Polinomio di Taylor

Def. f derivabile n volte in x_0

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$$

polinomio di Taylor di ordine n, di f centrato in x_0.

$x_0 = 0$ $$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

Mc Laurin

$f(x) = e^x$ $x_0 = 0$

$$T_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}$$

$$f(x) = \operatorname{sen} x \qquad x_0 = 0$$

$$T_n(x) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

$$f(x) = \cos x \qquad x_0 = 0$$

$$T_n(x) = \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!}$$

$$+ \dots$$

Relazione tra $f(x)$ e il suo $T_n(x)$.

<u>Teorema di Peano</u> $f:(a,b) \to \mathbb{R}$, $x_0 \in (a,b)$

f derivabile n volte in x_0.

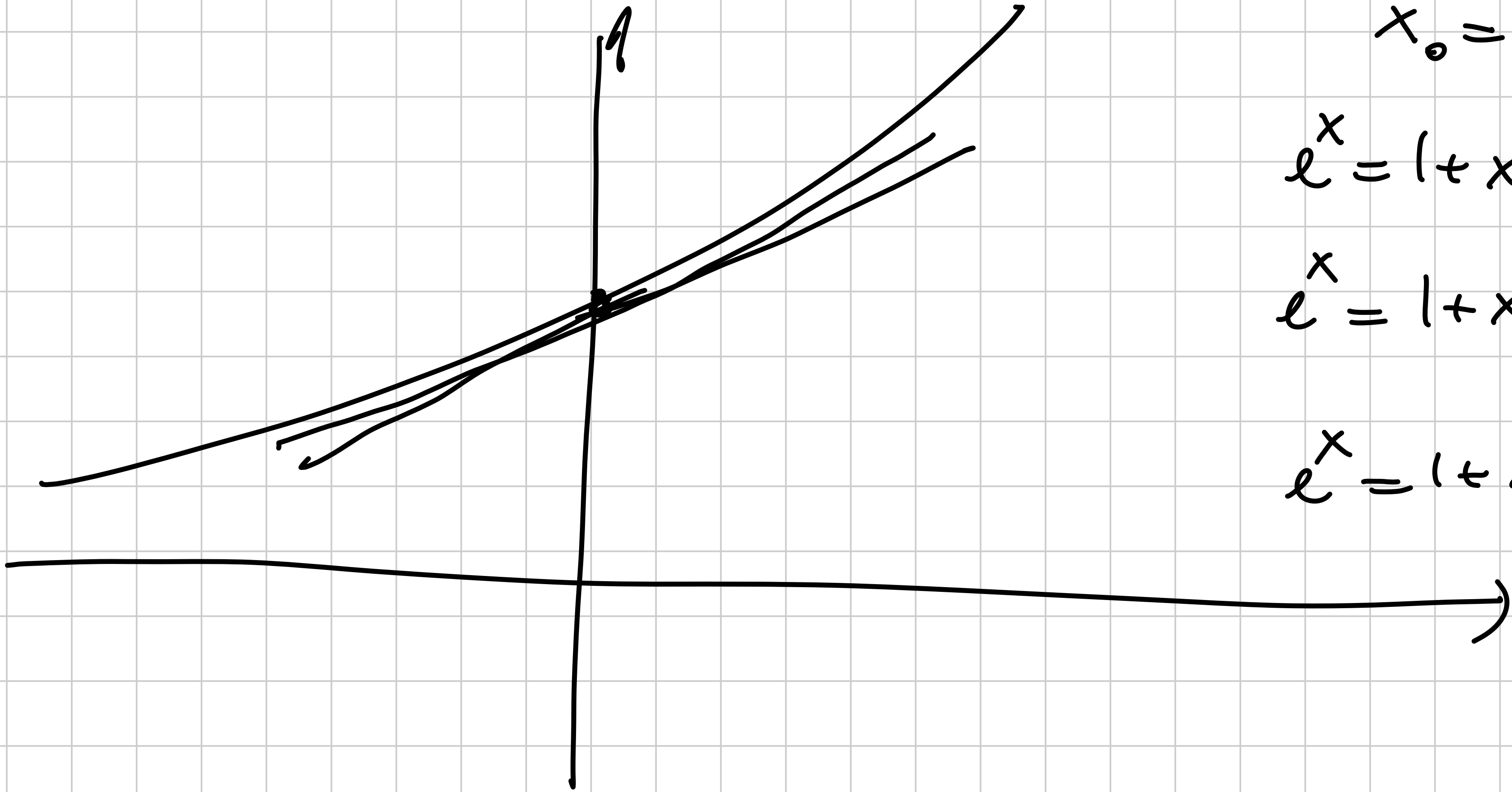
Allora

$$f(x) = T_n(x) + \underbrace{o\left((x-x_0)^n\right)}_{\text{resto di Peano}}, \quad x \to x_0$$

