ANALISI MATEMATICA I

Università di Padova

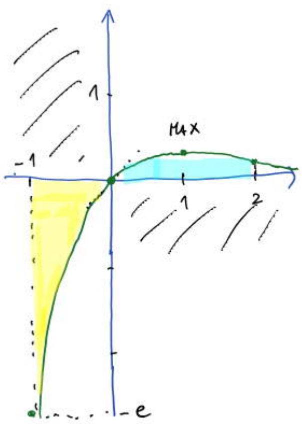
Lauree in Fisica e Astronomia - A.A. 2014/15

Lezione di martedì 02/12/2014

- $f(x) = x e^{-x}$ su $[-1, 2]$.

$$\int_{-1}^{2} f(x)\,dx = \int_{-1}^{2} \overset{F}{x}\, \overset{g}{e^{-x}}\,dx \quad f=1,\ G=-e^{-x} \quad = \left(-x e^{-x}\right|_{-1}^{2} - \int_{-1}^{2} 1\cdot(-e^{-x})\,dx = \left(-\frac{2}{e^2}\right) - (e) + \left(-e^{-x}\right|_{-1}^{2}$$

$$= -\frac{2}{e^2} - \cancel{e} + \left(-\frac{1}{e^2}\right) - (-\cancel{e}) = -3/e^2 \sim -0{,}4$$



f è C^∞; $f \geq 0 \iff x \geq 0$; $f(-1) = -e$

$f \sim_0 x$, $f \sim_{+\infty} x e^{-x}$ (infinitesimo rapido),

$f \sim_{-\infty} x e^{-x}$ (infinito rapido)

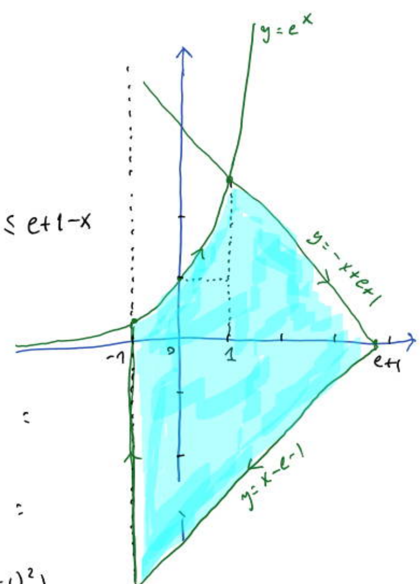
$f'(x) = 1\cdot e^{-x} + x\cdot(-e^{-x}) = (1-x)e^{-x} \geq 0 \iff x \leq 1$

$x=1$ max. assoluto, $f(1) = 1/e \sim 0{,}35$

$f''(x) = -e^{-x} + (1-x)(-e^{-x}) = -e^{-x}(2-x) = (x-2)\,e^{-x}$

$f'' \geq 0 \iff x \geq 2$: $x=2$ flesso, $f(2) = 2/e^2 \sim 0{,}3$, $f'(2) = -\frac{1}{e^2}$

- $S_2 = \{(x,y) : y \leq e^x,\ -1 \leq x \leq e+1-|y|\}$



$$x \leq e+1-|y| \iff |y| \leq e+1-x \iff \begin{cases} e+1-x \geq 0 \\ -(e+1-x) \leq y \leq e+1-x \end{cases}$$

$$\iff \begin{cases} x \leq e+1 \\ x-e-1 \leq y \leq -x+e+1 \end{cases}$$

$$e^x = -x+e+1 \iff x = 1$$

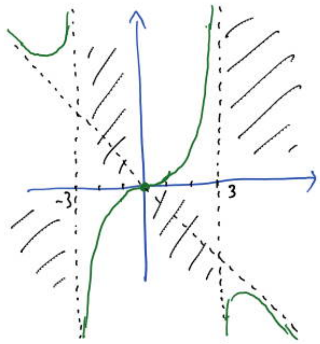
$$\text{Area}(S_2) = \int_{-1}^{1} e^x\,dx + \int_{1}^{e+1} (-x+e+1)\,dx + \int_{e+1}^{-1} (+x-e-1)\,dx =$$

