Limiti di successioni

$$|\log_b n|^\alpha \prec n^\beta \prec a^n \prec n! \prec n^n$$

scale infiniti

$$\lim_{n \to +\infty} \frac{n^3}{5^n} = 0$$

p.104 Libro

$$a^n < n^\beta$$

NO!

$$a_n = \left(1 + \frac{1}{n}\right)^n$$

$$\lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^n \qquad 1^\infty$$

- a_n è crescente $\qquad \left(1 + \frac{1}{n+1}\right)^{n+1} \geq \left(1 + \frac{1}{n}\right)^n$
- a_n è limitata $\qquad 1 < a_n < 3$

$\{a_n\}$ è crescente $\Rightarrow \exists$ limite finito

o $+\infty$

~~ma~~ poiché $\{a_n\}$ è limitata

$\Rightarrow \exists$ limite finito $\Rightarrow$ la succ. converge

$\exists \boxed{\lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^n =: e}$ numero di Nepero.

Si può dimostrare che

$e \notin \mathbb{Q}$ e $2 < e < 3$

es. $\lim_{n \to +\infty} n \log\left(1 + \frac{1}{n}\right) = \lim_n \log\left[\left(1 + \frac{1}{n}\right)^n\right]$

$= 1$

$$n \log\left(1 + \frac{1}{n}\right) = \frac{\log\left(1 + \frac{1}{n}\right)}{\frac{1}{n}} \xrightarrow[n \to +\infty]{} 1$$

$\log\left(1 + \frac{1}{n}\right) \to 0$, $\frac{1}{n} \to 0$

$\log\left(1 + \frac{1}{n}\right)$ e $\frac{1}{n}$ sono infinitesime dello stesso ordine $n \to +\infty$

$a_n \to 0$
$b_n \to 0$

$$\frac{a_n}{b_n} \longrightarrow 1$$

sono infinitesime dello stesso ordine

$$\Downarrow$$

$a_n \sim b_n$ (a_n asintotica a b_n)

Es. precedente

$$\log\left(1+\frac{1}{n}\right) \sim \frac{1}{n}$$

$$\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n =: e$$

Si può far vedere che vale anche per x

$n \to x$

$$\boxed{\lim_{x\to\pm\infty}\left(1+\frac{1}{x}\right)^x = e}$$

$$\lim_{x\to+\infty}\left(1+\frac{\alpha}{x}\right)^x = \lim_{y\to\substack{+\infty\\-\infty}}\left(\left(1+\frac{1}{y}\right)^y\right)^\alpha = e^\alpha$$

$\frac{\alpha}{x} = \frac{1}{y}$ $\quad x = \alpha y$ $\quad \alpha > 0$ $\quad y \to +\infty$

$\alpha \in \mathbb{R}$ $\quad \alpha < 0$ $\quad y \to -\infty$

Esercizio

$$\lim_{x \to 0} \frac{\log(1-x)}{x} = \lim_{x \to 0} \log\left[(1-x)^{1/x}\right] =$$

$$= \lim_{y \to \pm\infty} \log \underbrace{\left(1 - \frac{1}{y}\right)^{y}}_{\to e^{-1}} = -1 \qquad \boxed{\frac{1}{x} = y \quad x = \frac{1}{y}}$$

applico limite precedente con $\alpha = -1$

Es.

$$\lim_{x \to 0} x \log x \cdot \underbrace{\frac{\log(1-x)}{x}}_{\to -1} = 0$$

$$\lim_{x \to 0} x \log x = 0$$ (limite notevole)

$$\lim_{x \to 0} \frac{\log(1-x)}{x} = -1$$

$$\frac{0}{0}$$

$$\log(1-x) \longrightarrow 0$$

$$x \longrightarrow 0$$

$\log(1-x)$ e x sono infinitesime dello stesso ordine

$$\log(1-x) \qquad x$$

$$\left(1+\frac{1}{x}\right)^x \xrightarrow[x \to \pm\infty]{} e$$

$$\lim_{x \to 0} \frac{\log(1+x)}{x} = \lim_{x \to 0} \log(1+x)^{1/x} =$$

$$= \lim_{y \to \pm\infty} \log\left(1+\frac{1}{y}\right)^y = 1 \qquad \frac{1}{x} = y$$

$$\log(1+x) \sim x \quad , \quad x \to 0$$

sono asintotiche

$$\boxed{\lim_{x \to 0} \frac{e^x - 1}{x} = 1}$$

Infatti

$$e^x - 1 = y \qquad e^x = y + 1$$

$$x = \log(y+1)$$

$$\lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{y \to 0} \frac{y}{\log(y+1)} = 1$$

$$e^x - 1 \longrightarrow 0$$

$$x \longrightarrow 0$$

$$e^x - 1 \sim x \quad x \to 0$$

$$\lim_{x \to 0} \frac{a^x - 1}{x} = \log a$$

Fare!