<u>es.</u> $f(x) = e^x$ $\qquad x_0 = 0$

$$T_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!}$$

$$\boxed{e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \underset{x \to 0}{o}\left(x^3\right)}$$



$x_0 = 0 \qquad x \to 0$

$$e^x = 1 + x + o(x)$$

$$e^x = 1 + x + \frac{x^2}{2} + o(x^2)$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + o(x^3)$$

$\underline{\text{Ts.}}$ $f(x) = T_n(x) + o\left((x-x_0)^n\right)$

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

$\underline{\text{Dim.}}$ $n=0 \rightarrow T_0(x) = f(x_0)$

devo dim che ? $f(x) = f(x_0) + o(1)$?
def. di f. continua in x_0 $x \to x_0$

sì è vera

$n=1$ $T_1(x) = f(x_0) + f'(x_0)(x-x_0)$

devo dim. ? $f(x) = f(x_0) + f'(x_0)(x-x_0) + o(x-x_0)$?
sì (def. di derivata in x_0) $x \to x_0$

$n=2$ $T_2(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2$

? $f(x) = \underbrace{f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2}_{} + o\left((x-x_0)^2\right)$?

$T_2(x)$ $\quad x \to x_0$?

? $f(x) - T_2(x) = o\left((x-x_0)^2\right)$ $\quad x \to x_0$

quindi devo verificare che

$$\lim_{x \to x_0} \frac{f(x) - T_2(x)}{(x-x_0)^2} \overset{?}{=} 0$$

$T_2'(x)$

Uso teorema dell' Hôpital

$$\lim_{x \to x_0} \frac{f'(x) - \left(f'(x_0) + f''(x_0)(x-x_0)\right)}{2(x-x_0)} =$$

$$= \lim_{x \to x_0} \left(\frac{f'(x) - f'(x_0)}{2(x-x_0)} - \frac{f''(x_0)\cancel{(x-x_0)}}{2\cancel{(x-x_0)}} \right)$$

$$= \frac{1}{2} f''(x_0) - \frac{1}{2} f''(x_0) = 0 \quad \#$$

Applicazioni

es. $$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + o(x^3) \quad x \to 0$$

$$e^y = 1 + y + \frac{y^2}{2} + \frac{y^3}{3!} + o(y^3) \quad y \to 0$$

$y = \sqrt{x}$ $\quad x \to 0^+$

$\sqrt{x} = y \to 0$

$$e^{\sqrt{x}} = 1 + \sqrt{x} + \frac{1}{2}x + \frac{1}{3!}\sqrt{x^3} + o\left(\sqrt{x^3}\right) \quad x \to 0^+$$

questo si può fare solo se $f(x) \to 0 \quad x \to 0^+$

$e^{f(x)}$

<u>es.</u>

$e^{\sqrt{x}+5}$

e^{y} no polinomio di Mc Laurin con resto di Peano

$y = \sqrt{x} + 5$

$x \to 0$

$y \not\to 0$

$e^{5}\, e^{\sqrt{x}}$

$x \to 0$

$\sqrt{x} \to 0$

$\underline{\text{es.}}$ $\quad \operatorname{sen} x = x - \frac{x^3}{3!} + o(x^3) \qquad x \to 0$

