2010

Funzioni in più variabili

Es. $f(x) = x^2$ $\qquad f : \mathbb{R} \to \mathbb{R}$

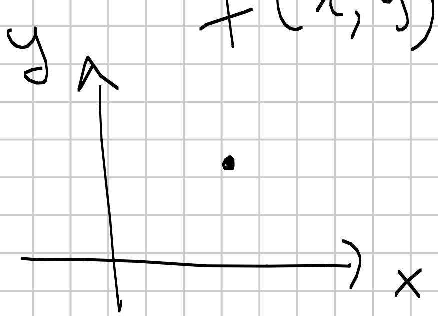
$x \longrightarrow x^2$

$f(x,y) = x^2 + y^2$ $\qquad (x,y) \longrightarrow x^2 + y^2 \in \mathbb{R}$

SCALARE

$\mathbb{R}^2 \to \mathbb{R}$

funzioni di due variabili a valori SCALARI




$f(x,y) = x^2 + y^2$

$$f(1,3) = 1^2 + 3^2 = 10$$

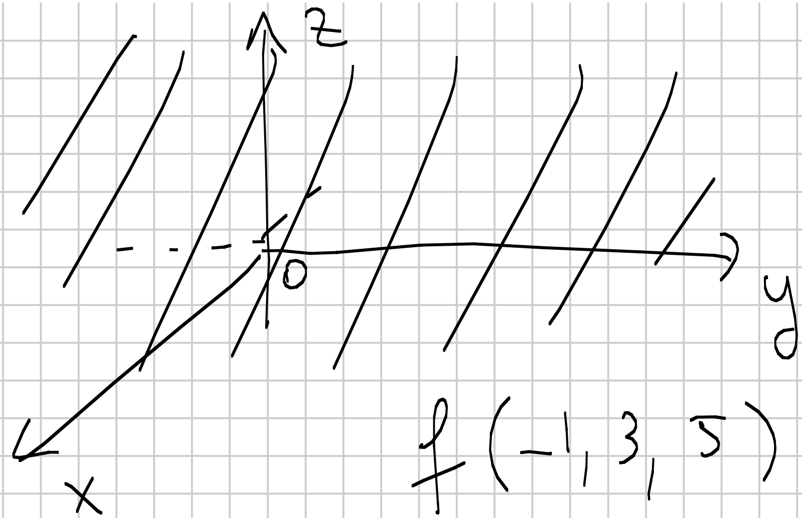
$$(1,3) \xrightarrow[f]{} 10 \qquad f: \mathbb{R}^2 \to \mathbb{R}$$

<u>es.</u> $f(x,y,z) = \operatorname{sen}(x+y) + \log z + \dfrac{x}{z}$

$$(x,y,z) \longrightarrow \operatorname{sen}(x+y) + \log z + \frac{x}{z} \in \mathbb{R}$$

$f: \mathbb{R}^3 \to \mathbb{R}$ Dominio di $f = \{ z > 0 \}$

$D = \{ (x,y,z) \in \mathbb{R}^3, \ z > 0 \} \subseteq \mathbb{R}^3$ dominio di $f$




tutti i punti sopra il piano xy

$(-1, 3, 5) \in D$

$$f(-1,3,5) = \sin(-1+3) + \log 5 + \left(\frac{-1}{5}\right)$$
$$= \sin(2) + \log 5 - \frac{1}{5} \in \mathbb{R}$$

$$f : D \subseteq \mathbb{R}^3 \longrightarrow \mathbb{R}$$

In generale consideriamo $f : \underline{\underline{\mathbb{R}^n}} \longrightarrow \mathbb{R}$

funzioni di $n$ variabili reali a valori <u>SCALARI</u>

n-pla $(x_1, x_2, x_3, \dots, x_n)$

$(x_1, x_2)$ $(x, y)$ $\mathbb{R}^2$

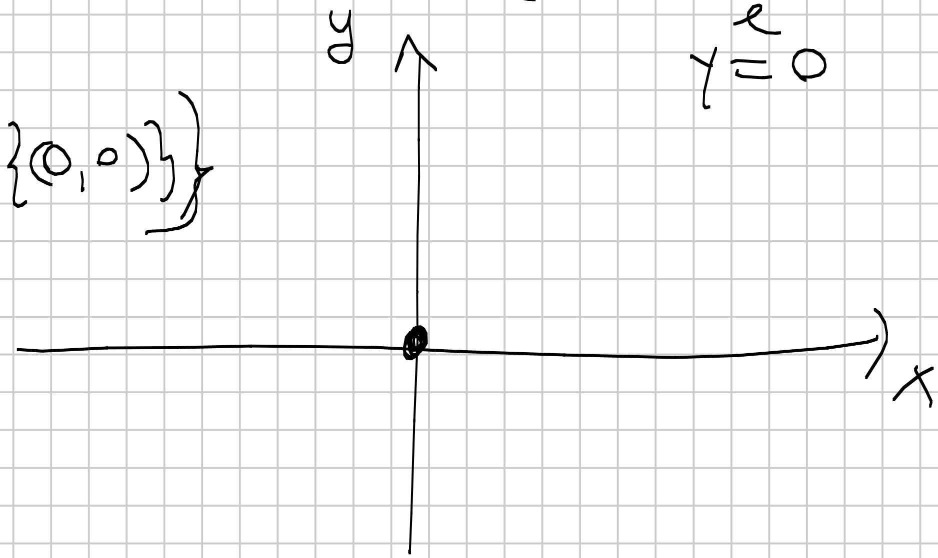
$(x_1, x_2, x_3)$ $(x, y, z)$ $\mathbb{R}^3$

