

1 Differential manifolds

In this section we present the notion of differential manifold from three different points of view, and show how to pass from one to the others.

1.1 Homeomorphisms and diffeomorphisms

Let $D \subset \mathbb{R}^m$ and $E \subset \mathbb{R}^n$ (any subsets of affine spaces of possibly different dimensions), and let $f : D \rightarrow E$ be a function. We say that f is a *homeomorphism* between D and E if f is bijective and bicontinuous (i.e. both f and f^{-1} are continuous). If there exists a homeomorphism between D and E , we say that D and E are *homeomorphic*: the meaning is that “even if D and E look different, in fact they are just a gentle —without breaks— deformation of each other and they have the same topological properties”: e.g. (simple) connectedness, and so on.

Homeomorphism

Examples. Let $D = [0, 1[\subset \mathbb{R}^1$, $E_1 = \{(x, y) \in \mathbb{R}^2 : 0 \leq x < 1, y = x^2\} \subset \mathbb{R}^2$ (a piece of parabola), $E_2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ (the unit circle) and $E_3 = \{(x, y) \in \mathbb{R}^2 : xy = 0, |x| \leq 1, |y| \leq 1\} \subset \mathbb{R}^2$ (a cross). Then $f_1 : D \rightarrow E_1$ given by $f_1(t) = (t, t^2)$ is a homeomorphism (the inverse is $f_1^{-1} : E_1 \rightarrow D$ given by the first projection $f_1^{-1}(x, x^2) = x$). On the other hand, $f_2 : D \rightarrow E_2$ given by $f_2(t) = (\cos 2\pi t, \sin 2\pi t)$ is not a homeomorphism: namely f_2 is bijective and continuous, but f_2^{-1} is not continuous (why?). • By what has been said, we have proved that D and E_1 are homeomorphic. But what about D and E_2 , or D and E_3 ? Couldn't there exist a homeomorphism between them, possibly hard to find? The answer is no. Namely, assume by absurd that $g : D \rightarrow E_2$ would be a homeomorphism, and let $P = g(\frac{1}{2})$: then g should induce a homeomorphism also between $D' = D \setminus \{\frac{1}{2}\}$ and $E'_2 = E_2 \setminus \{P\}$, but this is impossible since E'_2 is connected while D' is not. A similar argument shows that D and E_3 cannot be homeomorphic. • The whole $D = \mathbb{R}^n$ and the open unit ball $E = \{x \in \mathbb{R}^n : \|x\| < 1\}$ are homeomorphic: a homeomorphism is $f : D \rightarrow E$ given by $f(x) = \frac{1}{\|x\|+1}x$ (the inverse is $f^{-1} : E \rightarrow D$ given by $f^{-1}(y) = \frac{1}{1-\|y\|}y$). • Can $D = \mathbb{R}^2$ and $E = \mathbb{R}^3$ be homeomorphic? The answer is no. Namely, assume by absurd that $g : D \rightarrow E$ would be a homeomorphism, and let $P = g(0)$: then g should induce a homeomorphism also between $D' = D \setminus \{0\}$ and $E' = E \setminus \{P\}$, but this is impossible since E' is simply connected while D' is not. Similar arguments show that $\mathbb{R}^p$ and $\mathbb{R}^q$ cannot be homeomorphic unless $p = q$.

Now let us turn to differential calculus. Given a function $f : D \rightarrow E$, we know the notion of “being of class $\mathcal{C}^k$ ” when D is an open subset of $\mathbb{R}^m$. If D and E are open subsets of a same affine space $\mathbb{R}^n$, f is said to be a *diffeomorphism* of class $\mathcal{C}^k$ ⁽¹⁾ if it is

Diffeomorphism

⁽¹⁾if the class is unspecified, it is usually intended of class $\mathcal{C}^1$; for $k = 0$ one recovers the notion of homeomorphism.

a homeomorphism and both f and f^{-1} are of class $\mathcal{C}^k$. In this case we say that D and E are *diffeomorphic*: the meaning, going beyond the homeomorphism, is that “ D and E are just a very gentle and smooth —without neither breaks nor angles— deformation of each other and they have the same differential properties”.

An interesting weaker notion is the following one: if D and E are open subsets of $\mathbb{R}^n$ and $x_0 \in D$, f is said to be a *local diffeomorphism at x_0* if there exists an open neighborhood $U \subset D$ of x_0 such that $V = f(U)$ is an open neighborhood of $f(x_0)$ and $f|_U : U \rightarrow V$ is a diffeomorphism. Of course, if $f : D \rightarrow E$ is a diffeomorphism then it is also a local diffeomorphism at any point of D . To recover at which point f is a local diffeomorphism there is the following fundamental result, which is an easy consequence of Dini's Theorem for systems:

Local
diffeomorphism

Theorem 1.1.1. (Inverse Function Theorem) *f is a local diffeomorphism at x_0 if and only if $\det f'(x_0) \neq 0$.*

Proof. If f is a local diffeomorphism at x_0 , there exist an open neighborhood $U \subset \mathbb{R}^n$ of x_0 and an open neighborhood V of $y_0 = f(x_0)$ with $V = f(U)$ such that $f|_U : U \rightarrow V$ is a diffeomorphism: setting $g = (f|_U)^{-1} : V \rightarrow U$ one has $g \circ (f|_U) = \text{id}_U$ and $(f|_U) \circ g = \text{id}_V$; differentiating these equalities in x_0 we obtain $g'(y_0) \circ f'(x_0) = \text{id}_{\mathbb{R}^n}$ and $f'(x_0) \circ g'(y_0) = \text{id}_{\mathbb{R}^n}$, which yields $g'(y_0) = f'(x_0)^{-1}$: but this of course implies that $\det f'(x_0) \neq 0$. Conversely, we must prove that it is possible to invert the system $y = f(x)$ (in order to obtain $y_i = f_i(x_1, \dots, x_n)$ for $i = 1, \dots, n$) at least in a neighborhood of y_0 , i.e. that, setting $g_i(x, y) = y_i - f_i(x_1, \dots, x_n)$, it is possible to solve the system $g_1(x, y) = \dots = g_n(x, y) = 0$ with respect to the variables $x_1, \dots, x_n$ in a neighborhood of (x_0, y_0) : but, thanks to Dini's Theorem for systems, the condition $\det f'(x_0) \neq 0$ is exactly what is needed to this aim. $\square$

Let us make a couple of remarks about the notion of local diffeomorphism.

- In one variable, Theorem 1.1.1 simply says that if D is an open interval of $\mathbb{R}$, a function $f : D \rightarrow \mathbb{R}$ is a local diffeomorphism at some $x_0 \in D$ if and only if $f'(x_0) \neq 0$, which is a quite natural condition. For example, $f : \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = x^3$ is a homeomorphism but not a diffeomorphism (the inverse function is not derivable at 0), and it is a local diffeomorphism at any $x_0 \neq 0$. Other examples: the functions $g(x) = x^2$ and $h(x) = x(x^2 - 1)$ are not homeomorphisms since they are not injective, but g (resp. h) is a local diffeomorphism at any $x_0 \neq 0$ (resp. at any $x_0 \neq -\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}$).
- In two variables, let $D =]0, +\infty[\times \mathbb{R}$ and $E = \mathbb{R}^2 \setminus \{(0, 0)\}$, and consider the change of polar coordinates $f : D \rightarrow E$ given by $(x, y) = f(r, \theta) = (r \cos \theta, r \sin \theta)$. It is easy to compute that $\det f'(r, \theta) = r \neq 0$, so f is a local diffeomorphism at any $(r, \theta) \in D$.
- In the case of one variable we could add that *if $f : D \rightarrow \mathbb{R}$ is of class $\mathcal{C}^1$ and $f'(x) \neq 0$ for any $x \in D$ then not only is f a local diffeomorphism at any $x \in D$, but it is even a diffeomorphism of D on its image $E := f(D)$* : namely, since f' is a never vanishing continuous function then it must have constant sign, in other words f must be strictly monotone (increasing or decreasing), and in one variable this is equivalent to saying that f is a diffeomorphism. However, this is no longer true in the case of several variables: for example, the above change of polar coordinates has jacobian determinant always nonzero, but it is far from being a diffeomorphism, since it is not injective (note that $f(r, \theta) = f(r, \theta + 2k\pi)$ for any $k \in \mathbb{Z}$).

Can we give a meaning to the notion of “being of class $\mathcal{C}^k$ ” and of “diffeomorphism” also when D and E are not necessarily open? Yes: we say that $f : D \rightarrow E$ is of class $\mathcal{C}^k$ if it is induced by a function of class $\mathcal{C}^k$ in an open neighborhood U of D in $\mathbb{R}^m$, i.e. if there exists a function $\tilde{f} : U \rightarrow \mathbb{R}^n$ of class $\mathcal{C}^k$ such that $f(x) \in E$ for any $x \in D$. Now, we may say that $f : D \rightarrow E$ is a diffeomorphism of class $\mathcal{C}^k$ if it is both a homeomorphism and both f and f^{-1} are of class $\mathcal{C}^k$. See also Figure 1.1 for some visual examples.

Examples. (1) The above map $f : [0, 1[\rightarrow E_1$ given by $f(t) = (t, t^2)$ is a diffeomorphism. (2) Let $D = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 4, z > \sqrt{3}\}$ (what is it?) and E be the unit open ball in $\mathbb{R}^2$: then $f : D \rightarrow E$ given by $f(x, y, z) = (x, y)$ is a diffeomorphism (could you draw a picture for f ? what is f^{-1} ?)

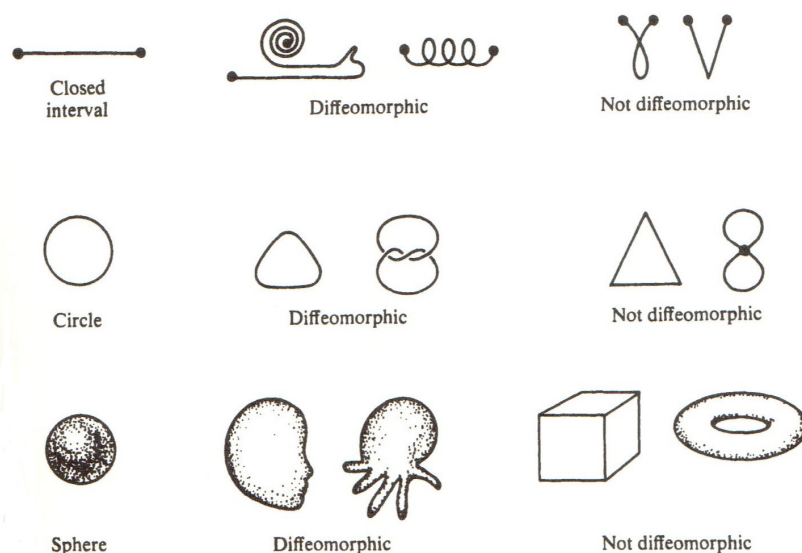


Figure 1.1: Diffeomorphism: yes or no (picture taken from V. Guillemin - A. Pollack *Differential Topology*, AMS 1974). Some more details about non-diffeomorphic objects: (1) the angle-shaped curve is homeomorphic to the closed interval, while the gamma-shaped curve is not; (2) the triangle is homeomorphic to the circle, while the figure-eight curve is not; (3) the cube is homeomorphic to the sphere, while the torus is not (namely the sphere is simply connected, the torus is not).

The nice feature of the Inverse Function Theorem 1.1.1 is to pass from an algebraic condition like $\det f'(x_0) \neq 0$ to an analytical property like the fact that f is a local diffeomorphism at x_0 . Now, with the above general notions of class $\mathcal{C}^k$ and of diffeomorphism in hand, it is quite natural to raise the following question, which generalizes the framework of Inverse Function Theorem: if f works between spaces of different dimensions and we know that the differential $f'(x_0)$ is only injective or only surjective (one says that f is *immersive* or *submersive* at x_0 , and if this happens in all points of D one says that f is an *immersion* or a *submersion*), can we hope that f is “locally injective” or “locally surjective” at x_0 ? The answer is yes, and the precise result is as follows.

Theorem 1.1.2. Let $D \subset \mathbb{R}^m$ be open and $f : A \rightarrow \mathbb{R}^n$ be differentiable in $x_0 \in D$.

- (i) (Immersion Theorem) If $m \leq n$, then f is *immersive* in x_0 if and only if “in a neigh-

borhood of x_0 , the function f identifies D with its image $f(D)$ ", i.e. there exists an open neighborhood $U \subset D$ of x_0 such that $f|_U$ is a diffeomorphism between the open U of $\mathbb{R}^m$ and the subset $f(U) \subset \mathbb{R}^n$.

- (ii) (Submersion Theorem) If $m \geq n$, then f is submersive in x_0 if and only if "in a neighborhood of x_0 , the function f behaves like a projection".⁽²⁾ In particular, if $y_0 = f(x_0)$, the "level set" $f^{-1}(y_0) = \{x \in D : f(x) = y_0\} \subset D$ is diffeomorphic, in a neighborhood of x_0 , to an open subset of $\mathbb{R}^{m-n}$.

Proof. Omitted. □

It is important to observe that the informations provided by the previous results are of *local* character, and therefore cannot exclude strange *global* behaviours of the function: for example let us have a look to the *figure eight* curve here below.

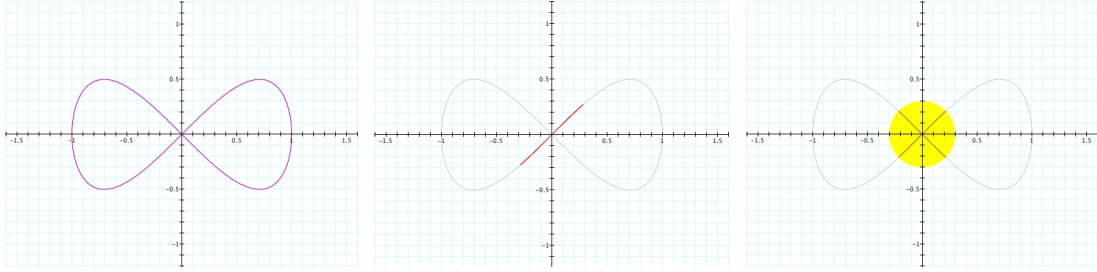


Figure 1.2: (a) The support of the *figure eight* curve. (b) The segment $] -\frac{\pi}{11}, \frac{\pi}{11}[\subset \mathbb{R}$, a neighborhood of $t_0 = 0$, is diffeomorphic to its image $\gamma(] -\frac{\pi}{11}, \frac{\pi}{11}[) \subset \mathbb{R}^2$, which is the piece of curve in red: the Embedding Theorem is fulfilled. (c) The purple-colored "x" that the yellow neighborhood of $(0,0)$ in $\mathbb{R}^2$ depicts on the *figure eight* curve cannot be diffeomorphic to an interval of $\mathbb{R}$.

Example. The *figure eight* curve $\gamma : \mathbb{R} \rightarrow \mathbb{R}^2$ given by $\gamma(t) = (\sin t, \sin t \cos t)$ (whose support $\Gamma = \gamma(\mathbb{R})$ is represented in Figure (a)) is an immersion: namely, for the curves it is enough to verify that the derived vector never vanishes, and in this case if $\gamma'(t) = (\cos t, \cos 2t)$ would be zero then $\cos t = \cos 2t = 0$, which is impossible. Therefore, thanks to the Immersion Theorem, there exists a neighborhood $] -\varepsilon, \varepsilon[$ of $t_0 = 0$ diffeomorphic to its image $\gamma(] -\varepsilon, \varepsilon[)$: this is true, and happens as soon as $\varepsilon < \pi$ (Figure (b)). On the other hand, the whole curve γ is far from being injective: in particular the point $(0,0) \in \Gamma$ can be reached, beyond than from $t_0 = 0$, also from all other values of the parameter t which are integer multiples of π ; moreover, despite of the fact that a small piece of Γ is diffeomorphic to an interval of $\mathbb{R}$, the whole curve Γ cannot be (Figure (c)), for the usual "cross argument".

For the sequel it is important to remark the following fact.

Proposition 1.1.3. Let D be an open subset of $\mathbb{R}^p$ and $h : D \rightarrow \mathbb{R}^q$ (with $p \leq q$) a function of class $\mathcal{C}^k$ which induces a $\mathcal{C}^k$ -diffeomorphism between D and $E = h(D)$. Then h is an immersion.

