ANALISI MATEMATICA I
Università di Padova
Laurea in Fisica e Astronomia - A.A. 2014/15
Lezione di mercoledì 03/12/2014

• $\int \frac{x+3}{x^2(x^2-2x+3)}\,dx$.

Seguendo il suggerimento, cerchiamo $A, B, C, D \in \mathbb{R}$ t.c. $\frac{x+3}{x^2(x^2-2x+3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2-2x+3}$

$$= \frac{(Ax+B)(x^2-2x+3) + x^2(Cx+D)}{x^2(x^2-2x+3)} = \frac{(A+C)x^3 + (-2A+B+D)x^2 + (3A-2B)x + 3B}{x^2(x^2-2x+3)}$$

$$\Longleftrightarrow \begin{cases} A+C=0 \\ -2A+B+D=0 \\ 3A-2B=1 \\ 3B=3 \end{cases} \qquad \begin{cases} B=1 \\ A=\frac{1+2B}{3}=1 \\ C=-A=-1 \\ D=2A-B=1 \end{cases}$$

. Dunque $\frac{x+3}{x^2(x^2-2x+3)} = \frac{1}{x} + \frac{1}{x^2} - \frac{x-1}{x^2-2x+3}$

$$\leadsto \int \frac{x+3}{x^2(x^2-2x+3)}\,dx = \int \frac{1}{x}\,dx + \int \overset{x^{-2}}{\frac{1}{x^2}}\,dx - \int \frac{x-1}{x^2-2x+3}\,dx =$$

$$= \log|x| + \frac{x^{-1}}{-1} - \frac{1}{2}\int \frac{2x-2}{x^2-2x+3}\,dx = \log|x| - \frac{1}{x} - \frac{1}{2}\log(x^2-2x+3) + K$$

$$= \log\left|\frac{x}{\sqrt{x^2-2x+3}}\right| - \frac{1}{x} + K.$$

Metodo di Hermite . Si debba calcolare $\boxed{\int \frac{A(x)}{B(x)}\,dx \ \text{ con } \deg A < \deg B}$.

(1) Decomporre $B(x)$ in fattori reali irriducibili

$$B(x) = (x-\alpha_1)^{m_1} \cdots (x-\alpha_r)^{m_r} \cdot (x^2+\beta_1 x+\gamma_1)^{n_1} \cdots (x^2+\beta_s x+\gamma_s)^{n_s}$$

(2) Si dimostra che si può scrivere

$$\frac{A(x)}{B(x)} = \frac{A_1}{x-\alpha_1} + \cdots + \frac{A_r}{x-\alpha_r} + \frac{B_1 x + C_1}{x^2+\beta_1 x+\gamma_1} + \cdots + \frac{B_s x + C_s}{x^2+\beta_s x+\gamma_s}$$ ← ADDENDI SEMPLICI!

$$+ \left(\frac{P(x)}{(x-\alpha_1)^{m_1-1} \cdots (x-\alpha_r)^{m_r-1} \cdot (x^2+\beta_1 x+\gamma_1)^{n_1-1} \cdots (x^2+\beta_s x+\gamma_s)^{n_s-1}} \right)'$$ DERIVATA!!

ove $\deg P(x) < \deg(\text{denominatore})$.

(3) Dunque $\int \frac{A(x)}{B(x)}\,dx$ diventa

$$\underbrace{\int \frac{A_1}{x-\alpha_1}\,dx + \cdots + \int \frac{B_s x + C_s}{x^2+\beta_s x+\gamma_s}\,dx}_{\text{LI SO FARE !}} + \underbrace{\frac{P(x)}{(x-\alpha_1)^{m_1-1}\cdots(x^2+\beta_s x+\gamma_s)^{n_s-1}}}_{\text{GIA' FATTO !}}$$

Es. L'esercizio fatto sopra:

$$\frac{x+3}{x^2(x^2-2x+3)} = \frac{A}{x} + \frac{Bx+C}{x^2-2x+3} + \left(\frac{D}{x}\right)', \text{ così come prima.}$$