$f : X \to \mathbb{R}$, $X \subseteq \mathbb{R}^n$ $X$ dominio della funzione

<u>Es.</u> $f(x,y) = \dfrac{x}{x^2+y^2}$ $\quad f : \mathbb{R}^2 \to \mathbb{R}$

$x^2+y^2 \neq 0$ $\quad x^2+y^2=0 \iff x=0$ e $y=0$ $\quad (0,0)$

$D = \{ (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\} \}$



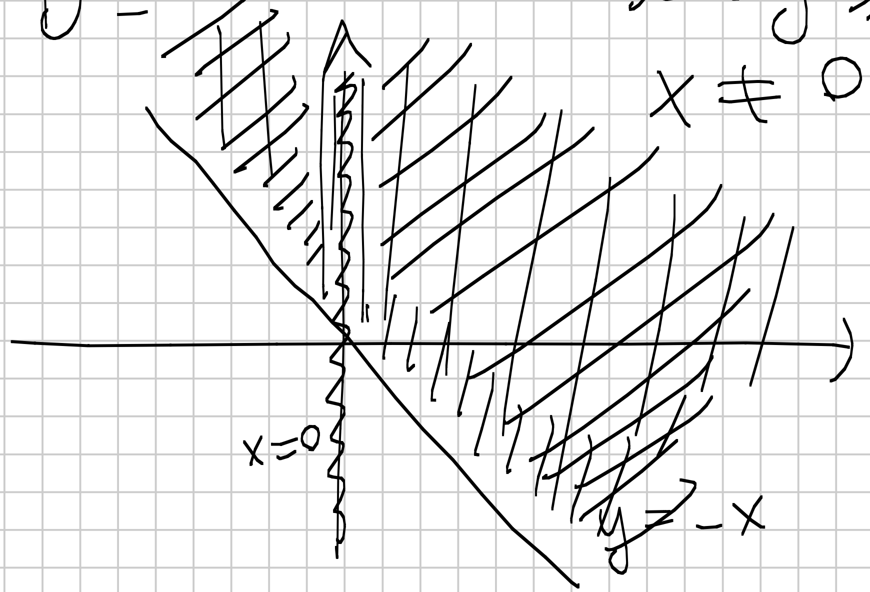
es. $f(x,y) = \dfrac{\sqrt{x+y}}{x}$ $\qquad (x,y) \to \mathbb{R}$

funzione di $\underline{\underline{2}}$ variabili

D – $\qquad x+y \geq 0$

$x \neq 0$

$$\begin{cases} y \geq -x \\ x \neq 0 \end{cases}$$



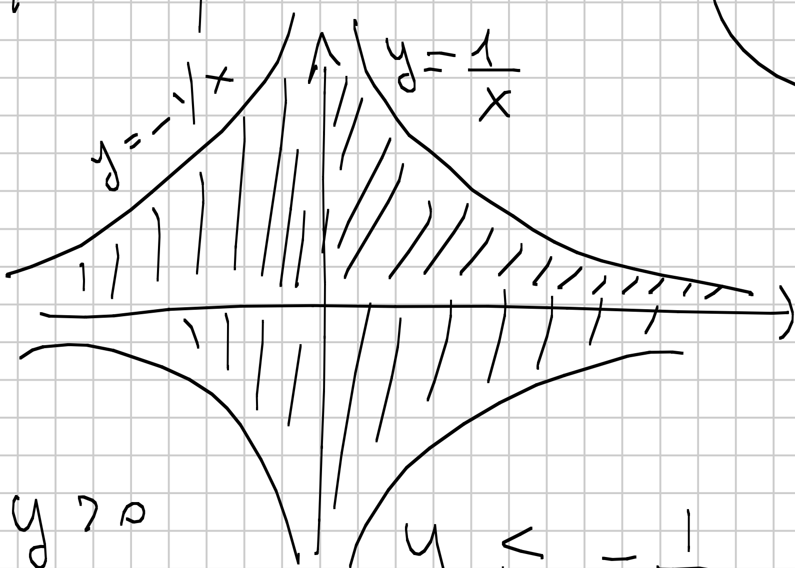

$y = -x$

$x = 0$

es. $f(x,y) = \arcsin(xy)$

$|xy| \leq 1$

$|y| \leq \frac{1}{|x|}$ se $x \neq 0$




$x = 0$ o $y = 0$

$x, y > 0$

$y \leq \frac{1}{x}$ $\quad$ $y = \frac{1}{x}$

$y > 0$
$x < 0$

$y \leq -\frac{1}{x}$

fate gli altri casi.