$$(\operatorname{sen} x)^2 = \left(x - \frac{x^3}{3!} + o(x^3) \right)^2 =$$

$$= x^2 + \frac{x^6}{(3!)^2} + (o(x^3))^2 - \frac{2}{3!} x^4 +$$

$$+ 2x\, o(x^3) - \frac{2}{3!} x^3 o(x^3) =$$

$$= x^2 - \frac{1}{3} x^4 + o(x^4)$$

$$x^6 = o(x^4)$$

$$x\, o(x^3) = o(x^4)$$

$$\frac{\cancel{x}\, o(x^3)}{x^{4\,3}} \to 0 \qquad x \to 0$$

$$(o(x^3))^2 = o(x^6)$$

$$\frac{o(x^6)}{x^6} \quad \text{è} \quad o(x^4)$$

devo dim che $o(x^4)$ è anche $o(x^6)$

$$\frac{o(x^6)}{x^4 \cdot x^2} \, x^2 \to 0$$

$$\to 0 \quad x \to 0$$

$$(\operatorname{sen} x)^2 = x^2 - \frac{1}{3} x^4 + o(x^2)$$

$x \to 0$

se avessi scritto

$$(\operatorname{sen} x)^2 = x^2 - \frac{1}{3}x^4 + \frac{x^6}{(3!)^2} + o(x^4)$$

$$f(x) = T_n(x) + o(x^n)$$

Limiti di funzioni con sviluppi di McLaurin

Strategia

$$\lim_{x\to x_0} \frac{h}{k} \qquad h = f + o(f) \quad x\to x_0$$

$$k = g + o(g)$$

$$\lim_{x\to x_0} \frac{f+o(f)}{g+o(g)} = \lim_{x\to x_0} \frac{f}{g}$$

Es. $\displaystyle \lim_{x\to 0} \frac{x + \operatorname{sen} x + \log(1+x)}{e^x - 1 + x^2}$

- <u>Hôpital</u>
- Limiti notevoli
- Sviluppi di McLaurin:

$$\operatorname{sen} x = x + o(x) \quad x \to 0$$

$$\log(1+x) = x + o(x)$$

$$e^x = 1 + x + o(x)$$

$$x^2 = o(x) \quad x \to 0$$

$\rightarrow$ $$\lim_{x \to 0} \frac{x + x + o(x) + x}{\cancel{1} + x + o(x) - \cancel{1}} = \lim \frac{3x + o(x)}{x + o(x)}$$

$$= \lim_{x \to 0} \frac{3\cancel{x}}{\cancel{x}} = 3$$

Es. $\displaystyle\lim_{x\to 0} \frac{\cos 2\sqrt{x} - e^{-2x}}{\operatorname{sen} x - x}$

• Hopital
Fare!

D. $\operatorname{sen} x - x =$

$\operatorname{sen} x = x + o(x)$, $x \to 0$

$= \cancel{x} + o(x) - \cancel{x} = o(x)$

non basta lo sviluppo al 1° ordine

$\operatorname{sen} x = x - \frac{x^3}{3!} + o(x^3)$, $x \to 0$

$\operatorname{sen} x - x = \cancel{x} - \frac{x^3}{3!} + o(x^3) - \cancel{x}$

$= -\frac{x^3}{3!} + o(x^3)$

N. $\cos 2\sqrt{x} - e^{-2x} = \cancel{1} - \cancel{2}x - \cancel{1} + \cancel{2}x + o(x)$

$\cos y = 1 - \frac{y^2}{2} + o(y^2) \quad y \to 0$

$y = 2\sqrt{x} \quad x \to 0 \quad 2\sqrt{x} = y \to 0$

$\cos 2\sqrt{x} = 1 - \frac{1}{2}\left(2\sqrt{x}\right)^2 + o\left(2\sqrt{x}\right)^2 =$

$= 1 - 2x + o(x) \quad x \to 0$

$o(2x) = o(x)$

$e^{y} = 1 + y + o(y) \quad y \to 0$

$e^{-2x} = 1 - 2x + o(x) \quad y = -2x \to 0$

$$\cos 2\sqrt{x} - e^{-2x} = \cancel{1} - \cancel{2}x - \cancel{1} + \cancel{2}x + o(x)$$

