Comparison principles for subelliptic equations of Monge-Ampère type

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Monge-Ampère equations (M-A)

In $\Omega \subseteq R^n$ open and bounded

- classical M-A $\qquad \det(D^2 u) = f(x)$
- Optimal transportation $\qquad \det(D^2 u) = \dfrac{f(x)}{g(Du)}$
- Prescribed Gauss Curvature $\qquad \det(D^2 u) = k(x)(1 + |Du|^2)^{\frac{n+2}{2}}$

References: e.g., P.-L. LIONS, Manuscripta Math. (1983)
I.J. BAKEL'MAN, book (1994),
C. GUTIERREZ, book (2001),
C. VILLANI, book (2003),
L. A. CAFFARELLI, Contemp. Math. (2004),
N.S. TRUDINGER, Intern. Congress Math., Eur. Math. Soc. (2006)

$$G(x, u, Du, D^2u) = -\det(D^2u) + H(x, u, Du) = 0$$

They are FULLY NONLINEAR DEGENERATE ELLIPTIC equations in the sense that $\forall\, X, Y \geq 0$, symmetric matrices

$$\det(X) \geq \det(Y), \quad \forall\, X - Y \geq 0.$$

So the Monge-Ampère equations are degenerate elliptic over CONVEX solutions.

VISCOSITY SOLUTIONS are a good notion for these equations if $H(x, r, p)$ is nondecreasing in r.

Known result

H. ISHII - P.L. LIONS, J. Diff. Eqs. (1990)

$$-\det(D^2u) + H(x,u,Du) = 0, \text{ in } \Omega \subseteq R^n \text{ bounded.}$$

Theorem

$H \geq 0$, H nondecreasing in u, and for all $R > 0$ there is L_R such that

$$|H^{1/n}(x,r,p) - H^{1/n}(x,r,p_1)| \leq L_R|p - p_1|, \quad \forall\, |r|, |p|, |p_1| \leq R.$$

Then the comparison principle holds between convex subsolutions and supersolutions.

Idea: $Y \geq 0$, $n \times n$ symmetric matrix

$$\boxed{-(\det\, Y)^{1/n} = \sup\{-\mathrm{tr}(MY), M \geq 0, \det M = n^{-n}\}}$$

Remarks

$$-\det(D^2 u) + H(x, u, Du) = 0.$$

- The principal part does NOT depend on x.
- For H not strictly increasing in u they perturb subsolutions to strict subsolutions.
- in R^n $\boxed{u \textit{ convex} \rightarrow \textit{locally Lipschitz}}$: weak assumption on H is enough.

Fully nonlinear subelliptic equations

Given a family of smooth vector fields $X_1, ..., X_m$ define

intrinsic (horizontal) gradient $D_{\mathcal{X}} u := (X_1 u, ..., X_m u)$,

symmetrized (horizontal) Hessian $(D^2_{\mathcal{X}} u)_{ij} := \dfrac{X_i X_j u + X_j X_i u}{2}$.

$$F(x, u, D_{\mathcal{X}} u, D^2_{\mathcal{X}} u) = 0$$

Initiated by Bieske, Manfredi, and others ($\sim$ 2002).

Example: the Heisenberg operator

In R^3 write (x, y, t), and take

$$X_1 u = u_x + 2yu_t, \; X_2 u = u_y - 2xu_t$$

$D_{\mathcal{X}} u(x) = (X_1 u, X_2 u), \quad m = 2, n = 3.$

Take the coefficients of X_1 and X_2 $\qquad \sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2y & -2x \end{bmatrix}.$

Then $\qquad D_{\mathcal{X}} u(x) = \sigma^T Du, \qquad D^2_{\mathcal{X}} u = \sigma^T D^2 u \, \sigma$

$$F(x, u, \sigma^T Du, \sigma^T D^2 u \, \sigma) = 0$$

Applications of Heisenberg geometry: L. CAPOGNA, D. DANIELLI, S.D. PAULS, J.T. TYSON (2007)

An alternative approach

Define

$$X_j = \sigma^j \cdot \nabla, \qquad \sigma_{ij} = \sigma^j_i, \qquad \sigma \quad n \times m \text{ matrix.}$$