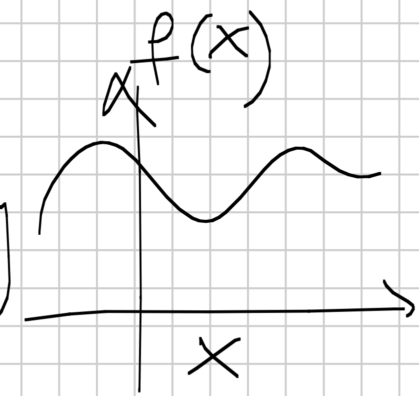
$f : X \subseteq \mathbb{R}^2 \to \mathbb{R}$ funzione di due variabili a valori scalari

$(x,y) \longrightarrow f(x,y)$

$\mathrm{graf}\, f = \{ (x, y, f(x,y)) \in \mathbb{R}^3 , (x,y) \in X \}$

$n=1$ $\quad f : \mathbb{R} \to \mathbb{R}$

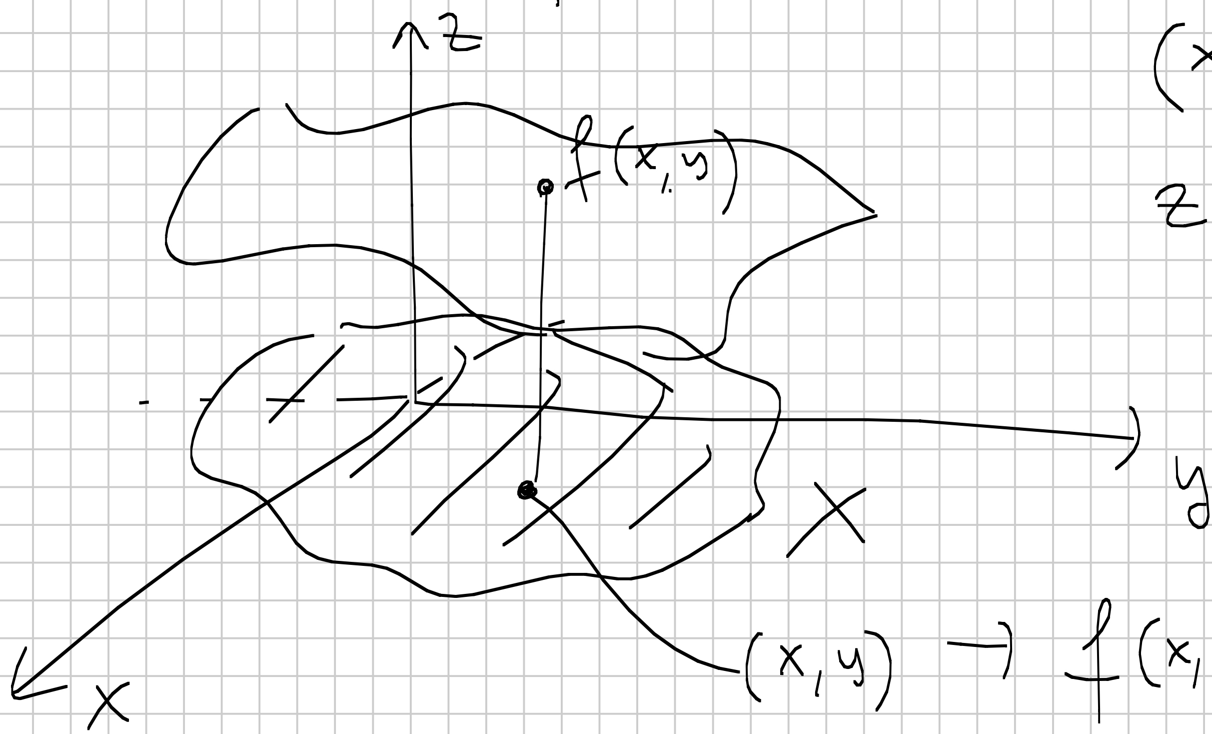
$\mathrm{graf}\, f = \{ (x, f(x)) \in \mathbb{R}^2 , x \in X \}$