$$= o(x)$$

$$\lim \frac{N}{D} = \lim \frac{o(x)}{-\frac{x^3}{3!} + o(x^3)}$$

$$\cos y = 1 - \frac{y^2}{2} + \frac{y^4}{4!} + o(y^4)$$

$$\cos 2\sqrt{x} = 1 - 2x + \frac{1}{4!}(2\sqrt{x})^4 + o(x^2)$$

$$= 1 - 2x + \frac{4 \cdot 4}{4!} x^2 + o(x^2)$$

$$\cos 2\sqrt{x} = 1 - 2x + \frac{2}{3}x^2 + o(x^2)$$

$$e^{-2x} = 1 - 2x + \frac{1}{2}(-2x)^2 + o(x^2) \qquad e^y = 1 + y + \frac{y^2}{2} + o(y^2)$$

$$= 1 - 2x + 2x^2 + o(x^2)$$

$$\cos 2\sqrt{x} - e^{-2x} = \cancel{1} - \cancel{2x} + \frac{2}{3}x^2 \cancel{-1} + \cancel{2x} - 2x^2 + o(x^2) = -\frac{4}{3}x^2 + o(x^2)$$

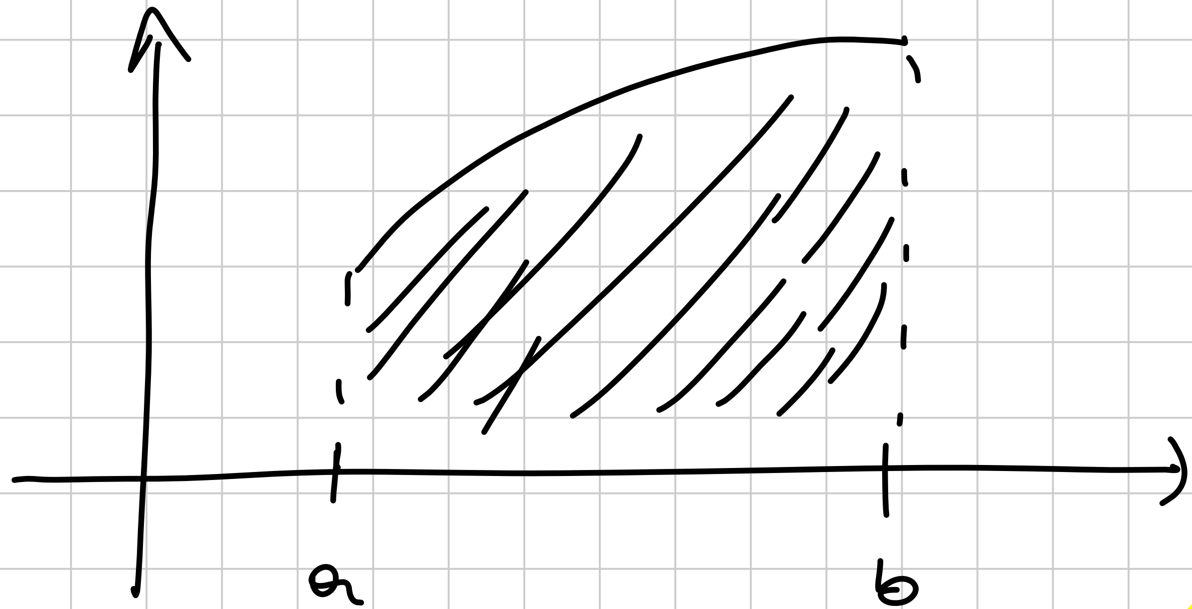
$$\frac{N.}{D.} = \frac{-\frac{4}{3}x^2 + o(x^2)}{-\frac{x^3}{6} + o(x^3)}$$