$$a^x = e^{x \log a}$$

$x \log a = y$

da limite precedente

$$\lim_{x \to 0} \frac{5^x - 7^x}{x} = \lim_{x \to 0} 7^x \frac{\left(\frac{5}{7}\right)^x - 1}{x}$$

$\log \frac{5}{7}$

$$= \log\left(\frac{5}{7}\right)$$

$5^x - 7^x$ è un infinitesimo di ordine 1 (x^1)

Un altro limite notevole $\alpha \in \mathbb{R}, \alpha > 0$

$$\lim_{x \to 0} \frac{(1+x)^\alpha - 1}{x} = \alpha$$

provare: $(1+x)^\alpha = e^{\alpha \log(1+x)}$

$$= \lim_{x \to 0} \frac{e^{\alpha \log(1+x)} - 1}{x} = \lim$$

$\alpha \log(1+x) = y$ $\quad \log(1+x) = \frac{y}{\alpha}$

$1 + x = e^{\frac{y}{\alpha}}$ $\quad x = e^{\frac{y}{\alpha}} - 1$

$$= \lim_{y \to 0} \frac{e^{y} - 1}{e^{\frac{y}{\alpha}} - 1} \cdot \frac{y}{y} \cdot \frac{\alpha}{\alpha} = \alpha$$

$\to 1$

$$\frac{y}{e^{\frac{y}{\alpha}} - 1} \quad \frac{\alpha}{\alpha} = \frac{\frac{y}{\alpha}}{\left(e^{\frac{y}{\alpha}} - 1\right)} \cdot \alpha$$

$\to 1$

$y \to 0$

$$\lim_{x \to 0} \frac{(1+x)^{\alpha} - 1}{x} = \alpha$$

$$\lim_{x \to 0} \frac{(1+x)^{1000} - 1}{x} = 1000$$

$$\lim_{x \to 0} \frac{\sqrt[5]{1+x} - 1}{x} = \frac{1}{5}$$

$$\lim_{x \to 0} \frac{\sqrt{1+x} - 1}{x} = \frac{1}{2}$$

Limiti di funzioni ottenuti da

$$\lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^n =: e$$

Criterio del <u>rapporto</u> per i limiti di successione.

$\{a_n\}$, $a_n > 0$ e $\exists \lim_n \frac{a_{n+1}}{a_n} = l \in \overline{\mathbb{R}}^*$

1) $l < 1 \Rightarrow \{a_n\}$ è strett. decrescente e $\lim_n a_n = 0$

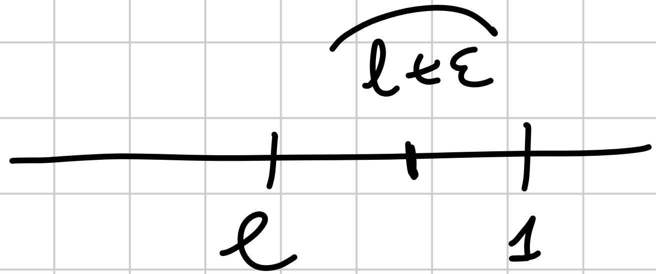
2) $l > 1 \Rightarrow \{a_n\}$ è strett. crescente e $\lim_n a_n = +\infty$

(oss. se $l = 1$ il criterio <u>non</u> dice <u>nulla</u>).