⁽²⁾ The precise meaning is as follows: there exist open neighborhoods $U \subset D$ of x_0 and $V \subset \mathbb{R}^n$ of $y_0 = f(x_0)$, an open subset $W \subset \mathbb{R}^{m-n}$ and a diffeomorphism $\beta : U \xrightarrow{\sim} V \times W$ such that, denoting by $\pi_V : V \times W \rightarrow V$ the projection $\pi_V(x_\lambda, x_\mu) = x_\lambda$, we have $f|_U = \pi_V \circ \beta$.

Proof. Let $t_0 \in D$ and $u_0 = h(t_0) \in E$. By hypothesis there exist an open neighborhood $V \subset \mathbb{R}^q$ of u_0 and a function $\psi : V \rightarrow D$ of class C^k such that $\psi|_{V \cap E} = h^{-1}|_{V \cap E}$. Since $\psi \circ h = id_D : D \rightarrow D$, and D is an open subset of $\mathbb{R}^p$, differentiating at t_0 we obtain $\psi'(u_0) \circ h'(t_0) = id_{\mathbb{R}^p}$, and this implies (that $\psi'(u_0)$ is surjective and that) $h'(t_0)$ is injective. $\square$

1.2 Differential manifolds: parameters, graphs, level sets

A *differential manifold* of dimension m in $\mathbb{R}^n$ (where $m \leq n$) is a subset M of $\mathbb{R}^n$ which “when observed close to any of its points, it seems like an open subset of $\mathbb{R}^m$ ”, i.e. any of its points has a open neighborhood (of the form $U \cap M$, with U open in $\mathbb{R}^n$) diffeomorphic to an open subset of $\mathbb{R}^m$. Let us stress once more on this fact: we do not demand M to be globally diffeomorphic to an open subset of $\mathbb{R}^m$ but only locally, i.e. at the neighborhood of any of its points.

Examples. (1) A line, a nondegenerate conic section (i.e. parabolas, hyperbolas, ellipses) are examples of differential manifolds of dimension 1 in $\mathbb{R}^2$. (2) The cylindrical helix $\gamma : \mathbb{R} \rightarrow \mathbb{R}^3$ with $\gamma(t) = (r \cos t, r \sin t, t)$, is a differential manifold of dimension 1 in $\mathbb{R}^3$. (3) In the cartesian plane, the subset $X = \{(x, y) : xy = 0\}$ (i.e. the union of the coordinate axes) is not a manifold: the problem is of course the “cross-point” $(0, 0)$, whose neighborhood cannot be diffeomorphic to an open interval as explained above. But if we throw this point away, things work: $X \setminus \{(0, 0)\}$ is a manifold (why?) (4) An affine plane, the paraboloidal surface $z = x^2 + y^2$, the cylinder $x^2 + y^2 = 1$, the spherical surface $x^2 + y^2 + z^2 = r^2$ are four examples of manifold of dimension 2 in $\mathbb{R}^3$.

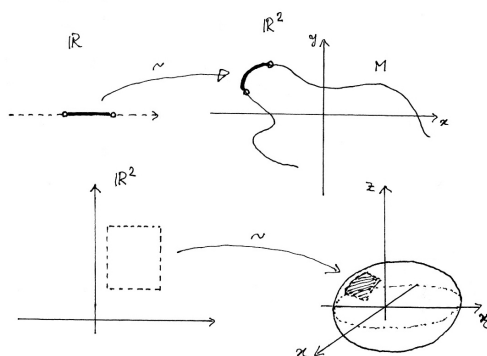


Figure 1.3: A manifold of dimension 1 (a curve) and of dimension 2 (an ellipsoid surface).

There are essentially three ways to describe a differential manifold:

- (1) *Parametric form*: describe locally M by means of m parameters.
- (2) *Graph form*: describe locally M as the graph of a function $\mathbb{R}^m \rightarrow \mathbb{R}^{n-m}$.
- (3) *Cartesian form*: describe locally M as the zero set of a submersive function $\mathbb{R}^n \rightarrow \mathbb{R}^{n-m}$.

The general principle is that the “graph form” provides a bridge between the other two: this is already well-shown in the basic example of *regular plane curves*, i.e. manifolds of dimension $m = 1$ in the plane $\mathbb{R}^2$, from which we start.

Regular plane curves

A subset Γ of the plane $\mathbb{R}^2$ is said to be a *regular (plane) curve* in $(x_0, y_0) \in \Gamma$ when one of the following three equivalent conditions are satisfied:

- (1) *Parametric form*: close to (x_0, y_0) , Γ is diffeomorphic to an open interval of $\mathbb{R}$.

There exist a neighborhood $U \subset \mathbb{R}^2$ of (x_0, y_0) , an open interval $I \subset \mathbb{R}$ and a diffeomorphism $\varphi : \Gamma \cap U \xrightarrow{\sim} I$ (called *local chart of Γ at (x_0, y_0)*); the inverse $\gamma = \varphi^{-1} : I \xrightarrow{\sim} \Gamma \cap U$, which is what one usually deals with in practice, is called *local parametrization of Γ at (x_0, y_0)*).

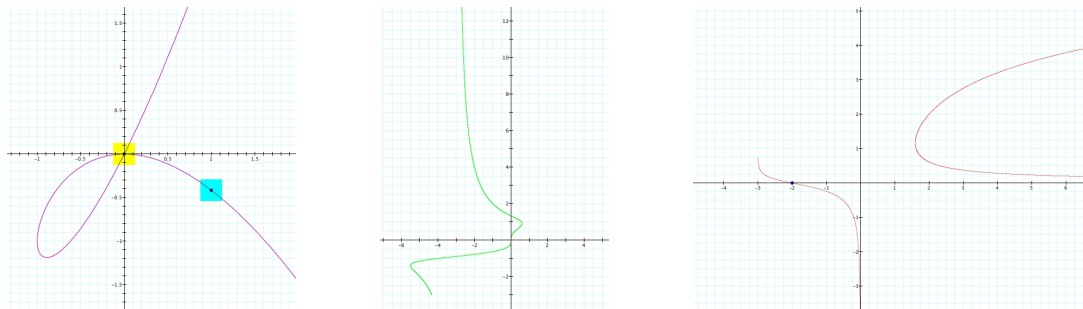
- (2) *Graph form*: close to (x_0, y_0) , Γ is the graph of a function of $\mathbb{R}$ to $\mathbb{R}$.

There exist neighborhoods $A \subset \mathbb{R}$ of x_0 and $B \subset \mathbb{R}$ of y_0 , and a function $\phi : A \rightarrow B$ (or $\psi : B \rightarrow A$) such that, setting $U = A \times B$, one has $\Gamma \cap U = \{(x, y) \in U : y = \phi(x)\}$ (or $\Gamma \cap U = \{(x, y) \in U : x = \psi(y)\}$).

- (3) *Cartesian form*: close to (x_0, y_0) , Γ is a level set of a submersion of $\mathbb{R}^2$ in $\mathbb{R}$.

There exist a neighborhood $U \subset \mathbb{R}^2$ of (x_0, y_0) and a function $g : U \rightarrow \mathbb{R}$ whose gradient is never vanishing in U such that, setting $\alpha = g(x_0, y_0)$, one has $\Gamma \cap U = \{(x, y) \in U : g(x, y) = \alpha\}$.

Now let us illustrate those conditions with three concrete examples, where we show that a subset of $\mathbb{R}^2$ which satisfies one of these, also satisfies the other two.



Examples. (1) (Curve in parametric form) As we said, in practice instead of dealing with a diffeomorphism $\varphi : \Gamma \cap U \xrightarrow{\sim} I$ one usually deals with a parametrization $\gamma : I \rightarrow \mathbb{R}^2$. If the derived vector $\gamma'(t)$ never vanishes, such γ is an immersion and hence (Immersion Theorem 1.1.2(i)) for any $t \in I$ it induces a diffeomorphism from a certain neighborhood J of t in I and its image $\gamma(J)$ in $\mathbb{R}^2$. The problem is: even if this happens for any $t \in I$, *who ensures us that γ induces a diffeomorphism between all of I and all of its image $\gamma(I) \subset \mathbb{R}^2$* ? For example, let us consider $\gamma : \mathbb{R} \rightarrow \mathbb{R}^2$ given by $\gamma(t) = (t(t-2), t^2(t-2))$ (Figure 1). The vector $\gamma'(t) = (2(t-1), t(3t-4))$ never vanishes, but it is clear that γ does not induce a diffeomorphism between all of $I = \mathbb{R}$ and all of its image $\Gamma = \gamma(I)$: the problem is caused by the point $\gamma(0) = \gamma(2) = (0, 0)$, where the support $\gamma(I)$ has a “self-intersection” (see also p. 6). In other words, Γ is not a regular curve in $(0, 0)$. On the other hand, close to any of its other points Γ is a regular curve: for

example, it is so in $(x_0, y_0) = (1, 1 - \sqrt{2}) = \gamma(t_0)$ with $t_0 = 1 - \sqrt{2}$ (see Figure 1 again). How to pass to a local description of Γ close to (x_0, y_0) in form of graph? Since $\gamma'(t_0) = (x'(t_0), y'(t_0)) = (-2\sqrt{2}, 5 - 2\sqrt{2})$, close to t_0 one could invert either the function $x(t)$ or the function $y(t)$: if we invert for example $x(t)$, it will be possible to write locally $t = t(x)$, and this yields the graph form $y = y(t(x)) = \phi(x)$. In this case we can complete the calculations: namely from $x = t(t - 2)$ we get $t = 1 \pm \sqrt{x+1}$, and around t_0 (which is < 1) it will be $t = 1 - \sqrt{x+1}$, and this gives $y = \phi(x) = t(x)^2(t(x) - 2) = x(1 - \sqrt{x+1})$. From $y = x(1 - \sqrt{x+1})$ we immediately find the cartesian form $y - x(1 - \sqrt{x+1}) = 0$; looking for a rational function, from $x\sqrt{x+1} = x - y$ we get $x^2(x+1) = (x-y)^2$, i.e. $g(x, y) = x^3 - y^2 + 2xy = 0$. Note that, not by chance, $\nabla g(x, y) = (3x^2 + 2y, 2(x-y))$ vanishes only in $(0, 0)$ and $(-\frac{2}{3}, -\frac{2}{3})$: of these two, only the first is in Γ , and it is already known as singular point. **(2) (Curve in graph form)** In the previous example, we noted that the passage from a parametric form to a graph or a cartesian form is rather delicate, because the parametrization could have a bad global behaviour. On the contrary, if we start from a graph form, for example if Γ is the graph of the function $x = \psi(y) = \frac{y^3(4-3y)}{\sqrt{y^8+2}}$ with $y > -3$ (Figure 2), we immediately get a parametric form with $\gamma :]-3, +\infty[\rightarrow \mathbb{R}^2$ given by $\gamma(y) = (\psi(y), y)$ (note that such parametrization induces a diffeomorphism between $I =]-3, +\infty[$ and $\Gamma = \gamma(I)$, with inverse $\varphi : \Gamma \rightarrow I$ given by $\varphi(x, y) = y$). As for the cartesian form, we get $g(x, y) = x - \psi(y) = 0$. **(3) (Curve in cartesian form)** Let us consider the curve Γ given in cartesian form by $g(x, y) = \log(x+3) - 2xy + e^y = 1$, and its point $(x_0, y_0) = (-2, 0)$ (Figure 3). We have $\nabla g(x, y) = (\frac{\partial g}{\partial x}(x, y), \frac{\partial g}{\partial y}(x, y)) = (\frac{1}{x+3} - 2y, -2x + e^y)$, hence $\nabla g(-2, 0) = (\frac{\partial g}{\partial x}(-2, 0), \frac{\partial g}{\partial y}(-2, 0)) = (-1, 5)$: by Dini's Theorem, at the neighborhood of $(-2, 0)$ from $g(x, y) - 1 = 0$ we could get one of the two variables x and y in function of the other. If for example we choose to get $x = x(y)$, we obtain a graph form (which on the other hand is not computable explicitly), from which we can also obtain a parametric form as seen above, i.e. $\gamma(y) = (x(y), y)$.

Let us consider a curve in cartesian form, i.e. a level curve $g(x, y) = \alpha$ regular at the neighborhood of one of its points (x_0, y_0) (hence $g(x_0, y_0) = \alpha$, and $\nabla g(x_0, y_0) \neq (0, 0)$). If g is of class C^r , we know that the polynomial function $P_{g,r}(x, y)$ of degree r which approximates $g(x, y)$ as best as possible at the neighborhood of (x_0, y_0) —in the sense that $|g(x, y) - P_{g,r}(x, y)| = o(\|(x - x_0, y - y_0)\|^r)$ — is its Taylor expansion until the r th order. As a consequence, the polynomial curve of degree r which approximates the level curve $g(x, y) = \alpha$ as best as possible at the neighborhood of (x_0, y_0) (one also talks about a *osculating polynomial curve*⁽³⁾ at (x_0, y_0)) is the curve of level α of $P_{g,r}(x, y)$: since $g(x_0, y_0) = \alpha$, such polynomial curve is

Osculating
polynomial curve

$$\sum_{\substack{k=1, \dots, r \\ j=0, \dots, k}} \frac{1}{(k-j)!j!} \left(\frac{\partial^k g}{\partial x^{k-j} \partial y^j}(x_0, y_0) \right) (x - x_0)^{k-j} (y - y_0)^j = 0.$$

For $r = 1$ one obtains the equation of the *affine tangent line* (which we will recover also later)

$$\frac{\partial g}{\partial x}(x_0, y_0) (x - x_0) + \frac{\partial g}{\partial y}(x_0, y_0) (y - y_0) = 0,$$

and for $r = 2$ the equation of the *osculating conic*

$$\begin{aligned} \frac{\partial g}{\partial x}(x_0, y_0) (x - x_0) + \frac{\partial g}{\partial y}(x_0, y_0) (x - x_0) + \frac{1}{2} \left(\frac{\partial^2 g}{\partial x^2}(x_0, y_0) (x - x_0)^2 + \right. \\ \left. + 2 \frac{\partial^2 g}{\partial x \partial y}(x_0, y_0) (x - x_0)(y - y_0) + \frac{\partial^2 g}{\partial y^2}(x_0, y_0) (y - y_0)^2 \right) = 0. \end{aligned}$$

⁽³⁾from the latin word *osculare*, which means “to kiss”.

Example. Let $g(x, y) = 2x^3y - 6x + y^2$, and let Γ be the level curve of g passing through $(x_0, y_0) = (0, 2)$: since $g(0, 2) = 4$, the curve Γ is then expressed by the cartesian equation $2x^3y - 6x + y^2 = 4$. The gradient $\nabla g(x, y) = (6(x^2y - 1), 2(x^3 + y))$ vanishes when $6x^2y - 6 = 2x^3 + 2y = 0$, i.e. only in $(-1, 1)$ which belongs to the level set $g(x, y) = g(-1, 1) = 5$. Hence all level sets $g(x, y) = \alpha$ with $\alpha \neq 5$ (and also Γ among these) are regular curves. The only 3rd order derivatives of g nonvanishing in $(0, 2)$ are $\frac{\partial g}{\partial x}(0, 2) = -6$, $\frac{\partial g}{\partial y}(0, 2) = 4$, $\frac{\partial^2 g}{\partial y^2}(0, 2) = 2$ and $\frac{\partial^3 g}{\partial x^3}(0, 2) = 24$: hence the affine tangent line to Γ at $(0, 2)$ is $-6(x - 0) + 4(y - 2) = 0$ (i.e. $y = \frac{3}{2}x + 2$), the conic osculating Γ at $(0, 2)$ is the parabola $-6(x - 0) + 4(y - 2) + \frac{1}{2}2(y - 2)^2 = 0$ (i.e. $x = \frac{1}{6}(y^2 - 4)$), while the cubic osculating Γ at $(0, 2)$ is $-6(x - 0) + 4(y - 2) + \frac{1}{2}2(y - 2)^2 + \frac{1}{6}24(x - 0)^3 = 0$ (i.e. $4x^3 + y^2 - 6x - 4 = 0$). Since g itself is a polynomial of degree 4, it obviously coincides with the osculating polynomial curves of degree ≥ 4 .