$$= \left(e^x\right|_{-1}^{1} + \left((e+1)x - \frac{x^2}{2}\right|_{1}^{e+1} + \left(\frac{x^2}{2} - (e+1)x\right|_{e+1}^{-1} =$$

$$= (e) - \left(\frac{1}{e}\right) + \left(\frac{(e+1)^2}{2}\right) - \left(e+\frac{1}{2}\right) + \left(e+3/2\right) - \left(-\frac{(e+1)^2}{2}\right)$$

$$= e^2 + 3e + 2 - 1/e \sim 16{,}8$$

• $\displaystyle\int \frac{x^3}{9-x^2}\,dx = -\int \frac{x(x^2-9)+9x}{x^2-9}\,dx = -\int\left(x+\frac{9x}{x^2-9}\right)dx = -\frac{x^2}{2}-\frac{9}{2}\log|x^2-9|+K.$



$f(x)\sim_0 x^3/9$. $f(x)\sim_{\pm\infty} -x$; $f(x) = -x\cdot\dfrac{1}{1-9/x^2}$ ($9/x^2 \to 0$) $=$

$= -x\left(1+9/x^2+81/x^4+o_{\pm\infty}(x^{-5})\right) = -x - 9/x + \cdots$ ($o_{\pm\infty}(1)$)

$\Rightarrow$ $y=-x$ è asintoto obliquo!

f è dispari; $f(x)=-x \iff \dfrac{x^3}{9-x^2} = -x \iff \cancel{x^3} = -9x+\cancel{x^3}$
$\iff x=0.$

$f'(x) = \dfrac{3x^2(9-x^2)-(-2x)x^3}{(9-x^2)^2} = \dfrac{x^2}{(9-x^2)^2}(27-x^2)$

$f'(x)=0 \iff x=0,\ x=\pm 3\sqrt{3}$, $f'(x)>0 \iff |x|<3\sqrt{3}$ $(x\neq\pm 3)$

$f(\pm 3\sqrt{3}) = \dfrac{\pm 81\sqrt{3}}{9-27} = \mp\dfrac{9\sqrt{3}}{2}$

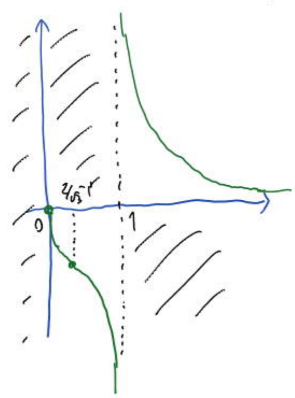
• $\displaystyle\int \frac{\sqrt{x}}{x-1}\,dx$ $x=t^2$, $dx = 2t\,dt$ $\displaystyle = \int \frac{t}{t^2-1}\cdot 2t\,dt = 2\int\frac{t^2}{t^2-1}\,dt = 2\int\frac{(t^2-1)+1}{t^2-1}\,dt$

$\displaystyle = 2\int\left(1+\frac{1}{t^2-1}\right)dt = 2\left(t+\int\frac{1}{t^2-1}\,dt\right)$

$\dfrac{1}{t^2-1} = \dfrac{A}{t+1}+\dfrac{B}{t-1} = \dfrac{(A+B)t+(B-A)}{t^2-1} \iff \begin{cases} A+B=0\\ -A+B=1\end{cases}$
$\iff A=-1/2,\ B=1/2$

$\displaystyle = 2\left(t+\frac{1}{2}\int\left(\frac{1}{t-1}-\frac{1}{t+1}\right)dt\right) = 2t+\log\left|\frac{t-1}{t+1}\right|+K$

$\displaystyle = 2\sqrt{x}+\log\left|\frac{\sqrt{x}-1}{\sqrt{x}+1}\right|+K$



Dominio $x\geq 0,\ x\neq 1$; $f(0)=0$. $f\geq 0 \iff x>1$

$f\sim_{0^+}\dfrac{\sqrt{x}}{-1} = -\sqrt{x}$; $f\sim_{+\infty}\dfrac{\sqrt{x}}{x} = \dfrac{1}{\sqrt{x}}$

$f'(x) = \dfrac{\frac{1}{2\sqrt{x}}(x-1)-\sqrt{x}}{(x-1)^2} = -\dfrac{x+1}{2\sqrt{x}(x-1)^2} < 0$ (f strett. decr.)