Then

- $D_{\mathcal{X}} u(x) = \sigma^T(x) Du$ and
- $D^2_{\mathcal{X}} u = \sigma^T(x) D^2 u\, \sigma(x) + Q(x, Du)$,

$$Q_{ij}(x,p) := \left[D\sigma^j\, \sigma^i + D\sigma^i\, \sigma^j \right](x) \cdot \tfrac{p}{2}$$

$$0 = F(x, u, \overbrace{\sigma(x)^T Du}^{D_{\mathcal{X}} u}, \overbrace{\sigma^T(x) D^2 u\, \sigma(x) + Q(x, Du)}^{D^2_{\mathcal{X}} u}) =: G(x, u, Du, D^2 u).$$

For

$$G(x,u,Du,D^2u) = F(x,u,\sigma^T(x)Du,\sigma^T(x)D^2u\,\sigma(x) + Q(x,Du)) = 0$$

can use standard viscosity theory if G is degenerate elliptic and strictly increasing in u.

Without strict monotonicity can prove COMPARISON PRINCIPLE if any subsolution can be perturbed to a STRICT subsolution.

see M. BARDI - P. MANNUCCI, On the Dirichlet problem for non-totally degenerate fully nonlinear elliptic equations, Commun. Pure Applied Anal. (2006).

Subelliptic Monge Ampère type equations

$$-\det(D^2_{\mathcal{X}}u) + H(x, u, D_{\mathcal{X}}u) = 0.$$

For $X_1, ..., X_m$ generators of the Heisenberg group

$$\boxed{-\det(\sigma^T(x)D^2u\,\sigma(x)) + H(x, u, \sigma^T(x)Du) = 0}$$

is a prototype fully nonlinear equation, see
J.J. MANFREDI, *Nonlinear Subelliptic Equations on Carnot Groups*, (2003), D. DANIELLI - N. GAROFALO - D.M. NHIEU, (2003), C.E. GUTIÈRREZ - A. MONTANARI, (2004).

Motivations

- D. DANIELLI - N. GAROFALO - D.M. NHIEU, (2003) propose a definition of HORIZONTAL Gauss curvature $k(x)$ in Carnot groups. The corresponding equation of prescribed curvature is

$$\det(D^2_{\mathcal{X}} u) = k(x)(1 + |D_{\mathcal{X}} u|^2)^{\frac{m+2}{2}}.$$

- Equations of the form

$$\text{"}\det(D^2_{\mathcal{X}} u) = \frac{f(x)}{g(D_{\mathcal{X}} u)}\text{"}$$

  are related to optimal transportation between Carnot groups or in sub-riemannian geometry:
  L. AMBROSIO-S. RIGOT (2004), A. FIGALLI-L. RIFFORD (2008)

- If $m = n$, Monge Ampere on vectorial fields (related with Riemannian geometry, T. Aubin, 1998)

Subelliptic Monge Ampère type equations

$$-\det(D^2_{\mathcal{X}}u) + H(x, u, D_{\mathcal{X}}u) = 0 \quad in\,\Omega$$

It is degenerate elliptic on $\boxed{\mathcal{X} - convex\ functions}$, i.e.

$$D^2_{\mathcal{X}}u \geq 0,$$

in the "viscosity" sense.

Some references on $\mathcal{X}$-convexity in Carnot groups
G. LU - J. MANFREDI - B. STROFFOLINI, (2004), D. DANIELLI - N. GAROFALO - D.M. NHIEU, (2003).
A survey of convexity in Carnot groups is in the book A. BONFIGLIOLI - E. LANCONELLI - F. UGUZZONI, (2007).

One of our main results

$$-\det(D^2_{\mathcal{X}}u) + H(x, u, D_{\mathcal{X}}u) = 0, \quad \text{in } \Omega \subseteq R^n \text{ bounded}$$

Theorem

$X_1, ..., X_m$ are the generators of a Carnot group on R^n.H nondecreasing in u. For all $R > 0$ there is L_R such that

$$|H^{1/m}(x, r, q) - H^{1/m}(x, r, q_1)| \leq L_R|q - q_1|, \ \forall \, |r|, |q|, |q_1| \leq R.$$

Let u $\mathcal{X}$-convex and subsolution, v supersolution. Then the comparison principle holds.