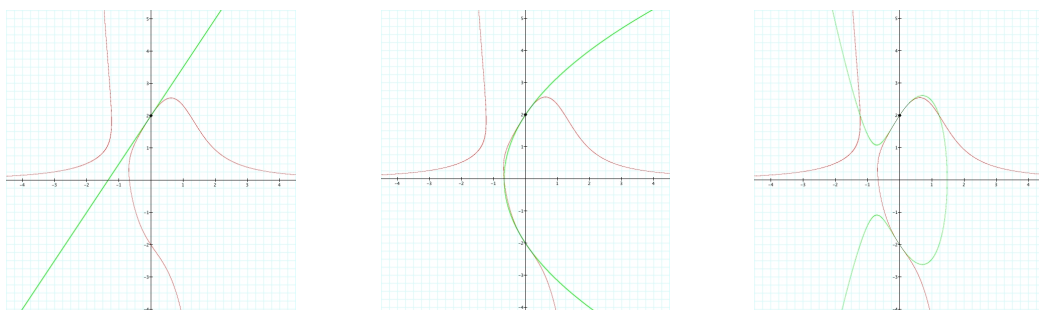


Figure 1.4: The line, conic and cubic (green) osculating the curve Γ (red) $2x^3y - 6x + y^2 = 4$ at the point $(0, 2)$.

Differential manifolds in the affine space

An affine set $M \subset \mathbb{R}^n$ is a *differential manifold* of class $\mathcal{C}^k$ of dimension m in the “ambient space” $\mathbb{R}^n$ (where $0 \leq k \leq \infty$ and $0 \leq m \leq n$) if it admits one of the following descriptions.

Differential manifold

- (1) (*Parametric form*: M is locally $\mathcal{C}^k$ -diffeomorphic to an open subset of $\mathbb{R}^m$)

For any $x_0 \in M$ there exist an open neighborhood $U \subset \mathbb{R}^n$ of x_0 , an open $V \subset \mathbb{R}^m$ and a $\mathcal{C}^k$ -diffeomorphism $\varphi : U \cap M \xrightarrow{\sim} V$.

The $\mathcal{C}^k$ -diffeomorphism $\varphi : U \cap M \xrightarrow{\sim} V$ is called *local chart* or *system of local coordinates* of M at x_0 . A family of local charts $\{\varphi_i : U_i \cap M \xrightarrow{\sim} V_i : i = 1, 2, \dots\}$ such that the open $U_1, U_2, \dots$ form a cover of M (i.e. $M \subset \bigcup_i U_i$) is called *atlas* of M , with an evident allusion of the term.

Local chart, atlas

If φ is a local chart of M , the inverse function $\gamma = \varphi^{-1} : V \rightarrow U \cap M \subset U$ is called *local parametrization* of M at x_0 . In practice, it is much more convenient to work with the parametrization γ than with the local chart φ , because the domain of γ is an open subset of $\mathbb{R}^m$ (and its codomain in fact $\mathbb{R}^n$), hence it can be differentiated in the usual sense. It is fundamental to observe (see Proposition 1.1.3) that γ is an *immersion*, i.e. that its differential $\gamma'(v) : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is injective for any $v \in V \subset \mathbb{R}^m$.

Parametrizzazione locale

This description “locally straightens M ”, by describing it as the subset of an affine space of smaller dimension. Hence, in this sense this description is the most adherent statement of the notion of

manifold as “a set locally isomorphic to an affine space of smaller dimension”, and it is necessary to use it in various situations.

- (2) (*Graph form*: M is locally the graph of a function $\mathbb{R}^m \rightarrow \mathbb{R}^{n-m}$ of class $\mathcal{C}^k$)

For any $x_0 \in M$ there exist an open neighborhood $U \subset \mathbb{R}^n$ of x_0 and m variables $x_\lambda = (x_{i_1}, \dots, x_{i_m})$ such that, written the remaining variables as $x_\mu = (x_{j_1}, \dots, x_{j_{n-m}})$ and denoted by $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^m$ the projection $\pi(x) = x_\lambda$, one has $U \cap M = \{x \in U : x_\mu = \phi(x_\lambda)\}$ for a suitable function $\phi = (\phi_{i_1}, \dots, \phi_{i_{n-m}}) : \pi(U) \rightarrow \mathbb{R}^{n-m}$ of class $\mathcal{C}^k$.

This is a particular case of the previous one, in the sense that to express locally M as a graph automatically provides a local parametrization for M . Once passed from “parameter of any nature” in (1) to “parameters-coordinates”, the description (2) provides a bridge between (1) and the next one (3): for more details see the proof of Proposition 1.2.1.

- (3) (*Cartesian form*: M is locally the zero set of a submersion $\mathbb{R}^n \rightarrow \mathbb{R}^{n-m}$ of class $\mathcal{C}^k$)

For any $x_0 \in M$ there exist an open neighborhood $U \subset \mathbb{R}^n$ of x_0 and a function $g = (g_1, \dots, g_{n-m}) : U \rightarrow \mathbb{R}^{n-m}$ of class $\mathcal{C}^k$ such that $U \cap M = g^{-1}(0_{n-m}) = \{x \in U : g_1(x) = \dots = g_{n-m}(x) = 0\}$ and that, in those points, g is submersive.

This description, where M is exhibited (locally, but in practice often globally) as the set of points of $\mathbb{R}^n$ which satisfy a given family of $n-m$ independent conditions $g_1(x) = 0, \dots, g_{n-m}(x) = 0$, is the most commonly used one. The vector-valued function g is called *defining function* of M , and its components $g_1, \dots, g_{n-m}$ are called *constraints*, with an evidente reference to Mechanics.

Defining function,
constraints

If M is a manifold of dimension m in $\mathbb{R}^n$, the number $n-m$ is called *codimension* of M .

Proposition 1.2.1. *The descriptions (1), (2) and (3) are equivalent, i.e. a subset $M \subset \mathbb{R}^n$ satisfies one of them if and only if it satisfies any of the other two of them.*

Proof. We shall show how the description (2) is equivalent both to (1) and to (3). Assume that M satisfies (1), i.e. that the local parametrization $\gamma : V \rightarrow U$ induces a diffeomorphism between V and its image $\gamma(V)$, and moreover that the latter is exactly $U \cap M$. Let $v_0 \in U$ be such that $\gamma(v_0) = x_0$: since γ induces a diffeomorphism between V and its image $\gamma(V)$, we know that γ is an immersion around v_0 , in particular is immersive at v_0 . Hence there exists a nonsingular minor (square submatrix) of order m of the matrix $\gamma'(v_0) \in M_{n,m}(\mathbb{R})$, say (just for notation convenience) the minor of the first m rows: by Theorem 1.1.1, the vector-valued function with components the first m among the n functions $x_i = \gamma_i(v_1, \dots, v_m)$ (with $i = 1, \dots, n$), is locally invertible around v_0 , and hence (denoting $x_\lambda = (x_1, \dots, x_m)$ and $x_\mu = (x_{m+1}, \dots, x_n)$), at the neighborhood of x'_0 we obtain inverse functions $v_\ell = \psi_\ell(x_\lambda)$ (with $\ell = 1, \dots, m$), from which, inserting the functions $\psi_1, \dots, \psi_m$ in place of the variables $v_1, \dots, v_m$ in the remaining $n-m$ functions $x_r = \gamma_r(v_1, \dots, v_m)$ (with $r = m+1, \dots, n$) we can express the variables x_μ in terms of the x_λ . In other words, we have been able to express $\gamma(V)$ in the graph form $x_\mu = \phi(x_\lambda)$, but in this case we also know that $\gamma(V) = U \cap M$: hence M satisfies (2).⁽⁴⁾

⁽⁴⁾This reasoning is more delicate than it could appear. The example of the previous paragraph, given by $\gamma : \mathbb{R} \rightarrow \mathbb{R}^2$ with $\gamma(t) = (t(t-2), t^2(t-2))$ should have suggested to be prudent: if we consider in it $x_0 = (0,0)$ and $v_0 = t_0 = 0$ (or $v_0 = t_0 = 2$) all arguments of the proof seem to work also (in fact, a neighborhood V of t_0 in $\mathbb{R}$ is identified diffeomorphically with its image $\gamma(V)$), but unfortunately in that case it happens that $\gamma(V) \neq U \cap \Gamma$ (namely $U \cap \Gamma$ is a “cross” centered in $(0,0)$ of the type of the one

If M satisfies (2) then we can write (always for notation convenience) $x_r = \phi_r(x_\lambda)$ (with $r = m+1, \dots, n$), and hence M satisfies (1) with parametrization $\gamma(x_\lambda) = (x_\lambda, \phi(x_\lambda))$ (note that the first m rows of the jacobian matrix $\gamma'(x_\lambda) \in M_{n,m}(\mathbb{R})$ form the identity matrix $\mathbf{1}_m$, hence γ is immersive) and satisfies also (3) with $g_r(x) = x_r - \phi_r(x_\lambda)$ (note that, setting $g = (g_1, \dots, g_{n-m})$, i.e. $g(x) = x_\mu - \phi(x_\lambda)$, the columns of indexes $m+1, \dots, n$ of the jacobian matrix $g'(x) \in M_{n-m,n}(\mathbb{R})$ form the identity matrix $\mathbf{1}_{n-m}$, hence g is submersive).

Finally, if M satisfies (3) then the jacobian matrix $g'(x) \in M_{n-m,n}(\mathbb{R})$ is submersive, and hence, applying Dini's Theorem for systems, from the vectorial equation $g(x_1, \dots, x_n) = 0_{n-m}$, i.e. the system $g_1(x_1, \dots, x_n) = \dots = g_{n-m}(x_1, \dots, x_n) = 0$, we may express $n-m$ variables among all the $x_1, \dots, x_n$ in function of the remaining m , and hence M satisfies (2). $\square$

As we said in the previous descriptions, and as it clearly appeared in the proof of Proposition 1.2.1, the *graph form* (2) mediates between the *parametric form* (1) and the *cartesian form* (3). While the passage from (2) to one of the other two forms is immediate (because the graph form provides at the same time both a parametric and a cartesian description), the inverse passages require care, because they require respectively to “express the m parameters $v_1, \dots, v_m$ in terms of m among the n cartesian coordinates $x_1, \dots, x_n$ (to go from (1) to (2)) or to “express $n-m$ among the n cartesian coordinates $x_1, \dots, x_n$ with respect to the remaining m (to go from (3) to (2)). It is then worth stressing once more how those two passages can be performed.

Proposition 1.2.2. *Let M be a differential manifold of dimension m in $\mathbb{R}^n$, and write $x = (x_\lambda, x_\mu) \in \mathbb{R}^n$ with $x_\lambda = (x_1, \dots, x_m)$ and $x_\mu = (x_{m+1}, \dots, x_n)$.⁽⁵⁾*

- (i) (From parametric form to graph form) *Let $\gamma : V \rightarrow \mathbb{R}^n$ (with V open subset of $\mathbb{R}^m$) be a local parametrization of M , and write $\gamma(v) = (x_\lambda(v), x_\mu(v))$ with $x_\lambda(v) = (\gamma_1(v), \dots, \gamma_m(v))$ and $x_\mu(v) = (\gamma_{m+1}(v), \dots, \gamma_n(v))$. If $\det x_\lambda'(v) \neq 0$ for a certain $v \in V$, we get a local description of M as a graph around $x = \gamma(v)$ by locally inverting the functions $x_\lambda(v)$ obtaining $v(x_\lambda)$, from which*

$$x_\mu = \phi(x_\lambda) = x_\mu(v(x_\lambda)), \quad \text{with} \quad \phi'(x_\lambda) = x_\mu'(v) x_\lambda'(v)^{-1}_{v=v(x_\lambda)}.$$

- (ii) (From cartesian form to graph form) *Let $g : U \rightarrow \mathbb{R}^{n-m}$ (with U open subset of $\mathbb{R}^n$) a local defining function of M , and write $g'(x) = (g'_{x_\lambda}(x), g'_{x_\mu}(x))$, where $g'_{x_\lambda}(x)$ and $g'_{x_\mu}(x)$ represent the jacobian matrices of g with respect to the variables x_λ and x_μ .⁽⁶⁾ If $\det g'_{x_\mu}(x) \neq 0$ for a certain $x \in U$, we get a local description of M as graph by locally making the variables x_μ explicit with respect to the x_λ , from which*

$$x_\mu = \phi(x_\lambda), \quad \text{with} \quad \phi'(x_\lambda) = -g'_{x_\mu}(x)^{-1} g'_{x_\lambda}(x)_{x=(x_\lambda, \phi(x_\lambda))}.$$

obtained in the picture by intersecting Γ with the yellow square, while $\gamma(V)$ is just one of the two arms of that “cross”) and hence we cannot transfer to $U \cap \Gamma$ what we have said about $\gamma(V)$.

⁽⁵⁾What will be said about the decomposition $x = (x_\lambda, x_\mu)$ could be obviously applied, following the cases, to any other decomposition of the n coordinates $x_1, \dots, x_n$ in two disjoint groups $x_\lambda = (x_{i_1}, \dots, x_{i_m})$ and $x_\mu = (x_{j_1}, \dots, x_{j_{n-m}})$.

⁽⁶⁾hence with $n-m$ rows and, respectively, m and $n-m$ columns.

Proof. We should prove only the statements about the jacobian matrices. (i) Differentiating the composite function $\phi(x_\lambda) = x_\mu(v(x_\lambda))$ we obtain $\phi'(x_\lambda) = x_\mu'(v(x_\lambda)) v'(x_\lambda)$; then recall that $v'(x_\lambda) = x_\lambda'(v)^{-1}_{v=v(x_\lambda)}$ (the jacobian matrix of the inverse is the inverse of the jacobian matrix). (ii) Follows from Dini's theorem for systems. $\square$

Let us put in evidence the most frequent examples of manifold.

- (a) **Graphs.** If V is an open subset of $\mathbb{R}^m$ and $\phi : V \rightarrow \mathbb{R}^p$ is a function of class $\mathcal{C}^k$, setting $n = m + p$ we have that the *graph* of ϕ

$$(1.1) \quad \Gamma_\phi = \{(x_\lambda, \phi(x_\lambda)) : x_\lambda \in V\} \subset \mathbb{R}^n$$

is manifold $\mathcal{C}^k$ of dimension m in $\mathbb{R}^n$, as it immediately follows from (2). A (global) parametrization of $M = \Gamma_\phi$ is a function $V \rightarrow \mathbb{R}^n$ such that $x_\lambda \mapsto (x_\lambda, \phi(x_\lambda))$ (namely, $M = \Gamma_\phi$ is $\mathcal{C}^k$ -diffeomorphic to V); a (global) defining function for $M = \Gamma_\phi$ is $g : \mathbb{R}^n \rightarrow \mathbb{R}^p$ given by $g(x_\lambda, x_\mu) = x_\mu - \phi(x_\lambda)$.

- (b) **Level sets of submersive functions.** Let W be an open subset of $\mathbb{R}^n$, $G : W \rightarrow \mathbb{R}^p$ a function of class $\mathcal{C}^k$, and a fixed $\alpha = (\alpha_1, \dots, \alpha_p) \in \mathbb{R}^p$. The *level set* of G at the value α is

$$(1.2) \quad X_{G,\alpha} = G^{-1}(\alpha) = \{x \in W : G(x) = \alpha\} \subset \mathbb{R}^n :$$

in other words, it is the set of solutions x in W of the system

$$\begin{cases} G_1(x) = \alpha_1, \\ \vdots \\ G_p(x) = \alpha_p \end{cases}$$

(where of course we mean that in the case $p = 1$ there is only one equation). If $X_{G,\alpha} \neq \emptyset$ and G is submersive in all points of $X_{G,\alpha}$ ⁽⁷⁾ then, setting $m = n - p$, we have that $M = X_{G,\alpha}$ is a manifold $\mathcal{C}^k$ of dimension m in $\mathbb{R}^n$, as it follows from (3) (a global defining function is $g : W \rightarrow \mathbb{R}^p$ given by $g(x) = G(x) - \alpha$). As it was said in the proof of Proposition 1.2.1, the submersivity of G in the points of $M = X_{G,\alpha}$ allows then to locally make explicit, from the system $G(x) = \alpha$, any family of p among the variables $x_1, \dots, x_n$ of $\mathbb{R}^n$ in terms of the remaining m , and hence to obtain in this way a local parametrization of $M = X_{G,\alpha}$.