Hp. $a_n > 0$

$$\frac{a_{n+1}}{a_n} \to \ell < 1$$

Ts. $a_n \to 0$ M



Dim. $0 \le \ell < 1$

$\forall \varepsilon > 0 \quad \exists\, n_0 : \quad \dfrac{a_{n+1}}{a_n} < \ell + \varepsilon \overset{M<1}{=:} , \quad \forall n > n_0$

scelgo ε t.c. $\ell + \varepsilon < 1 \qquad \ell + \varepsilon = M$

$$\frac{a_{n+1}}{a_n} < M \qquad a_{n+1} < M a_n \quad \forall n > n_0$$

$$0 < a_{n+1} < M a_n < M \cdot M a_{n-1} < M^2 \cdot M a_{n-2} <$$

$$< \cdots \cdots M^{n-n_0} a_{n_0}$$

$\downarrow$

$M < 1$

Ho trovato $\bot$

$$\underset{\downarrow\, 0}{0} < \underset{\searrow\, 0}{a_{n+1}} < M^{n-n_0} \cdot a_{n_0} = M^n \frac{a_{n_0}}{M^{n_0}}, \quad M^n \to 0$$

Dal teo. del confronto

$$a_n \to 0$$

Dim. 1) $\quad \ell < 1 \Longrightarrow a_n \to 0$

se $\ell > 1 \Rightarrow a_n \to +\infty$.

$a_n > 0$ $\qquad \dfrac{a_{n+1}}{a_n} \to \ell$

$\ell < 1 \qquad a_n \to 0$

$\ell > 1 \qquad a_n \to +\infty$

$\ell = 1 \qquad$ non so nulla

es. $\lim_n \dfrac{n^5}{3^n} = a_n$

$a_{n+1} = \dfrac{(n+1)^5}{3^{n+1}}$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^5}{3^{n+1}} \cdot \frac{\cancel{3^n}}{n^5} = \frac{1}{3}\left(\frac{n+1}{n}\right)^5$$

$\left(\frac{n+1}{n}\right) \to 1$

$1/3 < 1$

dal criterio del rapporto

$$\lim_{n \to +\infty} \frac{n^5}{3^n} = 0$$

Altro esempio

$$\lim_{n} \underbrace{\frac{n^n}{n!}}_{a_n} = +\infty \quad \text{(scala degli infiniti)}$$

Dim.

$$\frac{a_{n+1}}{a_n} = \underbrace{\frac{(n+1)^{n+1}}{(n+1)!}}_{a_{n+1}} \cdot \frac{n!}{n^n} = \frac{\cancel{n!}\,(n+1)^n\cancel{(n+1)}}{\cancel{(n+1)}\cdot\cancel{n!}\,n^n}$$

$$= \left(\frac{n+1}{n}\right)^n = \left(1 + \frac{1}{n}\right)^n$$

$$e > 1$$

Es. Quando non si può applicare il criterio del rapporto

$$a_n = \frac{1}{n} \longrightarrow 0$$

$$\frac{a_{n+1}}{a_n} = \frac{1}{n+1} \cdot n \to 1 \qquad l = 1$$

non ci dice niente

Formule da ricordare

$\{a_n\}$, $a_n > 0$ t.c. $\exists \lim_n \frac{a_{n+1}}{a_n}$

allora

$$\lim_n \frac{a_{n+1}}{a_n} = \lim_n \sqrt[n]{a_n} = \ell$$

es. $$\lim_n \sqrt[n]{\underbrace{n!}_{a_n}} = \lim_n \frac{(n+1)!}{n!} =$$

$$= \lim_n \frac{\cancel{n!}\,(n+1)}{\cancel{n!}} = +\infty$$

Si deduce che

$$\text{se } \lim_{n} \sqrt[n]{a_n} = \ell \begin{cases} \ell < 1 & a_n \to 0 \\ \ell > 1 & a_n \to +\infty \end{cases}$$