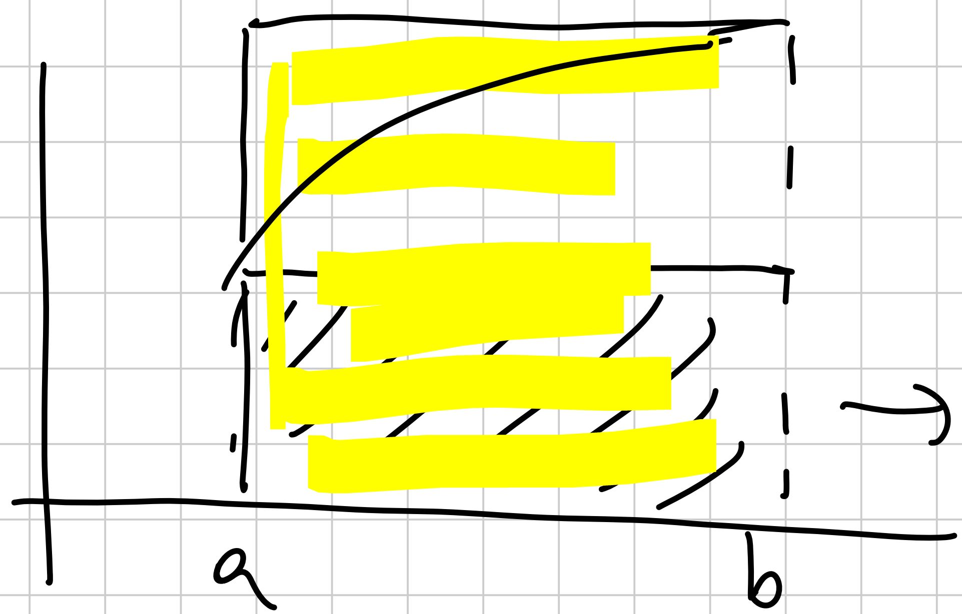
$$\lim \left(\quad \right) = \lim_{x \to 0} \frac{-\frac{4}{3}x^2}{-\frac{x^3}{6}} = \lim_{x \to 0^+} \frac{6 \cdot 4}{3x}$$

$$= +\infty$$

Definizione di integrale (Riemann)