Note that (a) and (b) are the cases where M is expressed globally respectively as *graph form* (2) and *cartesian form* (3). On the other hand, a global definition in *parametric form* (1), could not generate a manifold in the previous sense, but to rather a more general object called *parametric manifold*, where particular phenomena could happen as in the case of the figure-eight curve: we shall deal later with that (see p. 17).

⁽⁷⁾ $\alpha \in \mathbb{R}^p$ is said to be a *regular value* for G if $X_{G,\alpha} = \emptyset$ or if $X_{G,\alpha} \neq \emptyset$ and G is submersive in all points of $X_{G,\alpha}$, otherwise is said to be a *critical value*. A well-known result, due to Sard, ensure that “almost all” (i.e. —for those who know the Lebesgue integral, see also later on— all but a Lebesgue-null set, having measure zero) of the $\alpha \in \mathbb{R}^p$ are regular values for G .

Now let us study some closer the cases of manifolds of dimensions 0, 1, 2, $n - 1$ and n in $\mathbb{R}^n$, and introduce the weaker notion of “parametric manifold”.

- In the extremal cases, the manifolds of dimension 0 in $\mathbb{R}^n$ are the *discrete* subsets of $\mathbb{R}^n$, i.e. those formed by isolated points; and the manifolds of dimension n in $\mathbb{R}^n$ are the *open subsets* of $\mathbb{R}^n$.

- The manifolds of dimension 1 in $\mathbb{R}^n$, i.e. the subsets of $\mathbb{R}^n$ which are locally diffeomorphic to an open interval of $\mathbb{R}$, are said (*regular*) *curves* of $\mathbb{R}^n$.⁽⁸⁾ Hence the regular curves of $\mathbb{R}^n$ are (at least locally) parametric curves in the known sense; the converse is not always true, as in the case of the figure-eight curve (see examples below).

Curves

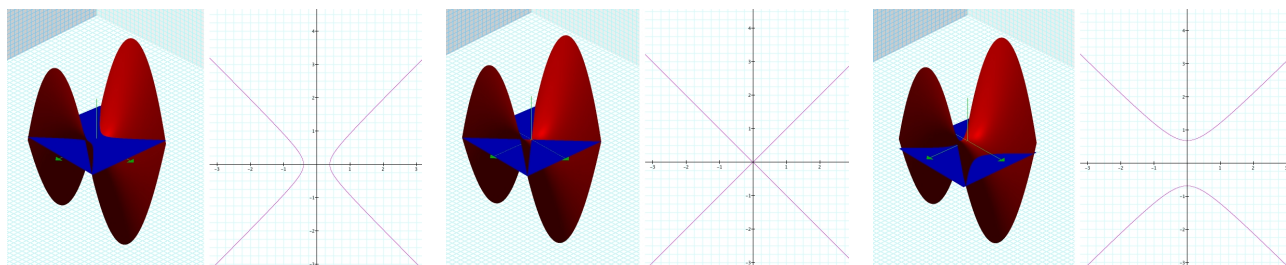


Figure 1.5: Level curves of $G(x, y) = x^2 - y^2$ visualized by means of the intersection between the graph of G (hyperbolic paraboloid surface) and the horizontal planes $z = \alpha$.

Examples. (1) From (1.1) we get that if $I \subset \mathbb{R}$ is an open interval, the graph of any function $\phi : I \rightarrow \mathbb{R}^{n-1}$ of class C^k is a regular curve in $\mathbb{R}^n$. For example the parabola $y = x^2$, graph of $\phi : \mathbb{R} \rightarrow \mathbb{R}$ given by $\phi(t) = t^2$, is a C^∞ curve of $\mathbb{R}^2$; the curve $\phi(t) = |t|$ is a curve C^0 of $\mathbb{R}^2$ (and C^∞ outside $(0, 0)$); the cylindrical helix $(x, y, z) = (r \cos t, r \sin t, t)$, seen as graph of $\phi : \mathbb{R} \rightarrow \mathbb{R}^2$ given by $\phi(t) = (r \cos t, r \sin t)$, is a curve C^∞ of $\mathbb{R}^3$. **(2) (Level curves)** The function $G : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $G(x, y, z) = x^2 - y^2$ has gradient $\nabla G = (2x, -2y)$, hence it is submersive everywhere excepted than $(0, 0)$. It follows that the level sets $X_{G, \alpha} = \{(x, y) \in \mathbb{R}^2 : G(x, y) = \alpha\}$ are manifold of dimension $2 - 1 = 1$ (i.e. curves) C^∞ of $\mathbb{R}^2$ for any $\alpha \neq 0$: in fact for $\alpha \neq 0$ the set $X_{G, \alpha}$ is a equilateral hyperbola, while for $\alpha = 0$ the set $X_{G, 0}$, defined by $x^2 - y^2 = 0$, is the union of the two bisecting lines $y = \pm x$ (which is also a regular curve as soon as $(0, 0)$ is removed). In Figure 1.5 it is possible to observe the evolution of such level curves by means of the l'intersection of the graph of G (the hyperbolic paraboloid surface $z = x^2 - y^2$) with the horizontal planes $z = \alpha$. **(3)** The circle $\mathbb{S}^1$ of center the origin and radius 1 in the cartesian plane, is a C^∞ regular curve: namely it is the level set of level 1 of the function $G : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $G(x, y) = x^2 + y^2$, which is submersive everywhere (excepted the point $(0, 0)$ which does not belong to $\mathbb{S}^1$). For a local parametrization of $\mathbb{S}^1$ around one of its points (x_0, y_0) it is enough, as already explained, to make locally explicit from $x^2 + y^2 - 1 = 0$ one of the two variables with respect to the other using Dini's Theorem: for examples, if $y_0 < 0$ we can make y explicit by writing $y = -\sqrt{1 - x^2}$ (with local parametrization $\gamma :]-1, 1[\rightarrow \mathbb{R}^2$ given by $\gamma(t) = (t, -\sqrt{1 - t^2})$), while around $(1, 0)$ we can make x explicit by writing $x = \sqrt{1 - y^2}$ (with local parametrization $\gamma :]-1, 1[\rightarrow \mathbb{R}^2$ given by $\gamma(t) = (\sqrt{1 - t^2}, t)$). Another usual local parametrization of $\mathbb{S}^1$ is $\gamma_{\theta_0} :]0, 2\pi[\rightarrow \mathbb{S}^1$ given by $\gamma_{\theta_0}(\theta) = (\cos(\theta + \theta_0), \sin(\theta + \theta_0))$, which does not include the point $(\cos \theta_0, \sin \theta_0)$.

⁽⁸⁾In the previous paragraph we dealt with the plane regular curves, i.e. the case $n = 2$.

If $N = (0, 1)$ represents the “North pole” of $\mathbb{S}^1$, a local chart of $\mathbb{S}^1$ is given by the *stereographic projection* $\varphi_N : \mathbb{S}^1 \setminus \{N\} \rightarrow \mathbb{R}$ given by $\varphi_N(x_0, y_0) = \frac{x_0}{1-y_0}$ (consider the half line issued from N and passing through (x_0, y_0) , then define $\varphi_N(x_0, y_0)$ as the x -abscissa of the point in which such half line meets the x -axis); its inverse $\varphi_N^{-1} : \mathbb{R} \rightarrow \mathbb{S}^1 \setminus \{N\}$ (which turns out to be $\varphi_N^{-1}(t) = (\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1})$) is a local parametrization of $\mathbb{S}^1$, which does not include the point N . Note that all these parametrizations are not global (there is always some point of $\mathbb{S}^1$ which is not included), and in fact a global parametrization of $\mathbb{S}^1$ does not exist: would there exist one by absurd, say $\gamma : I \rightarrow \mathbb{S}^1$ (where $I \subset \mathbb{R}$ is an open interval), since γ is a homeomorphism, taken any $t_0 \in I$ this would imply that $I \setminus \{t_0\}$ is homeomorphic to $\mathbb{S}^1 \setminus \{\gamma(t_0)\}$, but this is impossible because the latter is connected while the former is not. Hence an atlas will be formed by (the inverse functions of) more than one local parametrization, and for $\mathbb{S}^1$ two are enough (for example γ_0 and γ_π , or the two charts φ_N and φ_S , where the second is the stereographic projection from the “South pole” $S = (0, -1)$). (4) Let us show that the system $\begin{cases} y = 2x^2 + z^2 \\ 3x^2 + y^2 + 2z^2 = 1 \end{cases}$, defines a regular curve l in $\mathbb{R}^3$. Namely, defining $g = (g_1, g_2) : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ as $g(x, y, z) = (2x^2 - y + z^2, 3x^2 + y^2 + 2z^2 - 1)$ it holds $l = g^{-1}(0, 0)$. One has $g'((x, y, z)) = \begin{pmatrix} 4x & -1 & 2z \\ 6x & 2y & 4z \end{pmatrix}$, hence the rank of the differential is not maximal in the points which solve the system $x(4y+3) = xz = z(y+1) = 0$, i.e. those of the form $(0, \lambda, 0)$, $(0, -1, \mu)$, $(\nu, -\frac{3}{4}, 0)$ with $\lambda, \mu, \nu \in \mathbb{R}$; but none of these points belongs to l (check it). Hence l is a regular curve of $\mathbb{R}^3$. If we want to find a parametrization of the manifold l around e.g. its point $Q(\frac{1}{2}, \frac{1}{2}, 0)$, since $g'(Q) = \begin{pmatrix} 2 & -1 & 0 \\ 3 & 1 & 0 \end{pmatrix}$ we observe that the first minor of order 2 (relative to x and y) is nonsingular: therefore using Dini’s Theorem we can make x and y explicit as functions of z in an open neighborhood $J \subseteq \mathbb{R}$ of $z = 0$, with the condition $(x(0), y(0)) = (\frac{1}{2}, \frac{1}{2})$. The function $\psi : J \rightarrow l$ given by $\psi(z) = (x(z), y(z), z)$ is a parametrization of l around Q . (5) The support $\Gamma \subset \mathbb{R}^2$ of the *figure-eight* curve is not a regular curve: namely, as shown at p. 6, if $V \subset \mathbb{R}^2$ is an open neighborhood of $(0, 0)$ small enough then $\Gamma \cap V$ cannot be diffeomorphic to an open interval of $\mathbb{R}$. It is important to remark that this does not depend on the failure of injectivity of γ , but essentially on the form of Γ around its *singular* point $(0, 0)$ (in fact, $\Gamma \setminus \{(0, 0)\}$ is a regular curve): namely, using the trigonometric identities $\cos t = \frac{2\tau}{\tau^2+1}$ and $\sin t = \frac{\tau^2-1}{\tau^2+1}$ (with $\tau = \tan(\frac{t+\pi}{2})$), we can find a globally injective parametrization $\tilde{\gamma} : \mathbb{R} \rightarrow \mathbb{R}^2$, given by $\tilde{\gamma}(\tau) = (\frac{2\tau}{\tau^2+1}, \frac{2\tau(\tau^2-1)}{(\tau^2+1)^2})$ (note that $\tilde{\gamma}(\tau)$ “comes asymptotically” from $(0, 0)$ for $\tau \rightarrow -\infty$, passes again in $(0, 0)$ for $\tau = 0$, and finally “tends asymptotically” to it for $\tau \rightarrow +\infty$), but the problems in $(0, 0)$ are still there. The singularity of $(0, 0)$ emerges more clearly from the cartesian form of Γ . Namely from $x(t) = \cos t$ and $y(t) = \sin t \cos t$ we get $y^2 = (1-x^2)x^2$, i.e. Γ is the zero set of $g(x, y) = (x^2-1)x^2 + y^2$; the system $\frac{\partial g}{\partial x} = \frac{\partial g}{\partial y} = 0$ gives the solutions $(0, 0)$ (which belongs to Γ) and $(\pm \frac{\sqrt{2}}{2}, 0)$ (which does not), hence g is submersive in all points of Γ excepted $(0, 0)$.

- The manifolds of dimension 2 in $\mathbb{R}^n$, i.e. the subsets of $\mathbb{R}^n$ locally diffeomorphic to an open subset of $\mathbb{R}^2$, are called (*regular*) *surfaces* of $\mathbb{R}^n$. Surfaces

Examples. (1) From (1.1) we get that, if $A \subset \mathbb{R}^2$ is an open subset, the graph of any function $\phi : A \rightarrow \mathbb{R}^{n-2}$ of class C^k is a regular surface in $\mathbb{R}^n$. For example the elliptic paraboloid surface $z = ax^2 + by^2 + c$ or the hyperbolic paraboloid surface $z = ax^2 - by^2 + c$ (with $a, b > 0$) are surfaces C^∞ of $\mathbb{R}^2$. Since the paraboloid surface is already in graph form, a (global) parametrization of it is given by $\gamma : \mathbb{R}^2 \rightarrow \mathbb{R}^3$, $\gamma(x, y) = (x, y, ax^2 + by^2 + c)$; alternatively, if for example $a = b$ we can use cylindrical coordinates (ρ, θ, z) obtaining $\gamma : \mathbb{R}_{>0} \times]0, 2\pi[\rightarrow \mathbb{R}^3$, $\gamma(\rho, \theta) = (\rho \cos \theta, \rho \sin \theta, \rho^2 + c)$ (which does not include the points of the paraboloid surface on the half plane (x, z) with $x \geq 0$). (2) (*Level surfaces*) The function $G : \mathbb{R}^3 \rightarrow \mathbb{R}$ given by $G(x, y, z) = z^2 - x^2 - y^2$ has gradient $\nabla G = (-2x, -2y, 2z)$, hence it is submersive everywhere but in $(0, 0, 0)$.

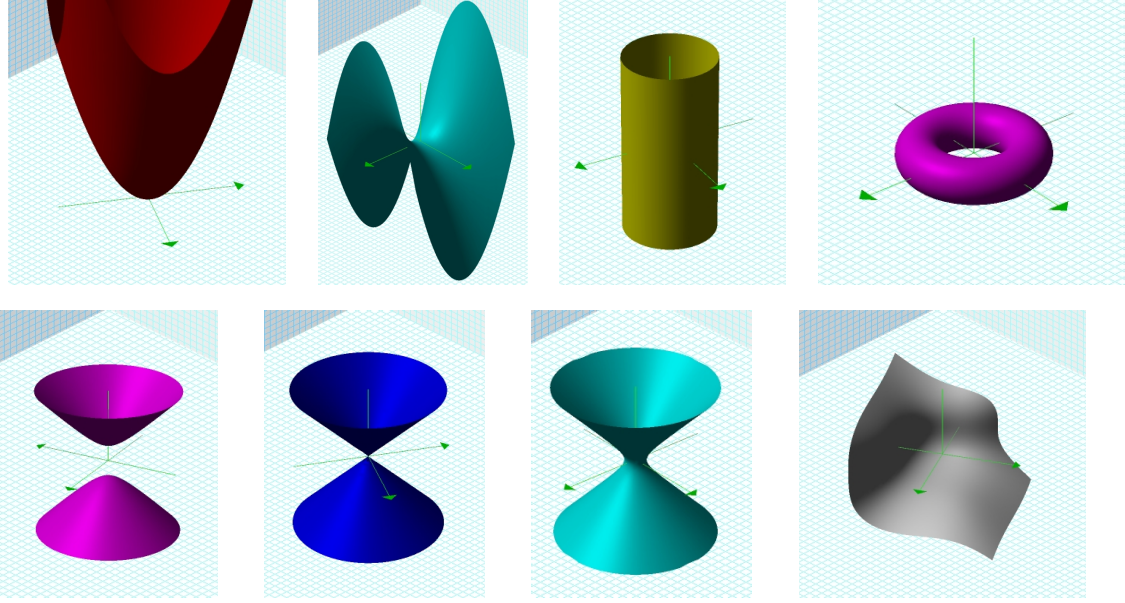


Figure 1.6: Examples of surfaces in $\mathbb{R}^3$. (a-b) Elliptic and hyperbolic paraboloid surface. (c) Cylinder. (d) Torus. (e-f-g) The locus $z^2 - x^2 - y^2 = \alpha$ is a one/two flaps hyperboloid surface when $\alpha \leq 0$; a bilateral cone when $\alpha = 0$. (h) The surface $x^3 + 2y^3 + 4z^3 - xy + 4 = 0$ of the exercise.

