorname{sen} 3x)}{(3x)}\, 3 = \lim_{y\to 0} \frac{\operatorname{sen} y}{y}\cdot 3$$

$$= 3$$

$$3x = y$$

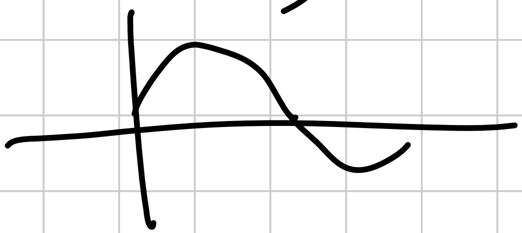
$$\lim_{x\to \pi^+} \frac{\operatorname{sen} x}{\sqrt{x-\pi}} = \lim_{y\to 0^+} \frac{\operatorname{sen}(y+\pi)}{\sqrt{y}} =$$

$$= \lim_{y\to 0^+} -\underbrace{\frac{\operatorname{sen} y}{\sqrt{y}\,\sqrt{y}}\,\sqrt{y}}_{1} = 0$$

$$\underline{x-\pi = y > 0}$$

$$x \to \pi^+$$

$$x = y + \pi$$



$$\lim_{x \to 0} \frac{\operatorname{sen} x}{x} = 1$$

$\operatorname{sen} x \longrightarrow 0$ $\quad$ $x \to 0$

$x \longrightarrow 0$

$\operatorname{sen} x \sim x \qquad x \to 0$

sen x e x sono <u>asintotiche</u> $x \to 0$

$$\lim_{x \to x_0} f(x) = 0$$

$f(x)$ è un infinitesimo per $x \to x_0$

$$\boxed{f(x) \underset{x \to x_0}{=} o(1)}$$

"o piccolo" di 1

$$\lim_{x \to x_0} f(x) = \ell \iff \lim_{x \to x_0} \left(f(x) - \ell \right) = 0$$

$$f(x) - \ell \underset{x \to x_0}{=} o(1) \Rightarrow f(x) \underset{x \to x_0}{=} \ell + o(1)$$

$$\lim_{x \to 0} \frac{\operatorname{sen} x}{x} = 1$$

$$\frac{\operatorname{sen} x}{x} - 1 \underset{x \to 0}{\longrightarrow} 0$$

$$\boxed{\frac{\operatorname{sen} x}{x} = 1 + o(1) \quad x \to 0}$$

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$$

$1 - \cos x$ e x^2 sono infinitesime dello stesso ordine

$$\lim_{x \to 0} \frac{(1-\cos x)}{x \cdot x} x = 0$$

$1/2$

$1 - \cos x \to 0$

$x \to 0$

$1 - \cos x$ è infinitesimo di ordine superiore rispetto a x

Altri limiti notevoli (scale di infiniti)

- $x^{\beta} \to +\infty$, $\beta > 0$ $\qquad$ $x^{5} \xrightarrow[x\to+\infty]{} +\infty$

- $\log_b x \to +\infty$ $\qquad$ $\log_3 x \xrightarrow[x\to+\infty]{} +\infty$

- $a^{x} \to +\infty$, $a > 1$ $\qquad$ $2^{x} \xrightarrow[x\to+\infty]{} +\infty$

x^{β}, $\log_b x$, a^{x} sono infiniti per $x \to +\infty$

$\beta > 0$, $b > 1$, $a > 1$

Es.

$$\lim_{x \to +\infty} \frac{x^2}{5^x} = \frac{\infty}{\infty}$$

$$\lim_{x \to +\infty} \frac{\log x}{x^3} = \frac{\infty}{\infty}$$

① $$\lim_{x \to +\infty} \frac{x^\beta}{a^x} = 0$$ (no dim.)