nel caso di funzioni di due variabili

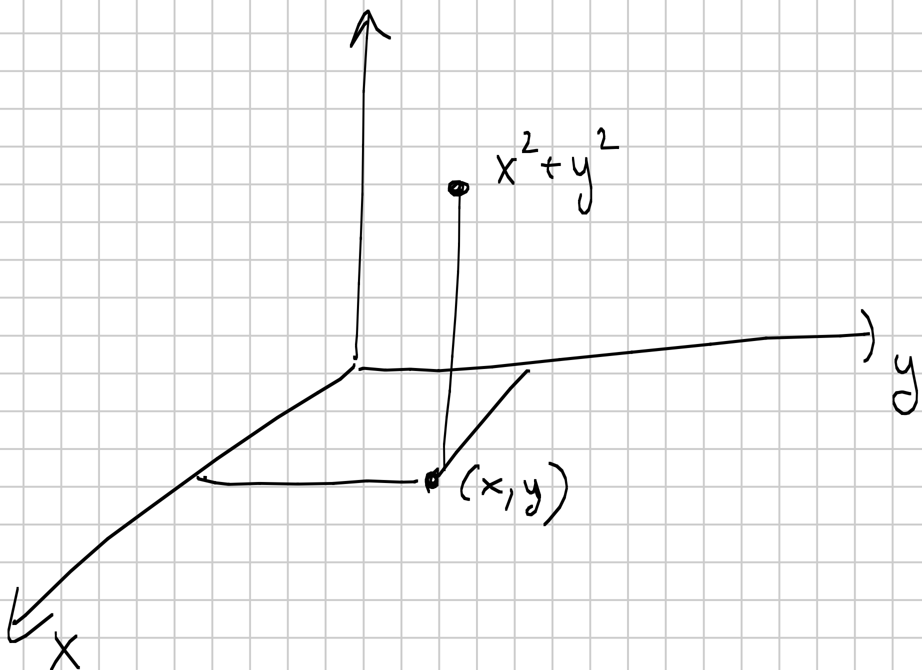
$(x, y) \in X$

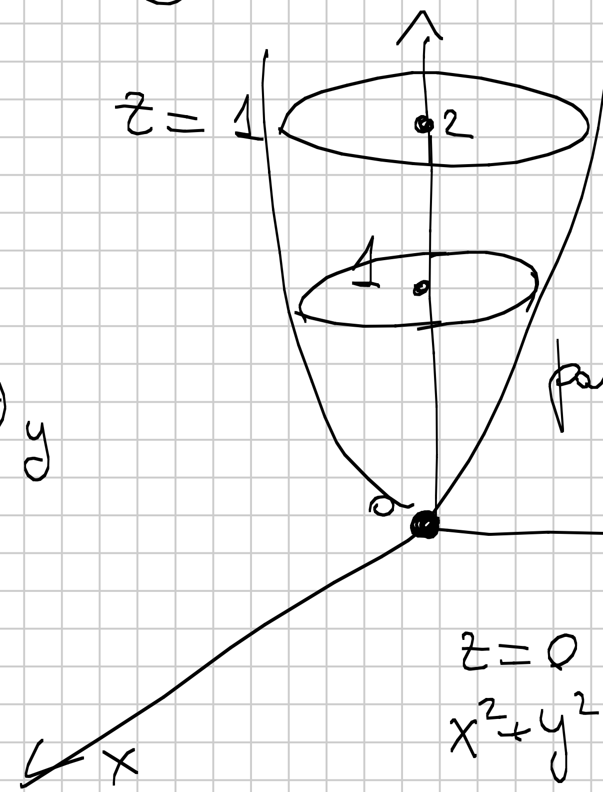
$z = f(x, y)$




$(x, y) \to f(x, y) = z$

Es. $f(x,y) = x^2 + y^2 = z$ Dominio $= \mathbb{R}^2$ ✓




$z = f(x,y)$

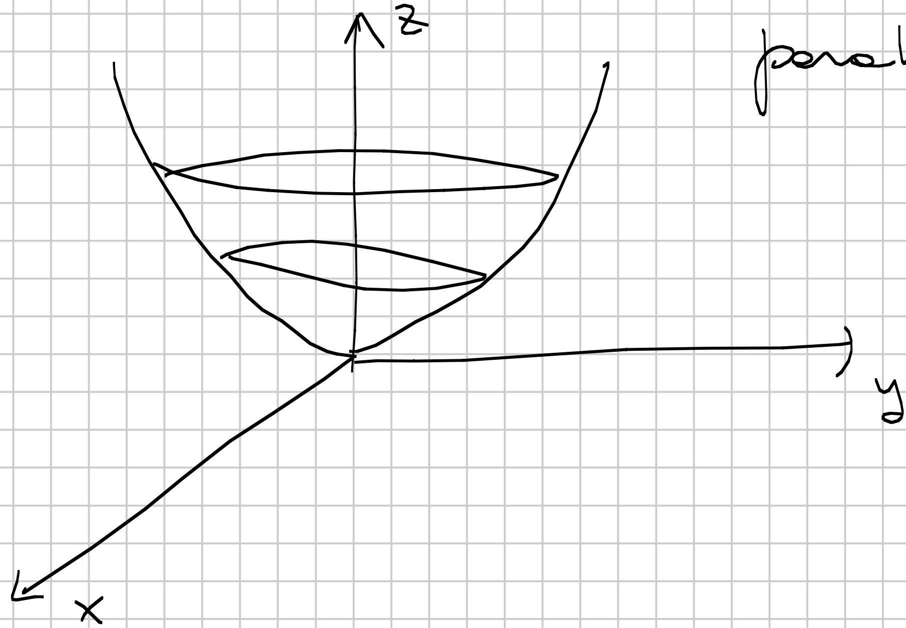
$x^2+y^2=1$

$x^2+y^2=2$

paraboloide

$z=0$

$x^2+y^2=0$




paraboloide

$$z = x^2 + y^2$$

$$f(x,y) = x^2 + y^2$$

$y = 0$

$$z = x^2$$

parabola

Altra classe: funzioni di più variabili a valori vettoriali.

es. $f(x,y) = \left(x+y,\ x^2+\frac{1}{y},\ \log xy\right)$

$(1,5) = (x,y)$

$f(1,5) = \left(1+5,\ 1^2+\frac{1}{5},\ \log(1\cdot 5)\right) = \left(6,\ \frac{6}{5},\ \log 5\right)$

$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3 \longrightarrow$ a valori VETTORIALI __es.__ $f(x) = \left( \operatorname{sen} x,\ x,\ \log \frac{1}{x} \right)$ $\quad \operatorname{Dom} f = \{x > 0\}$

$x = 1 \qquad f(1) = \left( \operatorname{sen} 1,\ 1,\ \log 1 \right)$

$f : \mathbb{R} \to \mathbb{R}^3$