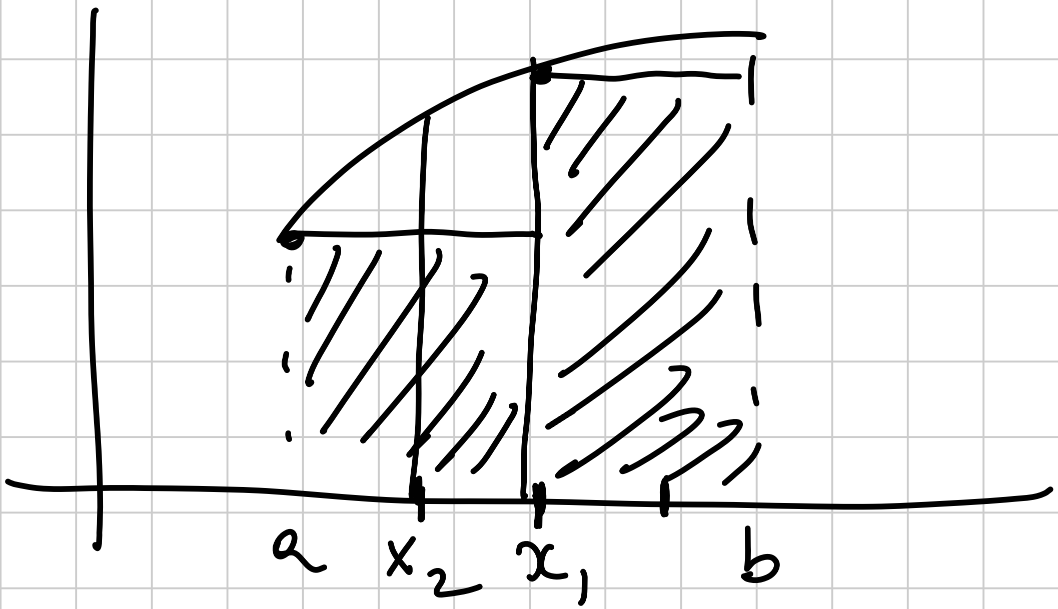
$f: [a, b] \to \mathbb{R}$

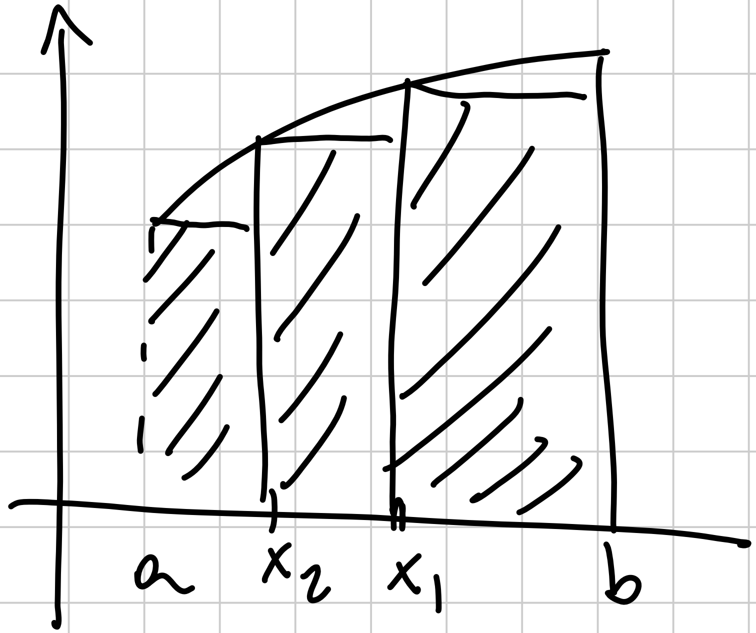




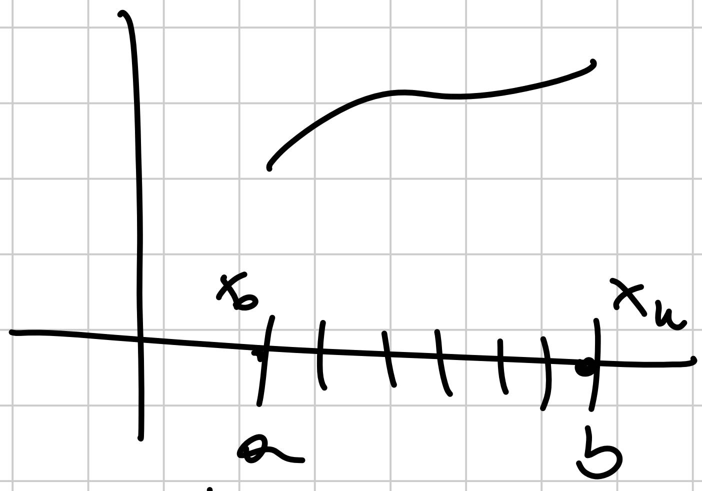
$(b-a)(M)$

$(b-a) \cdot m$ = Area rettangolino


a
x2
x1
b


a
x2
x1
b

Def. $[a,b]$ intervallo limitato



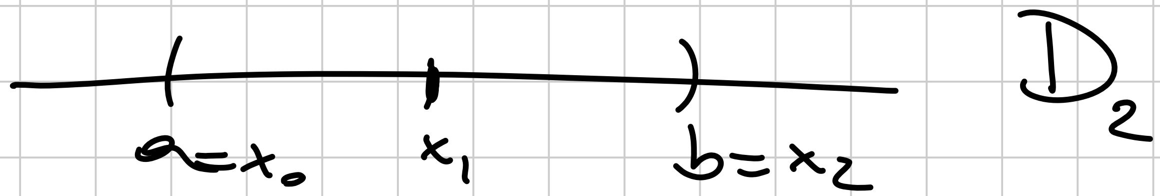
D una suddivisione se

$D = \{ x_i , \ i = 0, 1, \dots n \}$

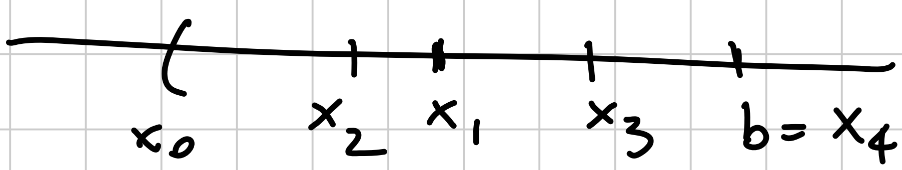
$a = x_0 < x_1 < x_2 < \dots < x_n = b$

$[a,b]$ è diviso in n intervallini

se ad una suddivisione D_2 aggiungo un punto, trovo un'altra suddivisione D_1. D_1 è più fine di D_2