$\forall \beta > 0$ e $a > 1$

Es.

$$\frac{x^2}{3^x} \xrightarrow[x \to +\infty]{} 0$$

a^x è infinito di ordine superiore rispetto a x^β

$$\frac{x^{1000}}{2^x} \xrightarrow[x \to +\infty]{} 0$$

(2) $$\lim_{x \to +\infty} \frac{|\log_b x|^{\alpha}}{x^{\beta}} = 0$$

$\left(\frac{\infty}{\infty}\right)$

$b > 1$

$\alpha > 0$

$\beta > 0$

es. $$\lim_{x \to +\infty} \frac{(\log x)^{1000}}{\sqrt{x}} = 0 \quad \frac{\infty}{\infty}$$

$\sqrt{x}$ è <u>infinito</u> di ordine <u>superiore</u> rispetto a $(\log x)^{1000}$

$$|\log_b x|^\alpha < x^\beta < a^x$$

Scala INFINITI $\quad x \to +\infty$

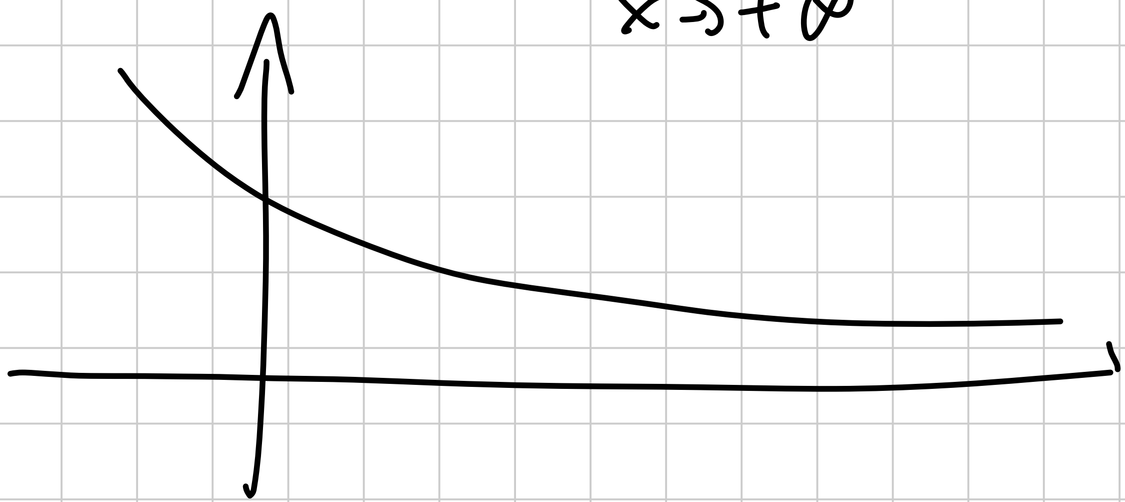
$$\lim_{x \to +\infty} \frac{(\log_5 x)^{\sqrt{2}}}{3^x} = 0$$

$$\frac{\operatorname{sen} x}{x} \xrightarrow[x \to 0]{} 1$$

ES $\lim_{x \to +\infty} \frac{a^x}{x^x} \overset{a>1}{=} \frac{\infty}{\infty}$

$a > 1$

$a < 1$



$\lim_{x \to +\infty} a^x = 0$

$\frac{a^x}{x^x}$ $\frac{0}{\infty} \to 0$

$\lim_{x \to +\infty} \frac{\left(\frac{1}{2}\right)^x}{x^x} = 0$

$$\lim_{x \to +\infty} \frac{a^x}{x^x} = 0$$

$$\frac{a^x}{x^x} = \left(\frac{a}{x}\right)^x = e^{\log\left(\frac{a}{x}\right)^x} =$$

$$= e^{\overset{\to +\infty}{x} \underbrace{\log\left(\frac{a}{x}\right)}_{\to -\infty}} \to 0$$

x^x è infinito di ordine superiore a a^x, $\forall a > 1$

$$|\log_b x|^\alpha \prec x^\beta \prec a^x \prec x^x$$

$$x \to +\infty$$

Altro limite notevole

es. $\lim_{x \to 0^+} x \cdot \log x = 0 \cdot \infty$

(3) $\lim_{x \to 0^+} x^{\beta} |\log_b x|^{\alpha} = 0$ $\quad \beta > 0$, $\alpha > 0$, $b > 1$

$\lim_{x \to 0^+} \sqrt{x} \left(\log x\right)^{1000} = 0$