In <u>generale</u> $f : \mathbb{R}^n \to \mathbb{R}^m$

es. $n = 2$, $m = 3$

es. $n = 1$, $m = 3$

funzioni di $n$ variabili a valori vettoriali

$F(P) \qquad P = (x_1, x_2, x_3)$

$F(x_1, x_2, x_3)$ $\qquad$ $F(x_1, x_2, x_3, t)$

LIMITI DI FUNZIONI IN PIÚ VARIABILI

**Def.** $\lim_{x \to x_0} f(x) = \ell$ $\qquad f : \mathbb{R} \to \mathbb{R}$

$\iff \forall\, V$ intorno di $\ell$ $\exists$

$x_0 \in \mathbb{R}$

$U_{x_0} = (x_0 - \delta,\ x_0 + \delta)$ $\qquad U$ intorno di $x_0$ t.c.

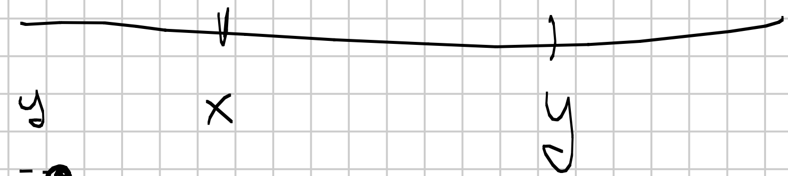
$f(x) \in V,\ \forall x \in U$.


$U = \{|y - x_0| < \delta\}$

Dobbiamo definire gli intorni in $\mathbb{R}^n$

distanza di due p.ti in $\mathbb{R}$ $\qquad x, y \in \mathbb{R}$

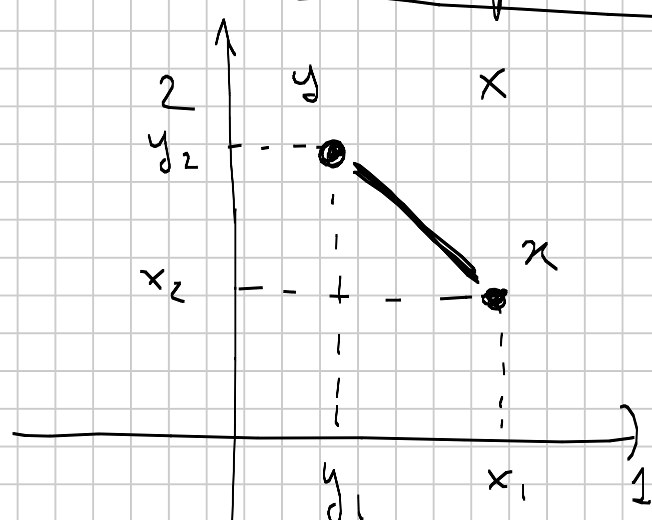
$d(x,y) = |x - y|$




<u>In generale</u> $\mathbb{R}^2$

$x = (x_1, x_2) \in \mathbb{R}^2$

$y = (y_1, y_2) \in \mathbb{R}^2$




$$d(x, y) = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}$$

Distanza tra due p.ti in $\mathbb{R}^2$

es.