è più fine di D_2

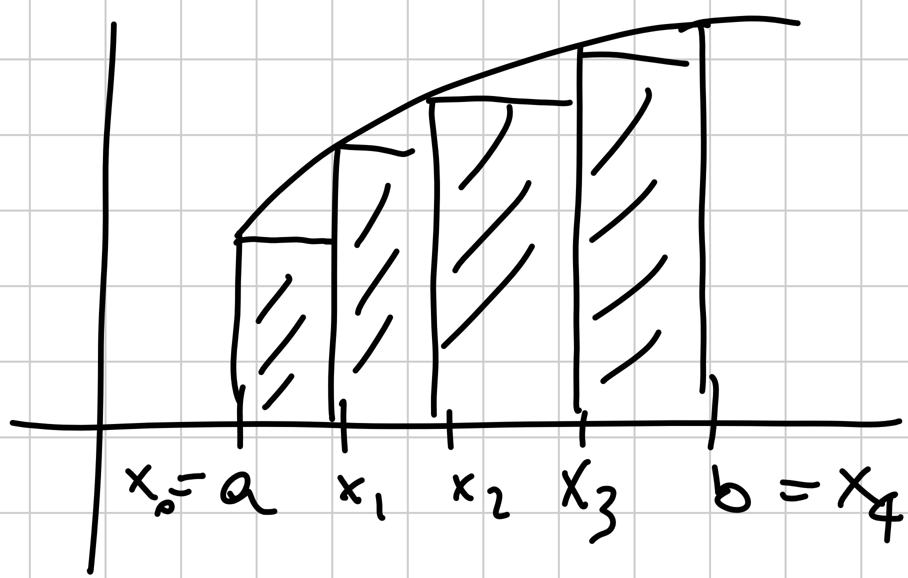


$f : [a, b] \to \mathbb{R}$ limitata

D suddivisione di $[a, b]$

$$m_i = \inf_{(x_{i-1}, x_i)} f$$

$$M_i = \sup_{(x_{i-1}, x_i)} f$$



$$\sum_{i=1}^{n} m_i(x_i - x_{i-1}) = s(D)$$

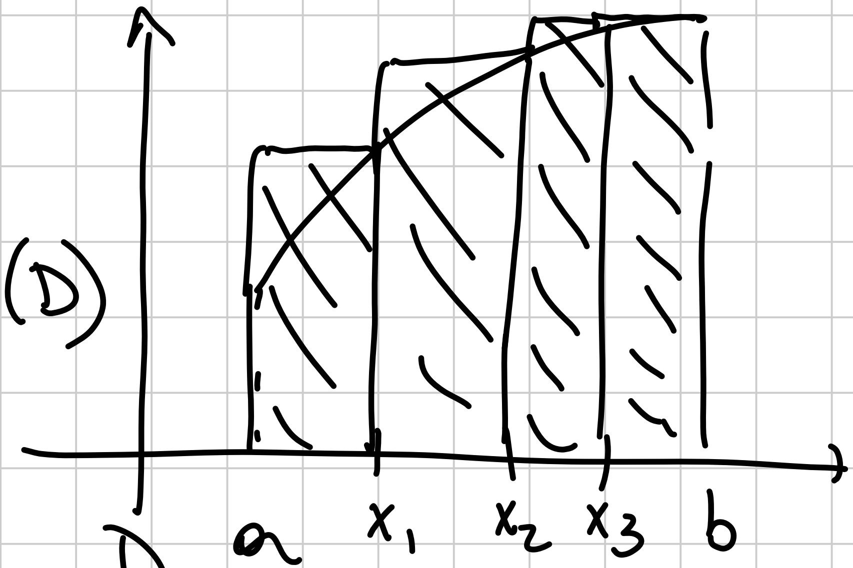
Somma inferiore rispetto alla suddivisione D

si può dim. che se D_1 è più fine di D_2

$$s(D_1) \geq s(D_2)$$

$$\sum_{i=1}^{n} M_i(x_i - x_{i-1}) = S(D)$$

Somma superiore rispetto a D



se D_1 è più fine di D_2

$$S(D_1) \leq S(D_2)$$

$s(D)$ al variare di D somme inferiori

$S(D)$ " " " somme superiori

$\sup_D s(D)$ è un numero finito

$\inf_D S(D)$ " " "