EXAMPLE

The assumptions of the comparison theorem cover the prescribed horizontal Gauss curvature equation in Carnot group

$$-\det(D^2_{\mathcal{X}}u) + k(x)(1 + |D_{\mathcal{X}}u|^2)^{(m+2)/2} = 0, \quad \text{in } \Omega,$$

for $k(x) > 0$.

In particular, we obtain the uniqueness of a viscosity solution of the PDE with prescribed boundary data.

New difficulties

- 1. The principal part of the operator

$$F(x,p,X) := -\det(\sigma^T(x) X \sigma(x) + Q(x,p))$$

depends on x and does not satisfy in general the standard structure conditions in viscosity theory.

- 2.

$$F(x,p,Y) := -\log\det\left(\sigma^T(x) Y \sigma(x) + Q(x,p)\right)$$

satisfies the structure conditions if

$$\sigma^T(x) Y \sigma(x) + Q(x,p) \geq \gamma I, \quad \gamma > 0.$$

We have to use uniformly $\mathcal{X}$-convex functions: for some $\gamma > 0$

$$D^2_{\mathcal{X}} u = \sigma^T(x) D^2 u \, \sigma(x) + Q(x, Du) \geq \gamma I,$$

in the "viscosity" sense.

$$-\det(D^2_{\mathcal{X}}u) + H(x, u, D_{\mathcal{X}}u) = 0.$$

- 1. H STRICTLY increasing in $u \to$ OK comparison principle.
- 2. H not decreasing in u (which is the most frequent in applications), we perturb a subsolution u to a STRICT subsolution.
- 3. u be $\mathcal{X}$-convex in Ω does this imply

$$|\sigma^T(x)\,Du| \le C \ \text{ in } \ \Omega_1 \subseteq \Omega\ ?$$

It is true in the Carnot groups: G. LU - J. MANFREDI - B. STROFFOLINI (2004), D. DANIELLI - N. GAROFALO - D.M. NHIEU (2003), V. MAGNANI (2006), M. RICKLY (2006), P. JUUTINEN - G. LU - J. MANFREDI - B. STROFFOLINI (2007).
If 3. holds then it is possible to construct a STRICT subsolution perturbing a subsolution without extra assumptions on H.

Comparison for general vector fields

$$-\det(D^2_{\mathcal{X}}u) + H(x, u, D_{\mathcal{X}}u) = 0, \quad \text{in } \Omega \subseteq R^n \text{ bounded}$$

Theorem

$H \in C(\overline{\Omega} \times R \times R^m)$, nondecreasing r; $H^{1/m}$ Lipschitz in q uniformly in x, r, $0 < C_o \leq H \leq C_1$, H satisfies the structure condition, $|x|^2$ uniformly $\mathcal{X}$-convex in Ω, i.e.,

$$\sigma^T(x)\sigma(x) + Q(x, x) \geq \eta\, I, \ \forall x \in \overline{\Omega}, \ \textit{for some } \eta > 0.$$

Then the comparison principle holds between $\mathcal{X}$-convex subsolutions and v supersolutions.

A model example of well-posedness

$$\begin{cases} -\det(D^2_{\mathcal{X}} u) + |D_{\mathcal{X}} u|^m = f(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

with $f \leq 0$, as P.L. Lions (ARMA,1985) in the Euclidean case;

$\Omega = \{\Phi(x) > 0\}$ uniformly $\mathcal{X}$-convex:

$$-D^2_{\mathcal{X}} \Phi(x) \geq \gamma I, \ \gamma > 0.$$

as Trudinger (2006) in the Euclidean case.

Theorem

$X_1, ..., X_m$ are the generators of a Carnot group on R^n.
Ω smooth and uniformly $\mathcal{X}$-convex.
Then there exists a unique solution of the Dirichlet problem, continuous in $\overline{\Omega}$.

Remark
The same result holds also if the PDE is replaced by

$$-\det(D^2_{\mathcal{X}} u) = f(x).$$