$x = (-1, 3)$

$y = (2, 1)$

$$d(x,y) = \sqrt{(2+1)^2 + (-2)^2} = \sqrt{9+4} = \sqrt{13}$$

prodotto scalare tra $x$ e $y$ : $\langle x, y \rangle$ $\quad (x \cdot y)$

$$\langle x, y \rangle \doteq x_1 y_1 + x_2 y_2$$

$x = (x_1, x_2)$

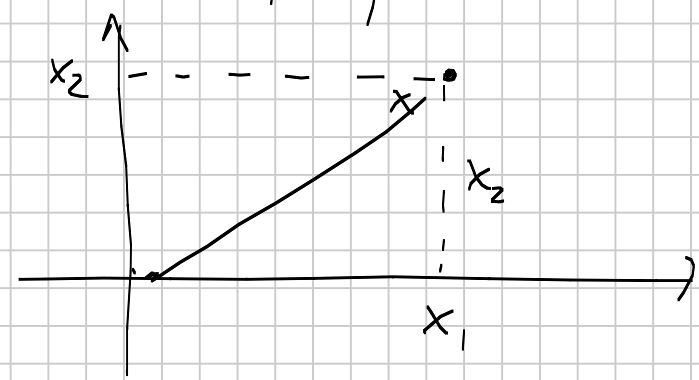
$y = (y_1, y_2)$

$$\langle (-1,3), (2,1) \rangle = (-1) \cdot 2 + 3 \cdot 1 = 1$$

**NORMA** di un elemento $x = (x_1, x_2) \in \mathbb{R}^2$

$$\|x\| = \sqrt{x_1^2 + x_2^2}$$

distanza del p.to $x$ dall'origine degli assi




Es. $x = (3, -8)$ $\|x\| = \sqrt{9 + 64} = \sqrt{73}$

In generale in $\mathbb{R}^n$

$$x = (x_1, x_2, \dots, x_n)$$
$$y = (y_1, y_2, \dots, y_n)$$

<u>Definizioni</u>

$$\langle x, y \rangle := x_1 y_1 + x_2 y_2 + \dots + x_n y_n$$ prodotto scalare tra $x$ e $y$

$$\| x \| := \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$$ NORMA di $x$

$$d(x, y) := \| x - y \| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2^2) + \dots + (x_n - y_n)^2}$$

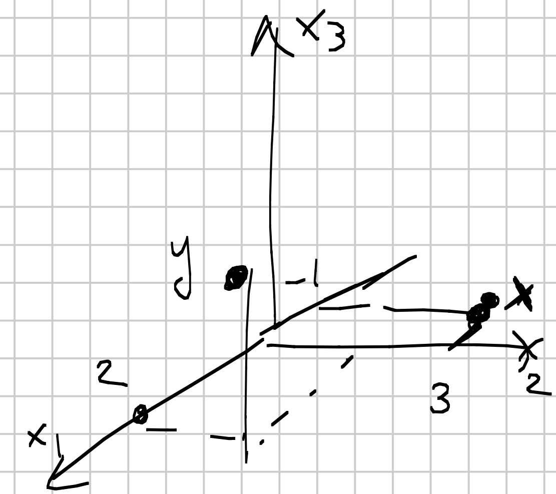
distanza tra $x$ e $y$

**Def.** Se $\langle x, y \rangle = 0$ si dice che $x$ e $y$ sono **ortogonali**

**Es.** in $\mathbb{R}^3$

$x = (-1, 3, 0)$

$y = (2, 1, 1)$




$$\langle x, y \rangle = -2 + 3 + 0 = 1$$

$$\|x\| = \sqrt{1 + 9 + 0} = \sqrt{10}$$

$$d(x,y) = \| x - y \| = \sqrt{(3)^2 + (1-3)^2 + (1-0)^2} =$$

$$= \sqrt{9 + 4 + 1} = \sqrt{14}$$

Definizione di <u>INTORNO</u> in $\mathbb{R}^2$

$x \in \mathbb{R}^2 \qquad x = (x_1, x_2)$

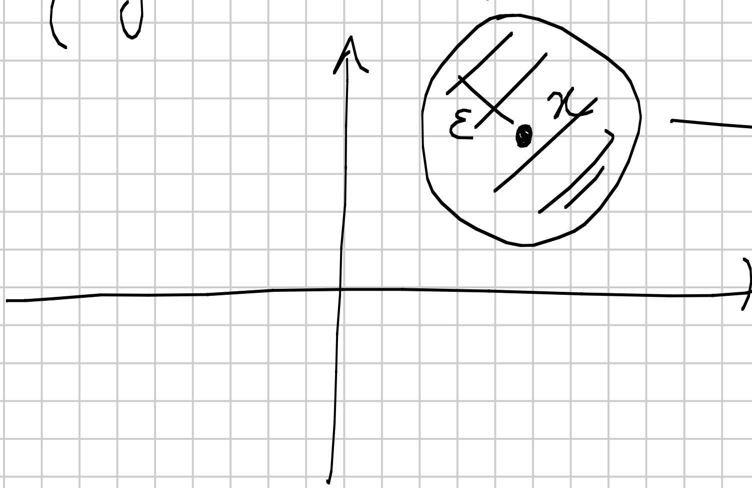
$$B_\varepsilon(x) = \{ y \in \mathbb{R}^2 : d(x,y) < \varepsilon \} =$$