__Def.__ $f : [a,b] \to \mathbb{R}$ è __integrabile__ secondo Riemann in $[a,b]$ se

$$\sup_D s(D) = \inf_D S(D)$$

e questo __numero__ è __l'integrale__ di f in $[a,b]$ che si indica con

$$\int_a^b f(x)\,dx$$

f = funzione integranda

$[a,b]$ dominio di integrazione

x = variabile muta

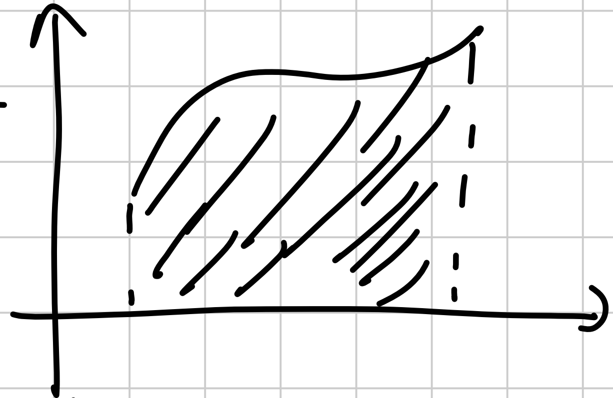
$$\int_a^b f(y)\,dy$$

$$\sum_k a_k \qquad \sum_n a_n$$

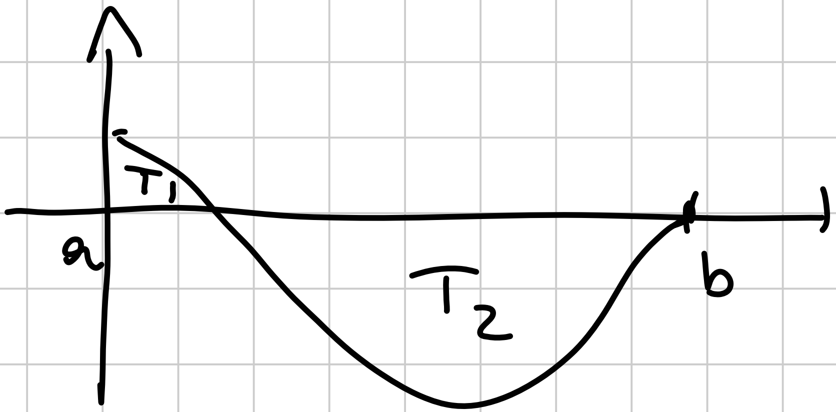
$$\int_a^b f(x)\,dx = \sup_D s = \inf_D S$$

$f \geq 0$, f continua

geometricamente

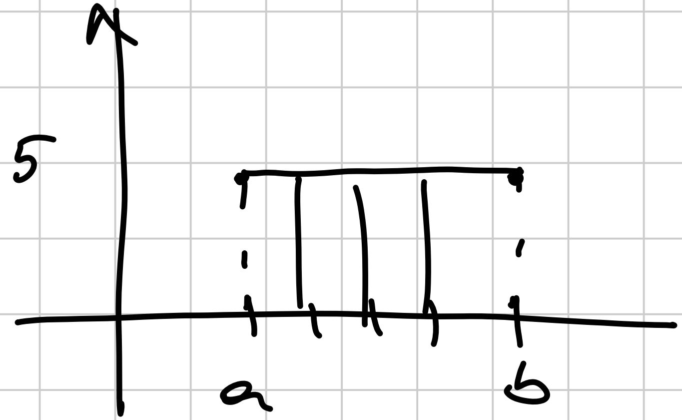


$$\int_a^b f(x)\,dx = \text{Area tra il grafico e l'asse delle } x$$



$$\int_a^b f(x)\,dx = \text{Area } T_1 - \text{Area } T_2$$

Es. $f(x) = 5$ in $[a, b]$



$$\int_a^b 5\,dx \overset{?}{=} (b-a) \cdot 5$$

$$s(D) = \sum_{i=1}^{n} 5\,(x_i - x_{i-1}) = 5\,(b-a)$$

$m_i = 5$

$\sup s(D) = 5(b-a)$

$$S(D) = \sum 5 \cdot (x_i - x_{i-1}) = 5\,(b-a)$$

$M_i = 5$

$\inf S(D) = 5(b-a)$

$f(x) = 5$ è integrabile e

$$\int_a^b 5 \cdot dx = 5(b-a)$$

$f(x) = C$ è integrabile $\displaystyle\int_a^b C\,dx = C(b-a).$