$$= \lim_{n} \frac{a_{n+1}}{a_n}$$

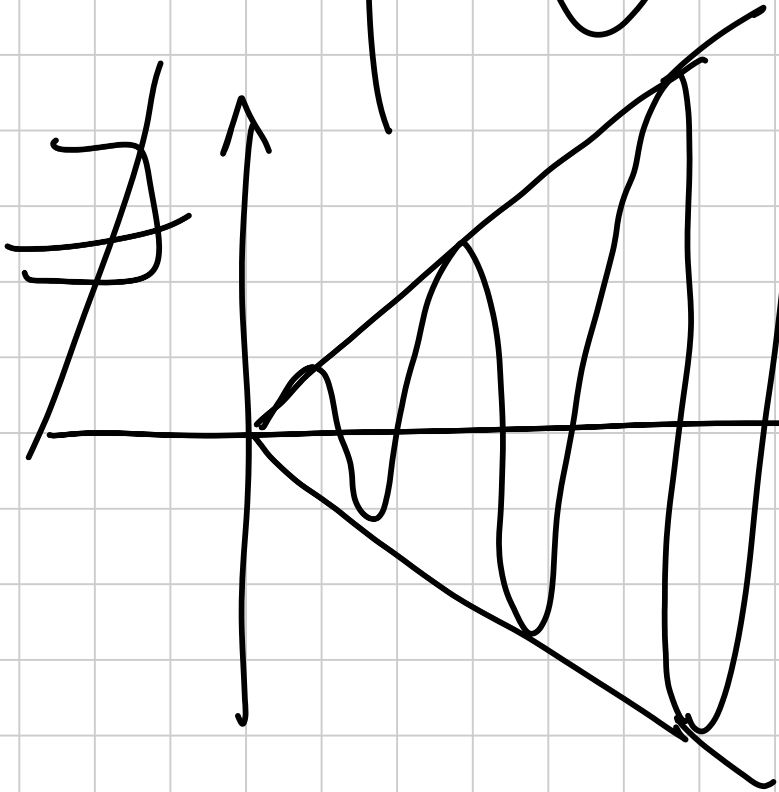
<u>Criterio della radice</u>

Rapporto tra limiti di funzioni e limiti di successioni.

Teorema ponte (tra funzioni e successioni)

$$\lim_{x \to +\infty} \operatorname{sen} x \ \nexists$$

$$\lim_{x \to +\infty} x \operatorname{sen} x \ \nexists$$



Teorema ponte.

$\lim_{x \to x_0} f(x) = l \in \mathbb{R}^* \iff \forall \{a_n\}$ successione t.c. $a_n \to x_0$ si ha $\lim_n f(a_n) = l$

Serve per dim. che $\nexists$ limite di una funzione.

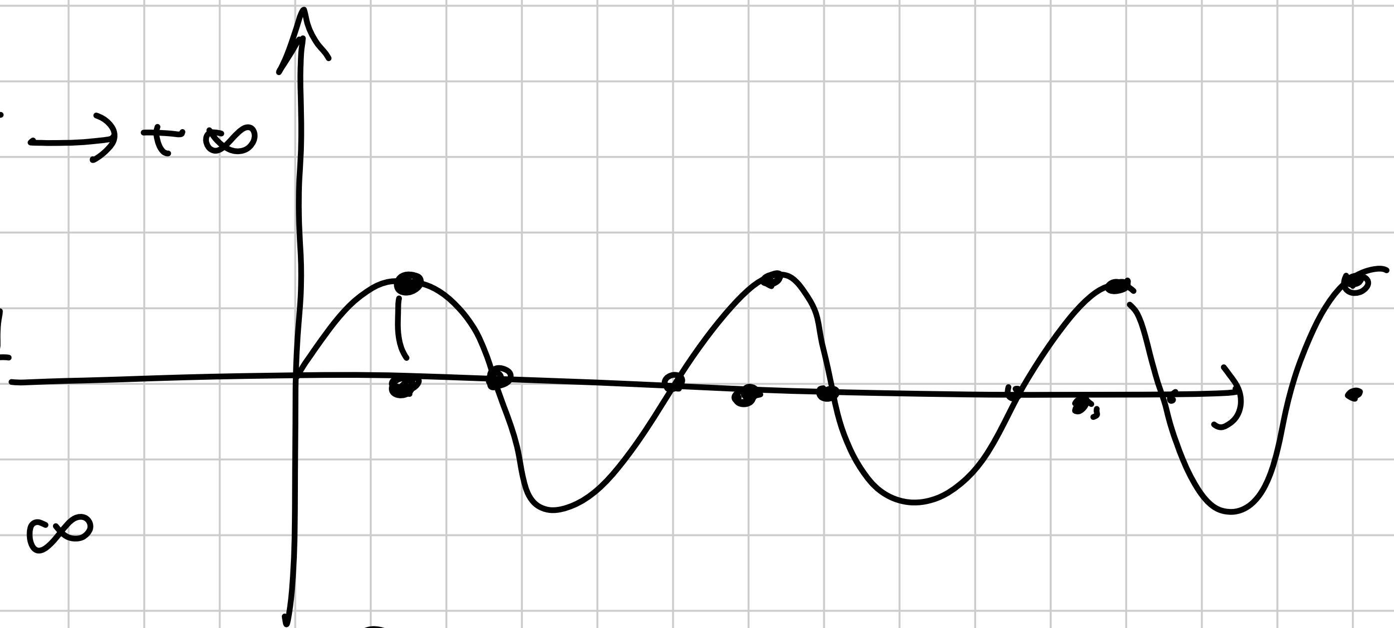
Infatti se trovo due successioni $\{a_n\}$ e $\{b_n\} \longrightarrow x_0$