It follows that the level sets $X_{G,\alpha} = \{(x, y, z) \in \mathbb{R}^3 : G(x, y, z) = \alpha\}$ are manifolds of dimension $3 - 1 = 2$ (i.e. surfaces) $\mathcal{C}^\infty$ of $\mathbb{R}^3$ for any $\alpha \neq 0$: for $\alpha > 0$, $X_{G,\alpha}$ is a rotation hyperboloid surface with two flaps, and for $\alpha < 0$ with one flap. In the particular case $\alpha = 0$ the set $X_{G,0}$, defined by $z^2 = x^2 + y^2$, is a bilateral cone with vertex $(0, 0, 0)$: excepted this vertex, also the cone is a $\mathcal{C}^\infty$ manifold. **(3)** The spherical surface $\mathbb{S}^2$ of the points of $\mathbb{R}^3$ having unitary distance from the origin is a regular surface of $\mathbb{R}^3$. For $\mathbb{S}^2$ we can argue in a way totally similar to the case of $\mathbb{S}^1$; moreover, using the spherical coordinates (ρ, θ, φ) , a local parametrization is $\gamma :]0, 2\pi[\times]0, \pi[\rightarrow \mathbb{R}^3$ given by $\gamma(\theta, \varphi) = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$ (also in this case, the points of $\mathbb{S}^2$ on the half plane (x, z) with $x \geq 0$ are not included). **(4)** Fixed two numbers $R > r > 0$, in the plane (x, z) let us consider the circle c of radius r centered in $(R, 0, 0)$, and rotate it completely around the z -axis: the so-described surface is called *torus*⁽⁹⁾. Using the angle θ of cylindrical coordinates of $\mathbb{R}^3$ and the usual angle α which describes the position of the point on c , we obtain a parametrization of the torus with $\gamma :]0, 2\pi[\times]0, 2\pi[\rightarrow \mathbb{R}^3$ given by $\gamma(\theta, \alpha) = (x(\theta, \alpha), y(\theta, \alpha), z(\theta, \alpha)) = ((R + r \cos \alpha) \cos \theta, (R + r \cos \alpha) \sin \theta, r \sin \alpha)$, which does not include the points of the torus on the half plane (x, z) with $x > 0$ and the internal ones on the plane (x, y) . **(5)** Let us show that the equation $g(x, y, z) = x^3 + 2y^3 + 4z^3 - xy + 4 = 0$ defines a regular surface X in $\mathbb{R}^3$: namely, solving the system $\frac{\partial g}{\partial x} = \frac{\partial g}{\partial y} = \frac{\partial g}{\partial z} = 0$ we obtain the points $(0, 0, 0)$ and $(\frac{\sqrt[3]{4}}{6}, \frac{\sqrt[3]{2}}{6}, 0)$, but no one of these belongs to X (i.e. satisfies the equation $g(x, y, z) = 0$). Hence g is submersive at any point of X . Now let us determine a parametrization of the manifold X around its point $P(0, 0, -1)$. Note that at P both $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$ vanish, while $\frac{\partial g}{\partial z}(P) = 12 \neq 0$: then by Dini's Theorem there exists an open neighborhood U of $(0, 0)$ in the plane (x, y) and a unique function $z(x, y)$ defined and derivable in U which satisfies the equation $f(x, y, z(x, y)) = 0$ and such that $z(0, 0) = -1$. Hence $\gamma : U \subseteq \mathbb{R}^2 \rightarrow X$ given by $\gamma(x, y) = (x, y, z(x, y))$ is a local parametrization of X around P .

⁽⁹⁾The name “torus” means “ring” (it comes from the latin word *torum*, and not from *taurus*, the bull).

• The manifolds of dimension $n - 1$ in $\mathbb{R}^n$ are also called *(regular) hypersurfaces* of $\mathbb{R}^n$: in the plane $\mathbb{R}^2$ they are the regular curves, in the space $\mathbb{R}^3$ the regular surfaces. Hence they are locally the *graph of a scalar function of $n - 1$ variables*, or *level sets of scalar functions* $g(x_1, \dots, x_n)$.

Hypersurfaces

• The examples of the figure-eight curve (see p. 6 and the examples of p. 14 and later) and of the plane curve $\gamma(t) = (t(t-2), t^2(t-2))$ (see p. 8) show that a parametric curve in the usual sense could not be a regular curve at all of its points. Note that the parametrizations used in both cases are immersions (namely the derived vector never vanishes), hence by the Immersion Theorem 1.1.2 they are “locally injective”: but this is not enough, because the definition of manifold is based on the set, not on its parametrizations. It is then worth introducing a notion generalizing to higher dimensions the one of affine parametric curves.

An affine set $M \subset \mathbb{R}^n$ is said to be a *parametric manifold $\mathcal{C}^k$ of dimension m* in the “ambient spazio” $\mathbb{R}^n$ if there exists an immersion (called *parametrization*) $\gamma : V \rightarrow \mathbb{R}^n$ of class $\mathcal{C}^k$, where V is an open subset of $\mathbb{R}^m$, such that $M = \gamma(V)$.

Parametric manifold, parametrization

It is clear that parametric manifolds of dimension 1 are (the supports of) the affine parametric curves which are “well parametrized”, i.e. with $\gamma : I \rightarrow \mathbb{R}^n$ such that $\gamma'(t) \neq 0$ for any $t \in I$: as we already said at p. 6 this is equivalent to being an immersion.

Any manifold is (locally) a parametric manifold, while the converse is not always true, as it happens for the “figure-eight curve”: because of its non-local definition, a parametric manifold could have points where it is not locally diffeomorphic to an open subset of $\mathbb{R}^m$: the parametrization $\gamma(v)$, during the evolution of the parameter $v \in V$, could create topologically unsuitable situations as self-intersections or asymptotic approaches.

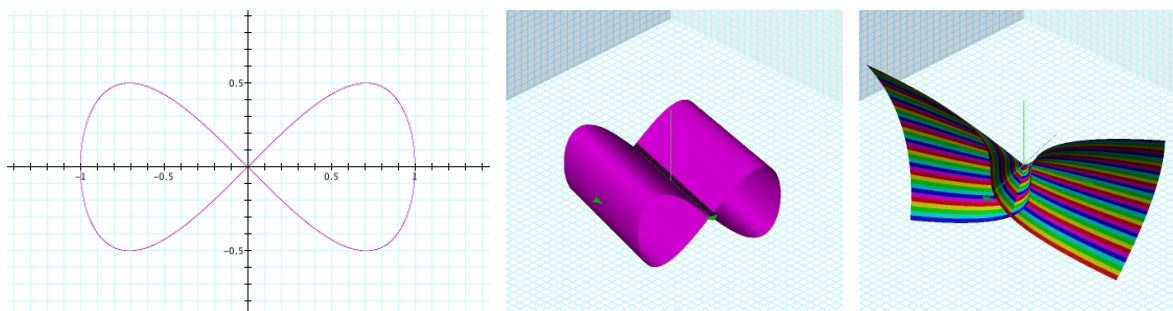


Figure 1.7: (a) The *figure-eight* curve, which (b), translated, gives a singular parametric surface. (c) Another singular parametric surface.

Examples. (1) The figure-eight curve (and other homeomorphic curves, as *Bernoulli's lemniscata*) is the most classical example, in dimension one, of singular parametric manifold: we refer to what has been said at p. 6 and in the examples of p. 14 and later. Moving the figure-eight curve (drawn in the (y, z) plane) along the y -axes we obtain an immediate example of parametric surface not regular at the points of the y -axis: it is parametrized by $\gamma : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ given by $\gamma(u, v) = (x(u, v), y(u, v), z(u, v)) = (\sin u, v, \cos u \sin u)$. **(2)** The function $\gamma : \mathbb{R}^2 \setminus \{(0, 0)\} \rightarrow \mathbb{R}^3$ given by $\gamma(u, v) = (u^2, 4u + v^3, 3uv)$ is an immersion (check it),

but its image is butterfly-shaped parametric surface in $\mathbb{R}^3$ (see Figure 1.7(c)), which is not regular at the points of self-intersection.

1.3 Tangent spaces

Let $M \subset \mathbb{R}^n$ be a manifold $\mathcal{C}^1$ of dimension m , and consider $x_0 \in M$. The *tangent space to M at x_0* is defined as the following subset of $\mathbb{R}^n$:

Tangent space

$$\mathbb{T}_{x_0} M = \{ \alpha'(t_0) : \alpha :]a, b[\rightarrow M \text{ derivable curve with } \alpha(t_0) = x_0 \},$$

i.e. the set of the tangent vectors at x_0 of affine parametric curves on M passing through x_0 . The following result shows that in fact $\mathbb{T}_{x_0} M$ is a m -dimensional vector subspace of $\mathbb{R}^n$, and how it can be computed both from a parametric or a cartesian form.

Proposition 1.3.1. *Let g be any defining function of class $\mathcal{C}^1$ of M at the neighborhood of x_0 , and $\gamma : V \rightarrow M$ any local parametrization of class $\mathcal{C}^1$ of M at the neighborhood of x_0 , where V is an open subset of $\mathbb{R}^m$ and $v_0 \in V$ such that $\gamma(v_0) = x_0$. then*

$$\begin{aligned} \mathbb{T}_{x_0} M &= \ker g'(x_0) = \{ u \in \mathbb{R}^n : g'(x_0)(u) = 0_{n-m} \} \\ &= \operatorname{im} \gamma'(v_0) = \{ \gamma'(v_0)(v) : v \in \mathbb{R}^m \}. \end{aligned}$$

In particular $\mathbb{T}_{x_0} M$ is a $\mathbb{R}$ -vector subspace of $\mathbb{R}^n$ of dimension m , which can be expressed both as the subspace orthogonal to the $n - m$ linearly independent gradients $\nabla g_1(x_0), \dots, \nabla g_{n-m}(x_0)$, or as the subspace spanned by the m linearly independent tangent vectors $\frac{\partial \gamma}{\partial v_1}(v_0), \dots, \frac{\partial \gamma}{\partial v_m}(v_0)$:

$$\mathbb{T}_{x_0} M = \langle \nabla g_1(x_0), \dots, \nabla g_{n-m}(x_0) \rangle^\perp = \left\langle \frac{\partial \gamma}{\partial v_1}(v_0), \dots, \frac{\partial \gamma}{\partial v_m}(v_0) \right\rangle.$$

Proof. Set $W_1 = \operatorname{im} \gamma'(v_0)$ and $W_2 = \ker g'(x_0)$. If we prove that (a) $W_1 \subset \mathbb{T}_{x_0} M$ and (b) $\mathbb{T}_{x_0} M \subset W_2$, we are done: namely it would follow that $W_1 \subset W_2$, and since both W_1 and W_2 are vector subspaces of dimension m of $\mathbb{R}^n$ (recall that γ is immersive at v_0 , and g submersive at x_0), this would yield $W_1 = W_2 (= \mathbb{T}_{x_0} M)$. The other statements are just explanations of the immediate consequences of that.

(a) Taken any $v \in \mathbb{R}^m$, since V is open there exists a $\varepsilon > 0$ such that the segment $\{v_0 + tv : t \in]-\varepsilon, \varepsilon[\}$ is contained in V . Let us consider the affine parametric curve $\varphi :]-\varepsilon, \varepsilon[\rightarrow \mathbb{R}^n$ given by $\varphi(t) = \gamma(v_0 + tv)$: then φ is a derivable curve with values in M (it is the restriction of γ to the segment), and it holds $\varphi(0) = x_0$; from $\varphi'(t) = \gamma'(v_0 + tv)(v)$ we get $\gamma'(v_0)(v) = \varphi'(0) \in \mathbb{T}_{x_0} M$. This shows that $W_1 \subset \mathbb{T}_{x_0} M$.

(b) Now let $\alpha :]a, b[\rightarrow M$ be any derivable curve with $\alpha(t_0) = x_0$. Since g vanishes on the points of M , it follows that $g \circ \alpha \equiv 0$; in particular, differentiating at t_0 , we obtain $g'(x_0)(\alpha'(t_0)) = 0$, i.e. $\alpha'(t_0) \in \ker g'(x_0)$. Hence $\mathbb{T}_{x_0} M \subset W_2$. $\square$

In concrete terms, $\mathbb{T}_{x_0} M$ is the vector subspace of dimension m in $\mathbb{R}^n$ parallel to the *affine tangent space to M at x_0* (of dimension m); the latter will then be

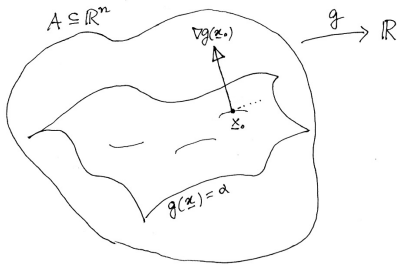
$$x_0 + \mathbb{T}_{x_0} M = \{ x_0 + \gamma'(v_0)(v) : v \in \mathbb{R}^m \} = \left\{ x \in \mathbb{R}^n : \begin{array}{l} \nabla g_1(x_0) \cdot (x - x_0) = 0 \\ \vdots \\ \nabla g_{n-m}(x_0) \cdot (x - x_0) = 0 \end{array} \right\}.$$

Let us show a remark about that, regarding manifolds in graph form.

Proposition 1.3.2. Let M be the graph of a function $\phi : A \rightarrow \mathbb{R}^{n-m}$ of class $\mathcal{C}^1$ (where A is an open subset of $\mathbb{R}^m$) and $x_0 = (x'_0, \phi(x'_0)) \in M$: then the affine tangent space to M at x_0 is the graph of the approximating affine function $\phi(x'_0) + \phi'(x'_0)(x_\lambda - x'_0)$.

Proof. Let us show the statemet in two ways. (i) the function $\gamma : A \rightarrow \mathbb{R}^n$ given by $\gamma(x_\lambda) = (x_\lambda, \phi(x_\lambda))$ is a global parametrization of M , and it holds $\gamma'(x'_0) = \begin{pmatrix} 1_m \\ \phi'(x'_0) \end{pmatrix}$: dunque $x_0 + \mathbb{T}_{x_0} M = \{x_0 + \gamma'(x'_0)(u) : u \in \mathbb{R}^m\} = \left\{ \begin{pmatrix} x_\lambda \\ x_\mu \end{pmatrix} = \begin{pmatrix} x'_0 \\ \phi(x'_0) \end{pmatrix} + \sum_{j=1}^m \lambda_j \begin{pmatrix} e_j \\ \frac{\partial \phi}{\partial x_\lambda j}(x'_0) \end{pmatrix} : \lambda_1, \dots, \lambda_m \in \mathbb{R} \right\}$. From the first m equations we get $\lambda_j = x_j - x_{0,j}$ (with $j = 1, \dots, m$), and inserting this information in the last $n - m$ gives $x_\mu = \phi(x'_0) + \sum_{j=1}^m \frac{\partial \phi}{\partial x_\lambda j}(x'_0)(x_j - x_{0,j}) = \phi(x'_0) + \phi'(x'_0)(x_\lambda - x'_0)$, as stated. (ii) A global defining function for M is $g : A \times \mathbb{R}^{n-m} \rightarrow \mathbb{R}^{n-m}$ given by $g(x_\lambda, x_\mu) = x_\mu - \phi(x_\lambda)$, and $g'(x'_0, \phi(x'_0)) = (-\phi'(x_\lambda), \mathbf{1}_{n-m})$: hence $x_0 + \mathbb{T}_{x_0} M = \left\{ x = \begin{pmatrix} x_\lambda \\ x_\mu \end{pmatrix} \in \mathbb{R}^n : (-\phi'(x_\lambda), \mathbf{1}_{n-m}) \begin{pmatrix} x_\lambda - x'_0 \\ x_\mu - \phi(x'_0) \end{pmatrix} = 0_{n-m} \right\} = \left\{ x = \begin{pmatrix} x_\lambda \\ x_\mu \end{pmatrix} \in \mathbb{R}^n : -\phi'(x_\lambda)(x_\lambda - x'_0) + x_\mu - \phi(x'_0) = 0_{n-m} \right\}$, i.e. once more $x_\mu = \phi(x'_0) + \phi'(x'_0)(x_\lambda - x'_0)$. $\square$