$f''(x) = -\dfrac{1}{2}\,\dfrac{1\cdot\sqrt{x}(x-1)^2-(x+1)\left(\frac{1}{2\sqrt{x}}(x-1)^2+2\sqrt{x}(x-1)\right)}{x(x-1)^4}$

$= \dfrac{3x^2+6x-1}{4x\sqrt{x}(x-1)^3}$ $f''(x)=0 \iff x = \dfrac{-3\pm\sqrt{12}}{3} = \dfrac{2}{\sqrt{3}}-1 \sim 0{,}2$
flesso

• $\int \arcsin(\alpha x)\, dx \quad (\alpha \in \mathbb{R})$.

$f = \frac{\alpha}{\sqrt{1-\alpha^2 x^2}}$, $G = x$

Se $\alpha = 0$: K.

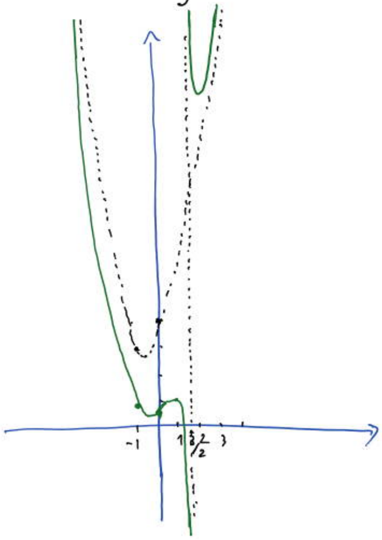
Sia $\alpha \neq 0$. $\int \overset{g}{1} \cdot \overset{F}{\arcsin(\alpha x)}\, dx = x \arcsin(\alpha x) - \int \frac{\alpha x}{\sqrt{1-\alpha^2 x^2}}\, dx$

$= x \arcsin(\alpha x) + \frac{1}{2\alpha} \int \left(\underbrace{-2\alpha^2 x}_{\varphi'(x)} \left(\underbrace{1-\alpha^2 x^2}_{\varphi(x)} \right)^{-1/2} \right) dx$

$= x \arcsin(\alpha x) + \frac{1}{2\alpha} \frac{(1-\alpha^2 x^2)^{1/2}}{1/2} + K = x \arcsin(\alpha x) + \frac{\sqrt{1-\alpha^2 x^2}}{\alpha} + K$

• $\int \frac{4x^3 - x - 1}{2x-3}\, dx$ $\quad 2x-3 = t,\ x = \frac{t+3}{2},\ dx = \frac{1}{2} dt$ $\quad = \int \frac{4 \cdot \frac{(t+3)^3}{8} - \frac{t+3}{2} - 1}{t} \cdot \frac{1}{2}\, dt = \frac{1}{4} \int \frac{t^3 + 9t^2 + 27t + 27 - t - 3 - 2}{t}\, dt$

$= \frac{1}{4} \int \left(t^2 + 9t + 26 + \frac{22}{t} \right) dt = \frac{1}{4} \left(\frac{t^3}{3} + \frac{9t^2}{2} + 26t + 22 \log|t| \right) + K$



$f(x) \sim_{\pm\infty} \frac{4x^3}{2x} = 2x^2$

$f(x) = \frac{1}{2x-3} \cdot (4x^3 - x - 1) = \frac{1}{2x} \cdot \frac{1}{1 - 3/2x} (4x^3 - x - 1)$

$= \left(2x^2 - \frac{1}{2} - \frac{1}{2x}\right) \cdot \left(1 + \frac{3}{2x} + \frac{9}{4x^2} + o_{\pm\infty}(x^{-2})\right)$