$f(a_n) \to l_1$

$f(b_n) \to l_2$

$\Longrightarrow \nexists \lim_{x \to x_0} f(x)$.

Es. $\lim_{x \to +\infty} \operatorname{sen} x \nexists$ $\qquad x_0 = +\infty$



$a_n = \frac{\pi}{2} + 2n\pi \to +\infty$

$f(a_n) = \operatorname{sen}(a_n) = 1 \to 1$

$b_n = n\pi \to +\infty$

$f(b_n) = \operatorname{sen} b_n = 0 \to 0$

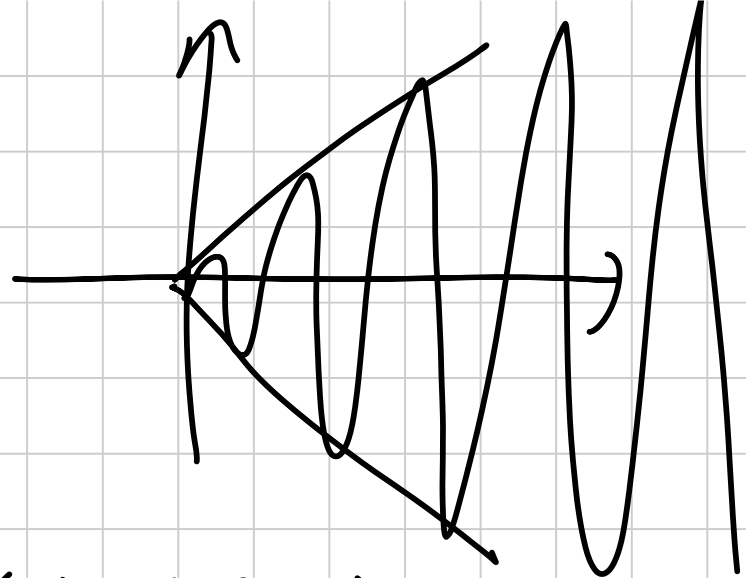
Ho trovato due successioni

a_n e $b_n \to +\infty$ t.c.

$\operatorname{sen}(a_n) \to 1$

$\operatorname{sen}(b_n) \to 0$ $\qquad \Rightarrow \nexists \lim_{x \to +\infty} \operatorname{sen} x$

Es. $\lim_{x \to +\infty} x \operatorname{sen} x \quad \nexists$



$f(x) = x \operatorname{sen} x$

trovo $a_n \to +\infty$

$b_n \to +\infty$

t.c.

$a_n \operatorname{sen} a_n \to \ell_1$

$b_n \operatorname{sen} b_n \to \ell_2$

$a_n = n\pi$

$b_n = \frac{\pi}{2} + 2n\pi$

$\ell_1 = 0$

$\ell_2 = +\infty$

finiamo qui

Esercizio

$$\lim_{n} \frac{n! + e^{5n}}{e^{6n} - n \operatorname{sen} n} =$$

$$= \lim_{n} \frac{n!\left(1 + \frac{e^{5n}}{n!}\right)}{e^{6n}\left(1 - \frac{n \operatorname{sen} n}{e^{6n}}\right)} = +\infty$$

$$\frac{e^{5n}}{n!} \to 0$$

$$\frac{n!}{e^{6n}} \to +\infty$$

$$(1 + \to 0)$$

$$\frac{n}{(e^{6})^{n}} \operatorname{sen} n \to 0$$

$$\lim_{n} \frac{a^{n+1} + n^2 + (-1)^{n+1}}{\pi^n - 2n^3 - 2\,\mathrm{sen}\, n} \qquad a>0$$

1) $a > 1$ "comanda" a^{n+1} di sopra

di sotto π^n

2) $a < 1$