Examples. (1) Assume that M is a surface in $\mathbb{R}^3$ graph of a function $\phi(x, y)$, i.e. that $M = \{(x, y, z) : z = \phi(x, y)\}$. Taken a point $P = (x_0, y_0, \phi(x_0, y_0)) \in M$ we then get $\mathbb{T}_P M = \{(u, v, w) : (\frac{\partial \phi}{\partial x}(x_0, y_0)u + \frac{\partial \phi}{\partial y}(x_0, y_0)v, -1) \cdot (u, v, w) = 0\} = \{(u, v, w) : w = -\frac{\partial \phi}{\partial x}(x_0, y_0)u - \frac{\partial \phi}{\partial y}(x_0, y_0)v : u, v \in \mathbb{R}\}$, and hence the affine tangent plane to M at P is $P + \mathbb{T}_P M = \{(x_0 + u, y_0 + v, \phi(x_0, y_0) + \frac{\partial \phi}{\partial x}(x_0, y_0)u + \frac{\partial \phi}{\partial y}(x_0, y_0)v) : u, v \in \mathbb{R}\}$. If we eliminate the parameters u and v we recover the known cartesian equation of the plane tangent to the graph, i.e. $z = \phi(x_0, y_0) + \frac{\partial \phi}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial \phi}{\partial y}(x_0, y_0)(y - y_0)$. **(2)** Let us show that the set $M = \{(x, y, z) : g(x, y, z) = x^2 + 2y^2 + 3z^2 = 21\}$ of $\mathbb{R}^3$ is a regular surface of $\mathbb{R}^3$: namely $\nabla g(x, y, z) = (2x, 4y, 6z)$ vanishes only at the origin $(0, 0, 0)$ which does not belong to M . (It is easy to realize that M is an ellipsoid surface.) The point $(1, 2, 2)$ is in M , hence $\mathbb{T}_{(1,2,2)} M = \ker g'((1, 2, 2)) = \{(u, v, w) \in \mathbb{R}^3 : \nabla g(1, 2, 2) \cdot (u, v, w) = 0\} = \{(u, v, w) \in \mathbb{R}^3 : u + 4v + 6w = 0\}$: it is the subspace spanned for example by the vectors $(0, 3, -2)$ and $(6, 0, -1)$. The affine tangent space will be $\{(1, 2, 2) + \lambda(0, 3, -2) + \mu(6, 0, -1) : \lambda, \mu \in \mathbb{R}\} = \{(1 + 6\mu, 2 + 3\lambda, 2 - 2\lambda - \mu) : \lambda, \mu \in \mathbb{R}\}$, or also $\{(x, y, z) : \nabla g(1, 2, 2) \cdot (x - 1, y - 2, z - 2) = 0\} = \{(x, y, z) : x + 4y + 6z = 21\}$. **(3)** Consider the set $M = \{(x, y, z) \in \mathbb{R}^3 : h(x, y, z) = x + y + z = -5, xy + xz + yz = 8\}$. M is a regular curve in $\mathbb{R}^3$: namely $h'((x, y, z)) = \begin{pmatrix} 1 & 1 & 1 \\ y+z & x+z & x+y \end{pmatrix}$ has always rank 2 in the points of M (the rank lowers to 1 in the points where $x = y = z$, which is impossible in M). Let us compute $\mathbb{T}_{(-2,-1,-2)} M$: it is $\{(u, v, w) : g'((-2, -1, -2))(u, v, w) = (0, 0)\} = \{(u, v, w) : u + v + w = 0, -3u - 4v - 3w = 0\} = \{(u, 0, -u) : u \in \mathbb{R}\}$, hence the tangent line to M at $(-2, -1, -2)$ is $\{(-2, -1, -2) + (u, 0, -u) : u \in \mathbb{R}\} = \{(u - 2, -1, -u - 2) : u \in \mathbb{R}\}$ or also $\{(x, y, z) : g'((-2, -1, -2))(x + 2, y + 1, z + 2) = (0, 0)\} = \{(x, y, z) : x + y + z + 6 = 0, 3x + 4y + 3z + 16 = 0\}$.



Proposition 1.3.1 illustrates the *geometric meaning of the gradient*: if A is an open subset of $\mathbb{R}^n$, $x_0 \in A$ and $g : A \rightarrow \mathbb{R}$ is a scalar differentiable function with $\nabla g(x_0) \neq 0_n$, setting $\alpha = g(x_0)$ one has that the level set $g(x) = \alpha$ is a regular hypersurface of A at the neighborhood of x_0 , and the gradient $\nabla g(x_0)$ represents the *orthogonal vector* to that hypersurface at x_0 , oriented towards the increasing values of g .

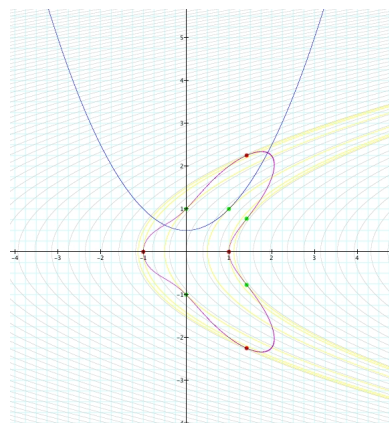
Geometric meaning of the gradient

1.4 Local and global extrema on manifolds

Let $M \subset \mathbb{R}^n$ be a manifold $\mathcal{C}^1$ of dimension m , and let $f : M \rightarrow \mathbb{R}$ a function. We now investigate how to determine possible *points of local extremum* of f on M ; moreover, if f is continuous and M is compact, Weierstrass Theorem ensures the presence of *points of global extremum*.

In the favorable case where we could represent both the manifold and the level sets of the function f of which we are looking for the extrema on M , there is a natural “graphic visualization” of the solution of such problem: it will be those points of M which, locally or globally on M , belong to level sets of f at minimal or maximal value. Of course, beyond the fact that this visualization be possible or not, the problem of concrete computation is still present.

Example. Let us look for extrema of the function $f(x, y) = 2x - y^2$ on the parabola M_1 given by $x^2 + 1 - 2y = 0$ (blue) and on the cartesian curve M_2 defined by $x^4 + y^4 - 4xy^2 = 1$ (porpora). The level curves of f are of type $f(x, y) = 2x - y^2 = k$ for $k \in \mathbb{R}$: it is a family of parallel parabolas (represented in gray) whose axis is the x -axis, with increasing values in the right-hand direction. The simple exam of the reciprocal positions of any of these two curves and of the level curves of f (the remarkable ones are in yellow) allows us to detect the points of the curves M_1 and M_2 which give local extrema for f , although we are still not able to compute them. Hence it is evident that on M_1 there is a unique point (in green) of local –in this case, even global– maximum and no points of local minimum for f . As for M_2 , the green (resp. red) points are of local maximum (resp. minimum) for f on M_2 ; moreover, since M_2 is compact, we already know that f will reach absolute extremes on it, and that the figure suggests that the absolute minimum is reached in the most-left red point of M_2 , and the absolute maximum in the most-right green points. Anyway, we will soon be able to compute precisely these points, and the computation will confirm these qualitative estimates.



If a point $x_0 \in M$ is a of local extremum for f in M , and $\varphi : I \rightarrow M$ is any derivable curve passing through x_0 (say with $\varphi(t_0) = x_0$), it is clear that t_0 will be a local extremum for the restriction $f \circ \varphi$: in particular $(f \circ \varphi)'(t_0) = f'(x_0)(\varphi'(t_0)) = 0$, i.e. $\varphi'(t_0) \in \ker f'(x_0)$. It is then natural to say that a point $x_0 \in M$ is *stationary* for f in M if

$$T_{x_0} M \subset \ker f'(x_0), \quad \text{i.e.} \quad \nabla f(x_0) \cdot u = 0 \quad \text{for any } u \in T_{x_0} M .$$

Note that, if M is an open subset of $\mathbb{R}^n$ (i.e. a manifold of maximal dimension n), then $T_{x_0} M = \mathbb{R}^n$, and we recover the definition of stationary point for functions defined on open subsets (i.e. such that $f'(x_0) = 0$).

Proposition 1.4.1. *Points of local extremum for f in M are stationary for f in M .*

Proof. The definition of stationary point has been constructed expressly for that, as it should be clear from what we have just said above here. $\square$

Hence we should first of all determine the stationary points for f in M .

Since this is a local problem, we may assume that M be defined (a) by a single parametrization, or (b) by a single defining function.

- (a) If M is defined by a single parametrization $\gamma : V \rightarrow M \subset \mathbb{R}^m$ where V is open in $\mathbb{R}^m$, the problem is immediately reduced to the known one of looking for points of local extremum for functions with open domain in an affine space:

Proposition 1.4.2. *Let $x_0 = \gamma(v_0)$ be a point of M . Then x_0 is a stationary point for f if and only if v_0 is so for $f \circ \gamma$; and is a point of local maximum (or minimum) for f if and only if v_0 is so for $f \circ \gamma$.*

Proof. From $(f \circ \gamma)'(v_0) = f'(x_0) \circ \gamma'(v_0)$ we get that $(f \circ \gamma)'(v_0) = 0$ (i.e. v_0 is stationary for $f \circ \gamma$) if and only if $T_{x_0}M = \text{im } \gamma'(v_0) \subset \ker f'(x_0)$ (i.e. x_0 is stationary for f). The rest of the statement is obvious. $\square$

Example. The parabola M_1 of equation $x^2 + 1 - 2y = 0$ of the previous example is easily parametrizable as graph by $\gamma(x) = (x, \frac{1}{2}(x^2 + 1))$ with $x \in \mathbb{R}$: therefore, to look for extrema of $f(x, y) = 2x - y^2$ on M_1 is equivalent to look for extrema of $F(x) := (f \circ \gamma)(x) = f(x, \frac{1}{2}(x^2 + 1)) = 2x - \frac{1}{4}(x^2 + 1)^2$ on $\mathbb{R}$. By deriving we obtain $F'(x) = 2 - x^3 - x = -(x-1)(x^2 + x + 2)$: since $F'(x) = 0$ for $x = 1$ and $F'(x) > 0$ for $x < 1$, we have that $x = 1$ is a point of strict absolute maximum for $F(x)$, with value $F(1) = 1$, and hence the point $\gamma(1) = (1, 1)$ is of absolute maximum for f on M_1 (with $f(1, 1) = F(1) = 1$), as the figure was showing.

- (b) Let us assume instead that M be described as the zero set of a single defining function $g : U \rightarrow \mathbb{R}^{n-m}$ with U open subset of $\mathbb{R}^n$ containing M , i.e. that $M = \{x \in U : g_1(x) = \dots = g_{n-m}(x) = 0\}$ with g submersive on M .

Henceforth we shall assume that $f : M \rightarrow \mathbb{R}$ be of class $\mathcal{C}^1$, which means that locally f is induced by a function of class $\mathcal{C}^1$ on an open subset of $\mathbb{R}^n$. Up to restricting the subset, we may assume that f be defined on the whole U . In particular, we may perform the partial derivatives of f , and for any $x_0 \in M$ we may consider the differential $f'(x_0) : \mathbb{R}^n \rightarrow \mathbb{R}$.

Theorem 1.4.3. (Lagrange Multipliers)⁽¹⁰⁾ *A point $x \in M$ is stationary for f in M if and only if the gradient $\nabla f(x)$ is generated by the $n - m$ gradients $\nabla g_1(x), \dots, \nabla g_{n-m}(x)$, i.e. if and only if there exist $n - m$ real numbers $\lambda_1, \dots, \lambda_{n-m}$ such that the $(2n - m)$ -tuple $(x, \lambda_1, \dots, \lambda_{n-m})$ satisfies the vectorial system*

$$\begin{cases} \nabla f(x) &= \lambda_1 \nabla g_1(x) + \dots + \lambda_{n-m} \nabla g_{n-m}(x) \\ g(x) &= 0, \end{cases}$$

⁽¹⁰⁾In the case where M is an open subset of $\mathbb{R}^n$ (i.e. when $m = n$), Theorem 1.4.3 reduces to the usual result for stationary points of functions with open domains.

which is equivalent to the system of $(2n - m)$ scalar equations

$$\begin{cases} \frac{\partial f}{\partial x_1}(x) &= \lambda_1 \frac{\partial g_1}{\partial x_1}(x) + \dots + \lambda_{n-m} \frac{\partial g_{n-m}}{\partial x_1}(x) \\ \vdots & \\ \frac{\partial f}{\partial x_n}(x) &= \lambda_1 \frac{\partial g_1}{\partial x_n}(x) + \dots + \lambda_{n-m} \frac{\partial g_{n-m}}{\partial x_n}(x) \\ g_1(x) &= 0 \\ \vdots & \\ g_{n-m}(x) &= 0. \end{cases}$$

Proof. A point $x \in M$ is stationary for f in M if and only if $T_x M \subset \ker f'(x)$, i.e. if and only if the gradient $\nabla f(x)$ is a vector orthogonal to the subspace $T_x M$, i.e. if and only if $\nabla f(x)$ is contained in the orthogonal subspace $T_x M^\perp = \{w \in \mathbb{R}^n : v \cdot w = 0 \text{ for any } v \in T_x M\}$. But $T_x M^\perp$ has dimension $n - m$,⁽¹¹⁾ hence it is generated by the $n - m$ linearly independent vectors $\nabla g_1(x), \dots, \nabla g_{n-m}(x)$, which are already known to be inside it: as a consequence, a point $x \in M$ is stationary for f in M if and only if the gradient $\nabla f(x)$ is generated by the $n - m$ gradients $\nabla g_1(x), \dots, \nabla g_{n-m}(x)$, but this is what we had to prove. $\square$

The existence of such a $(n - m)$ -tuple of numbers $(\lambda_1, \dots, \lambda_{n-m}) \in \mathbb{R}^{n-m}$ is equivalent to require that the matrix of $M_{n-m+1,n}(\mathbb{R})$ (i.e., with $n - m + 1$ rows and n columns) whose rows are formed by $\nabla f(x)$ and by $\nabla g_1(x), \dots, \nabla g_{n-m}(x)$, has rank $\leq n - m$ (and hence rank $= n - m$, since by submersivity the gradients $\nabla g_1(x), \dots, \nabla g_{n-m}(x)$ are independent), i.e. that *all its minors of order $n - m + 1$ be singular*: this gives the advantage of reducing the $2n - m$ unknowns to the n coordinates of x , which are in fact the only interesting things to determine.⁽¹²⁾

Once determined the stationary points for f in M , to determine the character of any of them it is necessary to reduce to the case of functions on an open domain (e.g. in order to use the Hessian criterion in the case of $\mathcal{C}^2$ regularity), and hence it will be necessary to find a local parametrization around each stationary point.