$= \underbrace{2x^2 + 3x + 4}_{\text{parabola asintotica!}} + o_{\pm\infty}(1)$ $\quad$ (IN ALTERNATIVA: $\lim_{x \to \pm\infty} \frac{f(x)}{x^2} = 2, \dots$)

$f(-1) = \frac{4}{5}$, $f(0) = 1/3$, $f(-1/2) = \frac{1}{4}$, $f(1) = -2$, $f(2) = 29$

$f'(x) = \frac{(12x^2 - 1)(2x-3) - 2(4x^3 - x - 1)}{(2x-3)^2} = \frac{16x^3 - 36x^2 + 5}{(2x-3)^2}$

$f'(-1) < 0$, $f'(0) > 0$, $f'(1) < 0$, $f'(2) < 0$, $f'(3) > 0$

Dunque ci sarà un min. loc. in $]-1,0[$ e in $]2,3[$, e un max. loc. in $]0,1[$.

Intersezioni con la parabola asintotica?

$f(x) = \frac{4x^3 - x - 1}{2x-3} = 2x^2 + 3x + 4 \iff \cancel{4x^3} - \cancel{x} - 1 = \cancel{4x^3} + \cancel{6x^2} + 8x - \cancel{6x^2} - 9x - 12$ impossibile.

• $\displaystyle\int \frac{1}{\sqrt{e^{2x}-4}}\,dx$ $\quad e^{2x}-4=t^2,\ e^{2x}=t^2+4,\ 2x=\log(t^2+4)$ $\quad x=\frac{1}{2}\log(t^2+4),\ dx=\frac{1}{2}\cdot\frac{1}{t^2+4}\cdot 2t\,dt$ $\quad = \displaystyle\int \frac{1}{\cancel{t}}\cdot\frac{1}{t^2+4}\cdot\cancel{t}\,dt$

$$= \frac{1}{4}\int \frac{1}{1+(t/2)^2}\,dt = \frac{1}{2}\int \frac{1/2}{1+(t/2)^2}\,dt = \frac{1}{2}\operatorname{arctg}(t/2)+K = \frac{1}{2}\operatorname{arctg}\left(\frac{\sqrt{e^{2x}-4}}{2}\right)+K.$$

E se invece studiassimo $\dfrac{1}{\sqrt{|e^{2x}-4|}}$, che ha dominio $x\neq\log 2$ (anziché solo $x>\log 2$)?

L'idea è la stessa: $|e^{2x}-4|=t^2 \overset{\sigma:=\operatorname{sign}(x-\log 2)}{\Longrightarrow} e^{2x}-4=\sigma t^2 \Rightarrow x=\frac{1}{2}\log(\sigma t^2+4) \Rightarrow$

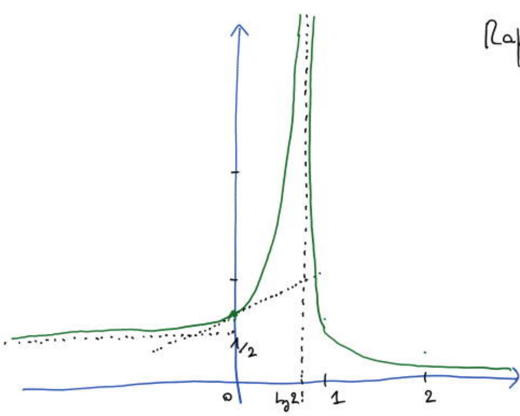
$dx=\frac{1}{\cancel{2}}\cdot\frac{1}{\sigma t^2+4}\cdot\cancel{2}\sigma t\,dt$, dunque $\displaystyle\int\frac{1}{\sqrt{|e^{2x}-4|}}\,dx=\int\frac{1}{\cancel{t}}\cdot\frac{1}{\sigma t^2+4}\cdot\sigma\cancel{t}\,dt =$

$= \sigma\displaystyle\int\frac{dt}{\sigma t^2+4}$. A questo punto per proseguire vanno distinti i casi:

- se $x>\log 2$ (ovvero $\sigma=1$) si ha $\displaystyle\int\frac{dt}{t^2+4}=\frac{1}{2}\operatorname{arctg}\left(\frac{\sqrt{e^{2x}-4}}{2}\right)+K$, come visto prima.
- se $x<\log 2$ (ovvero $\sigma=-1$) si ha $-\displaystyle\int\frac{dt}{4-t^2}=\int\frac{dt}{t^2-4}=\frac{1}{4}\int\left(\frac{1}{t-2}-\frac{1}{t+2}\right)dt$

$$=\frac{1}{4}\log\left|\frac{t-2}{t+2}\right|+K=\frac{1}{4}\log\left|\frac{\sqrt{4-e^{2x}}-2}{\sqrt{4-e^{2x}}+2}\right|+K$$

Nei due casi si ottengono primitive di forma decisamente diversa.



Rapido studio di $f(x)=\dfrac{1}{\sqrt{|e^{2x}-4|}}$:

Dominio $x\neq\log 2$, $f>0$ sempre.

$f(0)=\frac{1}{\sqrt{3}}\sim 0{,}6$, $\lim_{x\to\log 2} f(x)=+\infty$

$f(x)\sim_{+\infty}\frac{1}{\sqrt{e^{2x}}}=e^{-x}$;

$$f(x)=e^{-x}\cdot\frac{1}{\sqrt{1-4e^{-2x}}}=e^{-x}\left(1-4e^{-2x}\right)^{-1/2}$$
$$=e^{-x}\left(1-\frac{1}{2}(-4e^{-2x})+o_{+\infty}(e^{-2x})\right)$$
$$=e^{-x}+2e^{-3x}+o_{+\infty}(e^{-3x}).$$

A $-\infty$: $f(x) = \dfrac{1}{\sqrt{4-e^{2x}}} = \dfrac{1}{2}\cdot\left(1-\frac{1}{4}e^{2x}\right)^{-1/2} = \dfrac{1}{2}\left(1-\frac{1}{2}\left(-\frac{1}{4}e^{2x}\right)+\theta_{-\infty}(e^{2x})\right)$

$= \dfrac{1}{2}+\dfrac{1}{16}e^{2x}+\theta_{-\infty}(e^{2x}) = \dfrac{1}{2}+\theta_{-\infty}(1)$ dunque $y=\frac{1}{2}$ è asintoto orizz. a $-\infty$.

Intersezioni per $x<\log 2$? $f(x)=\frac{1}{2} \iff 4-e^{2x}=4$ impossibile.

In 0: $f(x) = \dfrac{1}{\sqrt{4-e^{2x}}} = (4-e^{2x})^{-1/2} = \left(3\cdot\dfrac{4-e^{2x}}{3}\right)^{-1/2} = \dfrac{1}{\sqrt{3}}\left(1+\dfrac{1-e^{2x}}{3}\right)^{-1/2}$ $\to 0$

$= \dfrac{1}{\sqrt{3}}\left(1-\dfrac{1}{2}\left(\dfrac{1-e^{2x}}{3}\right)+\theta_0\left(\dfrac{1-e^{2x}}{3}\right)\right)$.

Ora $\dfrac{1-e^{2x}}{3} = \dfrac{1-(1+2x+\theta_0(x))}{3} = -\dfrac{2x}{3}+\theta_0(x)$, dunque

retta tg

dunque $f(x) = \dfrac{1}{\sqrt{3}}\left(1-\dfrac{1}{2}\cdot\left(-\dfrac{2x}{3}\right)+\theta_0(x)\right) = \dfrac{1}{\sqrt{3}}+\dfrac{x}{3\sqrt{3}}+\theta_0(x)$

$f'(x) = -\frac{1}{\cancel{2}}\left|e^{2x}-4\right|^{-3/2}\cdot 6\cdot\cancel{2}e^{2x} = -6e^{2x}\left|e^{2x}-4\right|^{-3/2}$, dunque $f'(x)$ non si annulla mai, e $f'(x) \gtrless 0$ (ovvero f strett. cresc./decresc.) $\iff x \lessgtr \log 2$.