$$= \{ y \in \mathbb{R}^2 : \|x - y\| < \varepsilon \}$$

Intorno sferico di centro $x$ e raggio $\varepsilon$

$\downarrow$

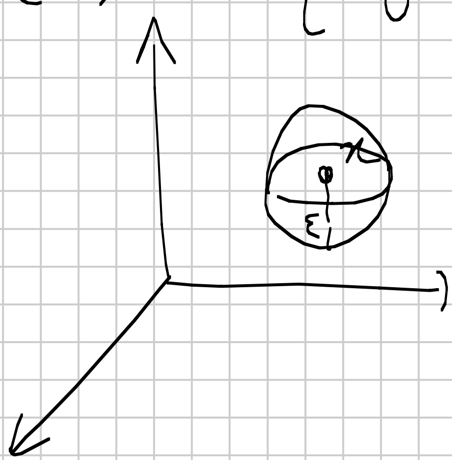
cerchio di raggio $\varepsilon$ e centro il p.to $x$.




$\longrightarrow B_\varepsilon(x)$

in $\mathbb{R}^3$ $\qquad x = (x_1, x_2, x_3)$

$$B_\varepsilon(x) = \{ y \in \mathbb{R}^3 : d(x, y) < \varepsilon \}$$




intorno sferico
è una SFERA
di raggio $\varepsilon$
e centro $x$

in $\mathbb{R}^n$

$$B_\varepsilon(x) = \{ y \in \mathbb{R}^n : d(x,y) < \varepsilon \} =$$

$$= \{ y \in \mathbb{R}^n : \|x - y\| < \varepsilon \}$$

distanza di y da x

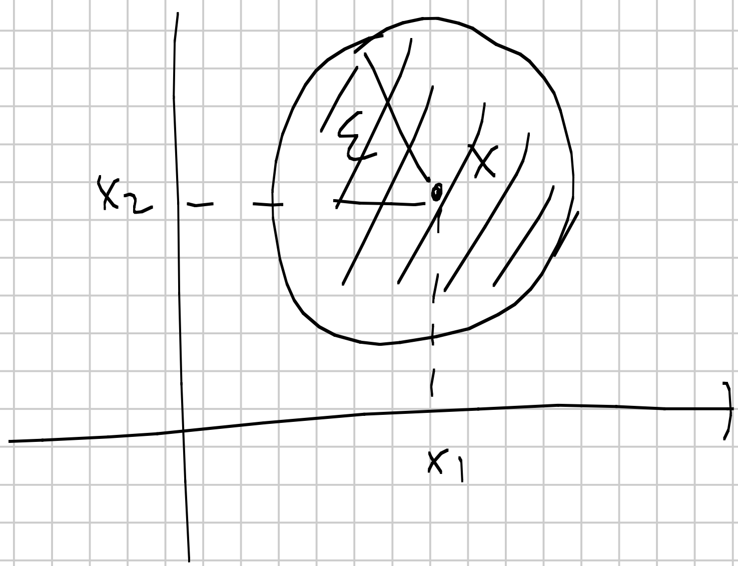
Intorno sferico in $\mathbb{R}^n$ di centro x e raggio $\varepsilon$

in $\mathbb{R}^2$

$$B_\varepsilon(x) = \{ y \in \mathbb{R}^2 : \|y - x\| < \varepsilon \}$$

$x = (x_1, x_2)$

$y = (y_1, y_2)$




$$\sqrt{(x_1-y_1)^2+(x_2-y_2)^2} < \varepsilon$$

$$(x_1-y_1)^2 + (x_2-y_2)^2 < \varepsilon^2$$

=

eq. del cerchio.

Quindi abbiamo dato la def. di intorno di un p.to $x$ in $\mathbb{R}^n$

tutte le def. date in $\mathbb{R}$ per la Topologia valgono in $\mathbb{R}^n$.

<u>es.</u> $x$ p.to <u>interno</u> di un insieme $E \subseteq \mathbb{R}^n$ se $\exists$ un intorno $U$ $(B_\varepsilon(x))$ tutto contenuto in $E$.

<u>es.</u> $x$ p.to di <u>accumulazione</u> per $E \subseteq \mathbb{R}^n$ $\Leftrightarrow$ $\forall\, U$ intorno di $x$ esistono punti $\neq x$

che appartengono a $E \cap U$.