Examples. (1) We start by treating the problem (seen in the pictorial visualization a little above) of studying the extrema of the function $f(x, y) = 2x - y^2$ on the cartesian curve M_2 defined by $x^4 + y^4 - 4xy^2 = 1$. Let us verify that M_2 is a regular curve: if $g(x, y) = x^4 + y^4 - 4xy^2$ then $M_2 = g^{-1}(1)$, and g is submersive in

⁽¹¹⁾In general one has the following result, for which we refer to any text of linear algebra: *Let b be a bilinear form on $\mathbb{R}^n$, and for a vector subspace $V \subset \mathbb{R}^n$ define the b -orthogonal $V^{\perp_b} = \{w \in \mathbb{R}^n : b(v, w) = 0 \text{ for any } v \in V\}$. If b is nondegenerate, i.e. if $(\mathbb{R}^n)^{\perp_b} = \{0\}$, then for any vector subspace $V \subset \mathbb{R}^n$ it holds $\dim V + \dim V^{\perp_b} = n$.*

⁽¹²⁾At a first glance it would seem that to replace the n scalar equations $\frac{\partial f}{\partial x_j}(x) = \lambda_1 \frac{\partial g_1}{\partial x_j}(x) + \dots + \lambda_{n-m} \frac{\partial g_{n-m}}{\partial x_j}(x)$ (with $j = 1, \dots, n$) by the equations expressing the fact that all minors of order $n - m + 1$ of the matrix of $M_{n-m+1,n}(\mathbb{R})$ whose rows are formed by $\nabla f(x)$ and by $\nabla g_1(x), \dots, \nabla g_{n-m}(x)$ be singular would cause a huge increasing of the number of equations: in fact, from a matrix in $M_{n-m+1,n}(\mathbb{R})$ one could extract $\binom{n}{n-m+1}$ minors of order $n - m + 1$. Nevertheless, since (as we said) we already know that this matrix has rank $n - m$, only $n - (n - m) = m$ of these equations are independent^(*), which joined to the $n - m$ equations “of constraints” $g_1(x) = \dots = g_{n-m}(x) = 0$ give a total of n equations, suitably destined to determine the n components of the point x . [Proof of (*): if a matrix $(m_{i,j}) \in M_{n-m+1,n}(\mathbb{R})$ has e.g. the minor made by the last $n - m$ rows and the first $n - m$ columns which is nonsingular, then this matrix has rank $n - m$ if and only if $m_{1,j} = 0$ for any $j = n - m + 1, \dots, n$, which are in fact m independent equations].

the points of M_2 (namely $dg(x, y) = (4(x^3 - y^2), 4y(y^2 - 2x)) : \mathbb{R}^2 \rightarrow \mathbb{R}$ is not surjective only in $(0, 0)$ and in $(\sqrt{2}, \mp\sqrt{8})$, and no one of these points belongs to M_2). The Lagrange method gives the system of the equations $\det \begin{pmatrix} 4(x^3 - y^2) & 4y(y^2 - 2x) \\ -2y & -2x \end{pmatrix} = 0$ (i.e. $xy(2 - x^2) = 0$) and $x^4 + y^4 - 4xy^2 = 1$: if $x = 0$ we obtain $y = \mp 1$ (hence the points $A_1(0, 1)$ and $A_2(0, -1)$); if $y = 0$ we obtain $x = \mp 1$ (hence the points $B_1(1, 0)$ and $B_2(-1, 0)$); if $x = \sqrt{2}$ we obtain $y^4 - 4\sqrt{2}y^2 + 3 = 0$, i.e. $y = \pm\sqrt{2\sqrt{2} + \sqrt{5}}$ or $y = \pm\sqrt{2\sqrt{2} - \sqrt{5}}$ (hence the points $C_1(\sqrt{2}, \sqrt{2\sqrt{2} + \sqrt{5}})$, $C_2(\sqrt{2}, -\sqrt{2\sqrt{2} + \sqrt{5}})$, $C_3(\sqrt{2}, \sqrt{2\sqrt{2} - \sqrt{5}})$ and $C_4(\sqrt{2}, -\sqrt{2\sqrt{2} - \sqrt{5}})$); finally, if $x = -\sqrt{2}$ we obtain $y^4 + 4\sqrt{2}y^2 + 3 = 0$, which has no real solutions. To understand the nature of any of these eight points we should use a local chart of M_2 around them: for example, let us check the nature of $A_2(0, -1)$. We have $dg(0, -1) = (-4, -4)$, hence from $g(x, y) \equiv 1$ we may make explicit e.g. $y(x)$ with $y(0) = -1$; by derivin gwith respect to x the identity $g(x, y(x)) = 1$ we have $4x^3 + 4y^3y' - 4y^2 - 8xyy' \equiv 0$, from which (setting $x = 0$) we obtain $-4y'(0) - 4 = 0$, hence $y'(0) = -1$; by deriving again we have $12x^2 + 12y^2(y')^2 + 4y^3y'' - 8yy' - 8yy' - 8x(y')^2 - 8xyy'' \equiv 0$, and hence (setting $x = 0$) we obtain $12 - 4y''(0) - 16 = 0$, which yields $y''(0) = -1$. The derivatives of the function $F(x) := f(x, y(x))$ are $F'(x) = 2 - 2yy'$ and $F''(x) = -2((y')^2 + yy'')$, which calculated in $x = 0$ with $y(0) = -1$ yield $F'(0) = 0$ (as expected) and $F''(x) = -4 < 0$: hence A_2 is a point of local maximum for f on M_2 , as it was clear from the figure.

(2) Let us determine the extrema of $f(x, y) = xy$ on $M = \{(x, y) \in \mathbb{R}^2 : x^2 + xy + y^2 = 1\}$ (showing also that f has absolute extrema on M). We verify that M is a regular curve: if $g(x, y) = x^2 + xy + y^2 - 1$ then $M = g^{-1}(0)$, and g is submersive in the points of M (namely $dg(x, y) = (2x + y, x + 2y) : \mathbb{R}^2 \rightarrow \mathbb{R}$ is not surjective only in $(0, 0) \notin M$). The Lagrange method yields the solutions $(x, y) = (\pm\frac{\sqrt{3}}{3}, \pm\frac{\sqrt{3}}{3})$ and $(\pm 1, \mp 1)$. To get a local chart around (for example) $A(1, -1)$, since $\frac{\partial g}{\partial y}(A) = -1 \neq 0$ we may make explicit $y = y(x)$: then $\frac{\partial g}{\partial x}(x, y(x)) = 2x + y(x) + xy'(x) + 2y(x)y'(x) \equiv 0$ (and so, since $y(1) = -1$, we get $y'(1) = 1$) and $\frac{\partial^2 g}{\partial x^2}(x, y(x)) = 2 + 2y'(x) + xy''(x) + 2(y'(x))^2 + 2y(x)y''(x) \equiv 0$ (hence $y''(1) = 6$). Considering then $\psi(x) := f(x, y(x)) = xy(x)$, we get $\psi'(x) = y(x) + xy'(x)$ (hence $\psi'(1) = 0$, as it was expected since A is stationary) and $\psi''(x) = 2y'(x) + xy''(x)$, which yields $\psi''(1) = 8 > 0$: hence $x = 1$ is a point of strict local minimum for ψ , and it follows that A is a point of strict local minimum for f on M . We also note that D is compact: namely it is closed, and it is also bounded since the equation (in x) $x^2 + yx + (y^2 - 1) = 0$ admits real solutions if and only if $|y| \leq \frac{2}{\sqrt{3}}$, and idem for x . Hence f admits *absolute* extrema on M , which will be $f(\pm\frac{\sqrt{3}}{3}, \pm\frac{\sqrt{3}}{3}) = \frac{1}{3}$ and $f(\pm 1, \mp 1) = -1$.

(3) Let us determine the extrema of $f(x, y, z) = x^2 - yz$ on $M = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 5, xy + xz + yz = 8\}$ which, as we have seen, is a regular curve in $\mathbb{R}^3$. The Lagrange method (which here says that the determinant of the matrix made by $\nabla f(x, y, z) = (2x, -z, -y)$, $\nabla g_1(x, y, z) = (1, 1, 1)$ and $\nabla g_2(x, y, z) = (y + z, x + z, x + y)$ be zero, beyond that $g_1(x, y, z) = g_2(x, y, z) = 0$) gives the solutions $A(1, 2, 2)$ and $B(\frac{2}{3}, \frac{4}{3}, \frac{4}{3})$. Observing $g'(A)$, at the neighborhood of A from the system $g_1(x, y, z) = g_2(x, y, z) = 0$ we may make x and y explicit in function of z , with $x(2) = 1$ and $y(2) = 2$; by deriving the identity $g_1(x(z), y(z), z) = g_2(x(z), y(z), z) \equiv 0$ twice with respect to z and setting $z = 2$ we obtain $x'(2) = 0$, $y'(2) = -1$, $x''(2) = 2$ and $y''(2) = -2$. Setting then $\psi(z) := f(x(z), y(z), z)$ (function of the variable z , defined in a neighborhood of $z = 2$ in $\mathbb{R}$), we must understand what is the character of $z = 2$ for ψ . By deriving we obtain $\psi'(z) = 2xx' - y'z - y$ and $\psi''(z) = 2(x')^2 + 2xx'' - y''z - 2y'$, hence $\psi'(2) = 0$ (obviously) and $\psi''(2) = 10$. We deduce that $z = 2$ is a strict local minimum for ψ , and hence A is a point of strict local minimum for f on M . Proceeding analogously, we obtain that also B is a point of strict local minimum for f on M .

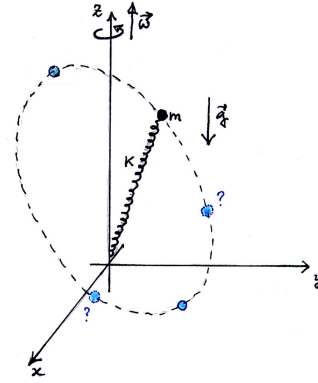
(4) Let us determine the extrema of $f(x, y, z) = 2y + z$ on $M = \{(x, y, z) \in \mathbb{R}^3 : x^2 + 2y^2 + 3z^2 = 21\}$. M is a manifold of dimension 2 (i.e., a *smooth manifold*) in $\mathbb{R}^3$: namely, if $g(x, y, z) = x^2 + 2y^2 + 3z^2 - 21$ we have that $g'(x, y, z) = (2x, 4y, 6z)$ has always rank 1 in the points of M (to have rank 0 it should be $x = y = z = 0$, which is impossible in M). It is also evident that M is compact (it is an ellipsoid surface), hence f will have absolute extrema on it. The

Lagrange method (which here yields $(0, 2, 1) = \lambda(2x, 4y, 6z)$ and $g(x, y, z) = 0$) gives the solutions $\pm D$, with $D = (0, 3, 1)$. Since $f(D) = 7 = -f(-D)$, it is already clear that the extrema of f in M are ± 7 . Just for exercise let us check directly that D is a point of maximum, by using a chart. Since $dg(D) = (0, 12, 6)$, from $g(x, y, z) = 0$ we may make $y = y(x, z)$ locally explicit in D , with $y(0, 1) = 3$; by deriving partially the identity $g(x, y(x, z), z) \equiv 0$ with respect to x and z we obtain $2x + 4y \frac{\partial y}{\partial x} = 0$, $4y \frac{\partial y}{\partial z} + 6z = 0$, $2 + 4(\frac{\partial y}{\partial x})^2 + 4y \frac{\partial^2 y}{\partial x^2} = 0$, $4(\frac{\partial y}{\partial z} \frac{\partial y}{\partial x} + y \frac{\partial^2 y}{\partial x \partial z}) = 0$ and $4(\frac{\partial y}{\partial z})^2 + 4y \frac{\partial^2 y}{\partial z^2} + 6 = 0$, da cui $\frac{\partial y}{\partial x}(0, 1) = 0$, $\frac{\partial y}{\partial z}(0, 1) = -\frac{1}{2}$, $\frac{\partial^2 y}{\partial x^2} y(0, 1) = -\frac{1}{6}$, $\frac{\partial^2 y}{\partial x \partial z}(0, 1) = 0$ and $\frac{\partial^2 y}{\partial z^2}(0, 1) = -\frac{1}{3}$. Setting $\psi(x, z) := f(x, y(x, z), z) = 2y(x, z) + z$ (a function of (x, z) , defined in a neighborhood of $(x, z) = (0, 1)$ in $\mathbb{R}^2$), we study the character of $(0, 1)$ for ψ . By deriving we obtain $\frac{\partial \psi}{\partial x} = 2 \frac{\partial y}{\partial x}$, $\frac{\partial \psi}{\partial z} = 2 \frac{\partial y}{\partial z} + 1$ (hence $\nabla \psi(0, 1) = (0, 0)$, as expected), $\frac{\partial^2 \psi}{\partial x^2} = 2 \frac{\partial^2 y}{\partial x^2}$, $\frac{\partial^2 \psi}{\partial x \partial z} = 2 \frac{\partial^2 y}{\partial x \partial z}$ and $\frac{\partial^2 \psi}{\partial z^2} = 2 \frac{\partial^2 y}{\partial z^2}$: hence $H_\psi((0, 1))$ is negative defined, and this confirms that D is a point of maximum.

Before ending this topic, it is natural to explain a little about the physical problem that led Lagrange to investigate how to determine the stationary points of functions on cartesian manifolds.

The Lagrange method and the equilibria of a system with fixed and smooth constraints. In Analytical Mechanics, if x represents the configuration of a mechanical system subject to a certain number of independent and time-fixed constraints (the so-called *manifold of constraints* $g_1(x) = \dots = g_{n-m}(x) = 0$ with $g'(x)$ submersive —i.e. $\nabla g_1(x), \dots, \nabla g_{n-m}(x)$ linearly independent— for any x satisfying the constraints) and smooth, in the sense that they can generate only normal reaction (in other words, frictions etc. are excluded), and $f(x)$ represent the potential energy associated to some conservative forces acting on the system, the Theorem of Lagrange Multipliers says that the equilibria

(i.e. the stationary points which satisfy the constraints) appear where the active force $F(x) = -\nabla f(x)$ is normal to the constraints, i.e. where it is possible to express it as linear combination of the gradients $\nabla g_1(x), \dots, \nabla g_{n-m}(x)$: namely this force is exactly balanced by the reaction of constraints, and there is no residual action. Such equilibria will then be stable or unstable depending on the fact that they are points of local maximum or minimum of the energy with respect to the constraints. • For example (see the figure), let us consider a point (x, y, z) of mass m constrained to move without frictions on the curve (ellipse) given by the intersection between the paraboloid surface $g_1(x, y, z) = 2z - x^2 - y^2 = 0$ and the plane $g_2(x, y, z) = x + y + z - 1 = 0$, and subject to gravity, to an elastic force of coefficient k with origin in $(0, 0, 0)$ and to an apparent centrifugal force due to a uniform rotation around the z -axis with angular speed ω : the potential energy is then $f(x, y, z) = mgz + \frac{1}{2}kz^2 + \frac{1}{2}(k - m\omega^2)(x^2 + y^2)$. The Lagrange condition is expressed in this case by saying that the determinant of the matrix made by ∇f , ∇g_1 and ∇g_2 must be zero, i.e. $(kz + mg + k - m\omega^2)(x - y) = 0$. If $x = y$, from the constraints $g_1 = g_2 = 0$ we get $(x, y, z) = (\mp\sqrt{2} - 1, \mp\sqrt{2} - 1, 3 \pm 2\sqrt{2})$, the obvious equilibria E_- and E_+ in the lower and upper extremities of the ellipse. On the other hand, the factor $kz + mg + k - m\omega^2$ vanishes if and only if $z_{\min} = 3 - 2\sqrt{2} \leq \frac{m\omega^2 - mg - k}{k} \leq z_{\max} = 3 + 2\sqrt{2}$, which is equivalent to $\omega_- \leq \omega \leq \omega_+$ with $\omega_\pm := \sqrt{g + (4 \pm 2\sqrt{2}) \frac{k}{m}}$ (a “tempered” rotation, in a suitable interval): in this case one has $z = z_0 = \frac{m(\omega^2 - g)}{k} - 1$, and from the constraints we get two more equilibria E_1 and E_2 , symmetrical with respect to the plane $x = y$. The study



of stability of the equilibria requires to understand the nature of the stationary points: to this aim we note that the constraint projects on the circle $x^2 + y^2 + 2x + 2y - 2 = 0$, of center $(-1, -1)$ and radius 2, hence can be parametrized by $\gamma(\theta) = (-1 + 2\cos\theta, -1 + 2\sin\theta, 3 - 2(\cos\theta + \sin\theta))$. By composing the energy $f(x, y, z)$ with $\gamma(\theta)$ we obtain $F(\theta) = (mg + k - m\omega^2)(3 - 2(\cos\theta + \sin\theta)) + \frac{1}{2}k(3 - 2(\cos\theta + \sin\theta))^2$, whose derivative $F'(\theta) = 2(\sin\theta - \cos\theta)(mg + k - m\omega^2 + 3k - 2k(\cos\theta + \sin\theta))$ as expected vanishes when $\sin\theta = \cos\theta$ (i.e. $\theta = \frac{\pi}{4}$ and $\theta = \frac{5\pi}{4}$, corresponding resp. to the equilibria E_- and E_+), and possibly when $\cos\theta + \sin\theta = \frac{m(g-\omega^2)}{2k} + 2$ (i.e. $\cos(\theta - \frac{\pi}{4}) = \frac{m(g-\omega^2)}{2\sqrt{2}k} + \sqrt{2}$, the two other values θ_1, θ_2 of θ symmetrical with respect to $\frac{\pi}{4}$, corresponding to the equilibria E_1 and E_2 which, as seen, exist if and only if $\omega_- \leq \omega \leq \omega_+$). Deriving again we get $F''(\theta) = 2[(\cos\theta + \sin\theta)(mg - m\omega^2 + 4k - 2k(\cos\theta + \sin\theta)) + 2k(\sin\theta - \cos\theta)^2]$: since $F''(\frac{\pi}{4}) = 2\sqrt{2}(mg - m\omega^2 + (4 - 2\sqrt{2})k)$, $F''(\frac{5\pi}{4}) = -2\sqrt{2}(mg - m\omega^2 + (4 + 2\sqrt{2})k)$ and $F''(\theta_j) = 4k(\sin\theta_j - \cos\theta_j)^2 \geq 0$ (for $j = 1, 2$), we see that for $\omega < \omega_-$ ("slow" rotation) E_- is stable, E_+ unstable while E_1 and E_2 do not exist; for $\omega_- \leq \omega \leq \omega_+$ ("tempered" rotation) both E_- and E_+ become unstable while E_1 and E_2 exist and are stable; finally, for $\omega > \omega_+$ ("rapid" rotation) E_- is unstable, E_+ becomes stable and E_1 and E_2 do not exist. We invite the reader to complete the study of stability in the critical values $\omega = \omega_{\mp}$, where the equilibria E_1 and E_2 coincide with E_- and E_+ .