Def. di <u>limite</u> per funzioni <u>in più</u> variabili

$f : X \subseteq \mathbb{R}^n \to \mathbb{R}$, $x_0 \in \mathbb{R}^n$ di accumulazione per $X$

$$\lim_{x \to x_0} f(x) = \ell \quad (f(x) \to \ell \text{ se } x \to x_0)$$

$\Longleftrightarrow$ $\forall\, V$ intorno di $\ell$ $\exists\, U$ intorno di $x_0$

t.c. $f(x) \in V$ se $x \in U \cap X$.

Esplicitamente:

Caso $\ell \in \mathbb{R}$ $\qquad V = (\ell-\varepsilon, \ell+\varepsilon)$

$x_0 \in \mathbb{R}^n$ $\qquad U = B_\delta(x_0) = \{x : \|x - x_0\| < \delta\}$

$$\boxed{\lim_{x \to x_0} f(x) = \ell \iff \forall \varepsilon > 0 \ \exists \delta : |f(x) - \ell| < \varepsilon \quad \forall x : \|x - x_0\| < \delta.}$$

Def. $f$ continua in $x_0 \in X$ se $\displaystyle\lim_{x \to x_0} f(x) = f(x_0)$

es. $f(x, y, z) = x + y + \operatorname{sen}(xy) + \sqrt{z - x}$

$f : X \subseteq \mathbb{R}^3 \longrightarrow \mathbb{R}$ $\qquad z \geq x$

Dove è definita è continua? Sì! come in una variabile.

## Esercizi

1) Determinare per quali $\alpha \in \mathbb{R}$ $\exists$ finito

2) calcolarlo per $\alpha = 1$

$$\int_0^{+\infty} \frac{x \arctan x}{x^\alpha (1+x)^{2\alpha}} \, dx =$$

$$= \int_0^1 \frac{x \arctan x}{x^\alpha (1+x)^{2\alpha}} \, dx + \int_1^{+\infty} \frac{x \arctan x}{x^\alpha (1+x)^{2\alpha}} \, dx$$

$$\int_0^1 \underbrace{\frac{x \arctg x}{x^{\alpha}(1+x)^{2\alpha}}}_{f(x)}\, dx$$

$f(x) > 0$

$x \to 0$

$$f(x) = \frac{x\,(x + o(x))}{x^{\alpha} \cdot 1} =$$

$$= \frac{x^2}{x^{\alpha}} + o\left(\frac{1}{x^{\alpha-2}}\right)$$

$$= \frac{1}{x^{\alpha-2}} + o\left(\frac{1}{x^{\alpha-2}}\right)$$

$$\int_0^1 \frac{1}{x^{\beta}}\, dx$$

converge sse $\beta < 1$

$\Updownarrow$

converge se $\alpha - 2 < 1$

$$\boxed{\alpha < 3}$$

Oss. se $\alpha < 0$ $f(x)$ è limitata tra $[0,1]$.

$$\int_1^{+\infty} \frac{x \operatorname{arctg} x}{x^\alpha (1+x)^{2\alpha}} \, dx$$

$x \to +\infty$

$$f(x) \sim \frac{x \frac{\pi}{2}}{x^\alpha x^{2\alpha}} = \frac{\pi}{2} \frac{1}{x^{3\alpha-1}}$$

$$\int_1^{+\infty} \frac{1}{x^\beta} \, dx \quad \text{conv.} \iff \forall \beta > 1$$

$\Longrightarrow$

converge $\iff 3\alpha - 1 > 1$

$\Longleftarrow 3\alpha > 2$

$\alpha > 2/3$

$\int_0^1 f(x)\,dx$ converge se $\alpha < 3$

$\int_1^{+\infty} f(x)\,dx$ converge se $\alpha > 2/3$

$\Rightarrow \int_0^{+\infty} f(x)\,dx$ converge $\forall \alpha \in \left(\frac{2}{3}, 3\right)$

Calcolarlo per $\alpha = 1$

$$\int_0^{+\infty} \frac{\cancel{x} \arctan x}{\cancel{x}(1+x)^2}\, dx = \int_0^{+\infty} \frac{\arctan x}{(1+x)^2} = \text{per parti}$$

derivare $\arctan x$ e integrare $\displaystyle\int \frac{1}{(1+x)^2}\, dx = \frac{-1}{(1+x)}$

$$\int \frac{\arctan x}{(1+x)^2}\, dx = \arctan x \left(\frac{-1}{1+x}\right) -$$

$$-\int \frac{-1}{(1+x)} \cdot \frac{1}{1+x^2}\, dx$$

$$\frac{1}{(1+x)} \cdot \frac{1}{1+x^2} = \frac{A}{1+x} + \frac{Bx+C}{1+x^2}$$

e si trova $A$, $B$, $C$ con questa proprietà