A frequent situation is when the function $f : D \rightarrow \mathbb{R}$ has compact domain $D \subset \mathbb{R}^n$, and we must search the absolute extrema of f on D , whose existence is ensured by Weierstrass Theorem. If f is differentiable and D can be decomposed into a *disjoint finite union* $D_1 \sqcup \dots \sqcup D_r$ of *manifolds of class $\mathcal{C}^1$* , an effective method is to *look, using the methods seen above, for the stationary points of f on any of the manifolds $D_1, \dots, D_r$, and then to compare the values of f in those points*: namely, if a point $x \in D$ is of absolute extremum for f on D , then necessarily it will be so also for f on the component D_{j_0} where it belongs, and hence it should appear among the stationary points of f on D_{j_0} .

Searching the
absolute extrema
on compact sets

- The simplest case of decomposable compact subset is the one of *compact manifold with boundary*, o i.e. a compact subset $X \subset \mathbb{R}^n$ locally diffeomorphic to an open subset of $\mathbb{R}^m$ (in its "internal points" $\dot{X}$, which form a manifold of dimension m in the usual sense) or to an open subset of the closed half space $\mathbb{H}^m = \{(v_1, \dots, v_m) \in \mathbb{R}^m : v_m \geq 0\}$ (in the points of its "boundary" ∂X which, taken by itself, is a manifold of dimension $m - 1$). For more details about manifolds with boundary we refer to Section 3.

Examples. The following examples of compact manifold with boundary are visualized in Figure 1.8. **(1)** Let us determine the extrema of $f(x) = x^3 - x$ on the compact interval $X = [-2, 0]$. X is a 1-manifold con bordo, con $\dot{X} =]-2, 0[$ (open interval, a 1-manifold in the usual sense) and $\partial X = \{-2, 0\}$ (discrete set, a 0-manifold). To detect the stationary points of f on $\dot{X}$ it is enough to derive: since $f'(x) = 3x^2 - 1 = 0$ for $x = \pm\frac{\sqrt{3}}{3}$, the unique stationary point in $\dot{X}$ is $x_1 = -\frac{\sqrt{3}}{3}$. On the other hand, all points of a 0-manifold are stationary by definition, hence in our case so are $x_2 = -2$ and $x_3 = 0$. Since $f(x_1) = \frac{2}{9}\sqrt{3}$, $f(x_2) = -6$ and $f(x_3) = 0$, the absolute minimum of f on X is -6 (in $x_2 = -2$) while the absolute maximum is $\frac{2}{9}\sqrt{3}$ (in $x_1 = -\frac{\sqrt{3}}{3}$). **(2)** Let us determine the extrema of $f(x, y) = x^2 - xy$ on the closed disk $\mathbb{D}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. $\mathbb{D}^1$ is a manifold with boundary, of dimension 2: note that the open disk $\mathbb{D}^1$ (an open subset of $\mathbb{R}^2$) is a 2-manifold in the usual sense, while the boundary $\partial\mathbb{D}^1 = \mathbb{S}^1$ is a regular curve. In the open subset $\mathbb{D}^1$ the stationary points of f can be detected by means of the partial derivatives: the only solution of $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$ is the origin $O(0, 0)$. For the boundary $\partial\mathbb{D}^1$ we can use the parametrization $\gamma(\theta) = (\cos\theta, \sin\theta)$ or the Lagrange method, but we better use the former (with Lagrange the computations turn out to be heavy):

we have $F(\theta) := f(\cos \theta, \sin \theta) = \cos^2 \theta - \cos \theta \sin \theta = \frac{1}{2}(\cos 2\theta - \sin 2\theta + 1)$, hence $F'(\theta) = -\sin 2\theta - \cos 2\theta$, hence $F'(\theta) = 0$ for $\tan 2\theta = -1$, i.e. $2\theta = -\frac{\pi}{4} + \mathbb{Z}\pi$, i.e. $\theta = -\frac{\pi}{8} + \mathbb{Z}\frac{\pi}{2}$. Therefore we have four points A_1, A_2, A_3, A_4 , of the unitary circle with arguments $\theta_1 = -\frac{\pi}{8}$, $\theta_2 = \frac{3\pi}{8}$, $\theta_3 = \frac{7\pi}{8}$ and $\theta_4 = \frac{11\pi}{8}$. Since the only points on which the extrema of f on $\mathbb{D}^1$ can be reached are the five ones above (i.e. O, A_1, A_2, A_3 and A_4), on which the function has value $f(O) = 0$, $f(A_1) = f(A_3) = \frac{1}{2}(\sqrt{2} + 1) > 0$ and $f(A_2) = f(A_4) = -\frac{1}{2}(\sqrt{2} - 1) < 0$, the absolute maximum of f on $\mathbb{D}^1$ is $\frac{1}{2}(\sqrt{2} + 1)$ (in A_1 and A_3) and the absolute minimum is $-\frac{1}{2}(\sqrt{2} - 1)$ (in A_2 and A_4). **(3)** Let us look for the highest and lowest point of $X = \{(x, y, z) : z = x^2 + y^2, x + y + 2z - 5 \leq 0\}$, i.e. the elliptic paraboloid surface which are above or on the plane $x + y + 2z - 5 = 0$. Note that X is compact since it projects horizontally on the disk $\{(x, y) : x^2 + y^2 - \frac{1}{2}x - \frac{1}{2}y + \frac{5}{2} \leq 0\}$ (hence x and y are bounded, and the same for $z = x^2 + y^2$). The question is obviously equivalent to looking for the absolute extrema of $f(x, y, z) = z$ on X , which is a manifold with boundary with $\dot{X} = \{(x, y, z) : z = x^2 + y^2, x + y + 2z - 5 < 0\}$ (the points of the paraboloid surface which stay below the plane, manifold of dimension 2 in the usual sense) and $\partial X = \{(x, y, z) : z = x^2 + y^2, x + y + 2z - 5 = 0\}$ (the ellipse intersection between the paraboloid surface and the plane, a regular curve). $\dot{X}$ can be parametrized by its projection $D = \{(x, y) : x^2 + y^2 - \frac{1}{2}x - \frac{1}{2}y + \frac{5}{2} < 0\}$, with $\gamma(x, y) = (x, y, x^2 + y^2)$, hence we must find the stationary points of $F(x, y) := f(x, y, x^2 + y^2) = x^2 + y^2$ staying in D : since $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = 0$ if and only if $x = y = 0$, in $\dot{X}$ we find the origin $O(0, 0, 0)$. For ∂X , using Lagrange we obtain the system given by $2x - 2y = 0$, $z = x^2 + y^2$ and $x + y + 2z - 5 = 0$, whose solutions are the points $A(\frac{\sqrt{21}-1}{4}, \frac{\sqrt{21}-1}{4}, \frac{11-\sqrt{21}}{4})$ and $B(-\frac{\sqrt{21}+1}{4}, -\frac{\sqrt{21}+1}{4}, \frac{11+\sqrt{21}}{4})$. Since $f(O) = 0$, $f(A) = \frac{11-\sqrt{21}}{4} \sim 1,6$ and $f(B) = \frac{11+\sqrt{21}}{4} \sim 3,9$, the highest point of X is B and the lowest one is O .

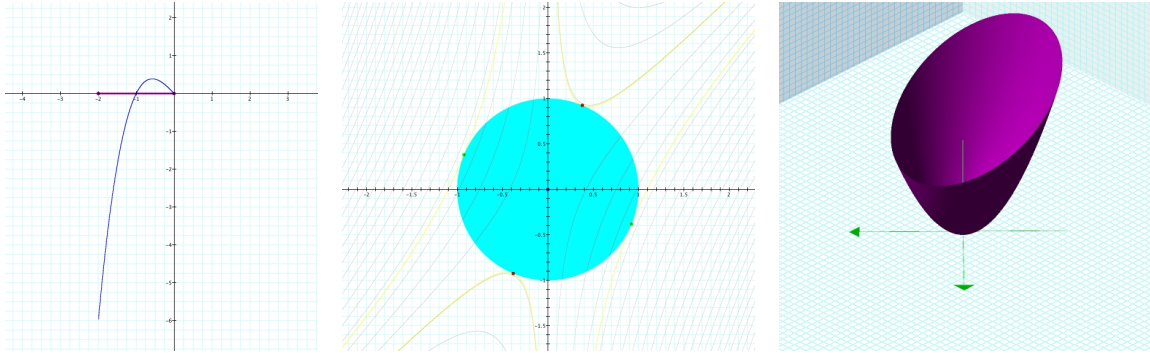


Figure 1.8: (a) The closed segment $[-2, 0]$ and the graph of $f(x) = x^3 - x$. (b) The closed disk $\mathbb{D}^1$ and the level curves of $f(x, y) = x^2 - xy$. (c) The sector X of the paraboloid surface $z = x^2 + y^2$ below the plane $x + y + 2z - 5 = 0$.

- More generally, the decomposition of the compact could give rise to several manifolds but the procedure remains unchanged, as the following examples show.

Examples. The following examples are visualized in Figure 1.9. **(1)** Let us show that $D = \{(x, y) \in \mathbb{R}^2 : 4x^2 + \frac{4}{3}y^2 \leq 1, \sqrt{3}x - y \geq 0\}$ is compact, and compute the absolute extrema of $f(x, y) = x + y$ on D . In fact, D is the plane region (including the boundary) enclosed inside the ellipse $4x^2 + \frac{4}{3}y^2 = 1$ (centered in the origine, with half-axes $\frac{1}{2}$ along x and $\frac{\sqrt{3}}{2}$ along y) and below the line $y = \sqrt{3}x$, hence it is closed, and also bounded (if $(x, y) \in D$ then $|x| \leq \frac{1}{2}$ and $|y| \leq \frac{\sqrt{3}}{2}$): hence it is compact, and f admits absolute extrema on it. Now, D can be decomposed into its interior $\dot{D} = \{(x, y) \in \mathbb{R}^2 : 4x^2 + \frac{4}{3}y^2 < 1, \sqrt{3}x - y > 0\}$ (an open subset of $\mathbb{R}^2$, a manifold of dimension 2), the open segment $D' = \{(x, y) \in \mathbb{R}^2 : 4x^2 + \frac{4}{3}y^2 < 1, \sqrt{3}x - y = 0\}$ (a regular curve, a manifold of dimension 1: note that it is necessary to remove the extremities to have a manifold in

the classical sense!), the elliptic arc $D'' = \{(x, y) \in \mathbb{R}^2 : 4x^2 + \frac{4}{3}y^2 = 1, \sqrt{3}x - y > 0\}$ (idem, another manifold of dimension 1), and the two points $\pm A$ con $A(\frac{\sqrt{2}}{4}, \frac{\sqrt{6}}{4})$ (which form together a manifold of dimension 0). The usual method ($\dot{D}$ is open) shows that $\dot{D}$ does not contain stationary points for f ; parametrizing D' with $\phi(t) = (t, \sqrt{3}t)$ (where $t \in]-\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}[$) we have $(f \circ \phi)(t) = (\sqrt{3} + 1)t$, without stationary points; for D'' the Lagrange method yields $C(-\frac{1}{4}, -\frac{3}{4})$, with $f(C) = -1$; finally, we have $f(\pm A) = \pm \frac{\sqrt{6} + \sqrt{2}}{4}$. Since $-1 < -\frac{\sqrt{6} + \sqrt{2}}{4}$, we deduce that the maximum for f on D is $\frac{\sqrt{6} + \sqrt{2}}{4}$ (in A) and the minimum is -1 (in C).

(2) Consider the solid cone of vertex $V(0, 0, 2)$ and base the ellipse $x^2 + \frac{1}{4}y^2 = 1$ on the plane (x, y) , and let X be its portion of points with $x \geq 0$ and $y \geq 0$. We aim to determine the absolute extrema of $f(x, y, z) = xy(z + 1)$ on X , which is compact (why?). To this aim, we can decompose X into the disjoint union of 12 manifolds, as follows: the discrete set of points $\{O(0, 0, 0), V(0, 0, 2), A(1, 0, 0), B(0, 2, 0)\}$ (a 0-manifold), the six edges without extremities given by five segments and one arc of ellipse X_1 (six regular curves, i.e. six 1-manifolds), the four faces without edges given by two triangles, a quarter of elliptic region X'_2 and a sector of conic surface X''_2 (four regular surfaces, i.e. four 2-manifolds) and the portion X_3 of solid cone without surface (an open subset of $\mathbb{R}^3$, i.e. a 3-manifold). We immediately note that the function is ≥ 0 on X , and that f vanishes precisely on the planes $x = 0$ and $y = 0$: therefore we already know that the absolute minimum of f on X is 0, reached in all of these points. We are left with checking the manifolds X_1 , X'_2 , X''_2 and X_3 (by the way the only ones, not by chance, to which we assigned a name). On the open subset X_3 it is enough to study the system $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0$, which has no solutions therein. X'_2 can be identified to the open subset $U = \{x^2 + \frac{1}{4}y^2 < 1, x > 0, y > 0\}$ of the plane (x, y) , on which f has value $F(x, y) := f(x, y, 0) = xy$: but also here, the system $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = 0$ we do not find any solution in U . For $X_1 = \{(x, y, z) : x^2 + \frac{1}{4}y^2 = 1, z > 0, x > 0, y > 0\}$ the Lagrange method yields the system given by $(y^2 - 4x^2)(z + 1) = 0$ with constraints $z = 0$ and $4x^2 + y^2 = 4$ (with $x > 0$ and $y > 0$) which provides the only solution $A(\frac{\sqrt{2}}{2}, \sqrt{2}, 0)$. Finally, let us parametrize X_2 as follows: the generic point of the base contour (X_1) is $P_\alpha = (\cos \alpha, 2 \sin \alpha, 0)$ with $0 < \alpha < \frac{\pi}{2}$, and the open segment joining V and P_α is given by $(0, 0, 2) + t((\cos \alpha, 2 \sin \alpha, 0) - (0, 0, 2)) = (t \cos \alpha, 2t \sin \alpha, 2(1 - t))$ with $0 < t < 1$, hence X_2 is parametrized by $\gamma :]0, 1[\times]0, 1[\rightarrow \mathbb{R}^3$ with $\gamma(t, \alpha) = (t \cos \alpha, 2t \sin \alpha, 2(1 - t))$ (removing the parameters (t, α) we easily find the cartesian equation $(z - 2)^2 = 4x^2 + y^2$ con $0 < z < 2$). But, setting $F(t, \alpha) := f(\gamma(t, \alpha)) = t^2(3 - 2t) \sin 2\alpha$, the system $\frac{\partial F}{\partial t} = \frac{\partial F}{\partial \alpha} = 0$ has no solutions in $]0, 1[\times]0, 1[$. Therefore the absolute maximum of f on X must be reached only in the point A , and it is $f(A) = 1$.

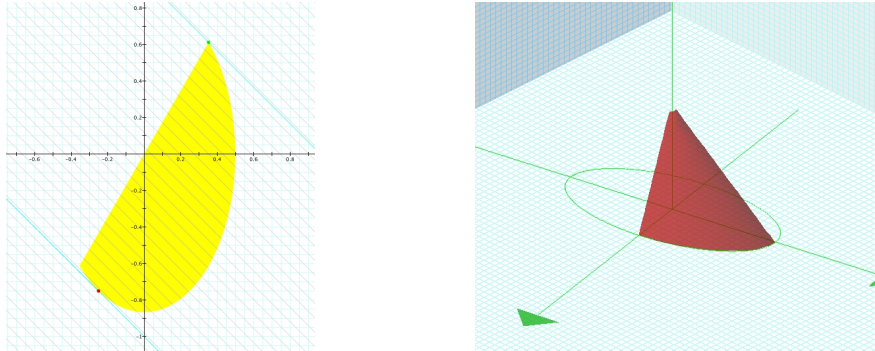


Figure 1.9: (a) The compact D and the level curves of $f(x, y) = x + y$. (b) The cone portion X .