Es. $$\frac{3x^6-8x^5+18x^4-37x^3+36x^2-32x+8}{(x-1)^3(x^2+2)^2} = f(x)$$

$$= \frac{A}{x-1} + \frac{Bx+C}{x^2+2} + \left(\frac{Dx^3+Ex^2+Fx+G}{(x-1)^2(x^2+2)}\right)'$$

Dopo pazienti calcoli e confronto di coefficienti a sx e dx (in particolare bisogna sviluppare la derivata, ovvero

$$\frac{(3Dx^2+2Ex+F)(x-1)^{\cancel{2}}(x^2+2) - (Dx^3+Ex^2+Fx+G)\left(2\cancel{(x-1)}(x^2+2)+2x(x-1)^{\cancel{2}}\right)}{(x-1)^{\cancel{4}\,3}\cdot(x^2+2)^2},$$

e fare denominatore comune con $\frac{A}{x-1} + \frac{Bx+C}{x^2+2}$)

si dovrebbe ottenere $(A,B,C,D,E,F,G) = (2,1,-1,0,0,3,-1)$

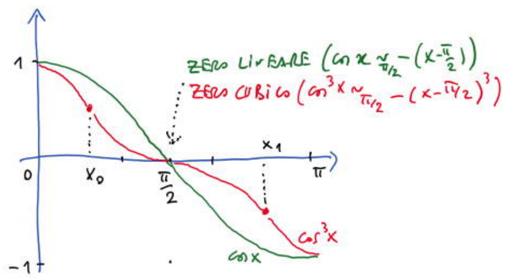
$$\rightsquigarrow f(x) = \frac{2}{x-1} + \frac{x-1}{x^2+2} + \left(\frac{3x-1}{(x-1)^2(x^2+2)}\right)'$$

$$\rightsquigarrow \int f(x)\,dx = 2\log|x-1| + \int \frac{x-1}{x^2+2}\,dx + \frac{3x-1}{(x-1)^2(x^2+2)}$$

$$= 2\log|x-1| + \frac{1}{2}\log(x^2+2) + \frac{1}{\sqrt{2}}\operatorname{arctg}\left(\frac{x}{\sqrt{2}}\right) + \frac{3x-1}{(x-1)^2(x^2+2)} + k$$

• $S_4 = \{(x,y):\ 2(1-x) \le y \le 1+\cos^3 x\ ,\ y+2\sin x \ge 0,\ x \le \pi,\ y+1 \ge 0\}$

Iniziamo da uno studio preliminare di $f(x)=\cos^3 x$ in $[0,\pi]$, che è l'intervallo d'interesse per l'esercizio.



$f'(x) = -3\sin x \cos^2 x \le 0$

$f''(x) = -3(\cos x \cdot \cos^2 x + \sin x \cdot 2(-\sin x)\cos x)$

$= -3\cos x\,(3\cos^2 x - 2)$

$f''(x)=0$ per $x=\pi/2$, $x=x_0=\arccos(\sqrt{2/3}) \sim 0{,}61$,

$x=x_1=\arccos(-\sqrt{2/3}) = \pi - x_0 \sim 2{,}53$

$f''(x)>0$:

	0 – x_0	x_0 – $\pi/2$	$\pi/2$ – x_1	x_1 – π
$\cos x$	+	+	−	−
	+	−	−	+
f''	−	+	−	+
f	$\cap$	$\cup$	$\cap$	$\cup$
	FLESSO	FLESSO	FLESSO	

$f(\pi/2) = f'(\pi/2) = 0$

$f(x_0) = \frac{2\sqrt{2}}{3\sqrt{3}} \sim 0{,}54$

$f'(x_0) = -2/\sqrt{3} \sim -1{,}2$

$f(x_1) = -f(x_0)$

$f'(x_1) = f'(x_0)$

$\int \cos^3 x\,dx = \int \cos x \cdot \cos^2 x\,dx =$

$\int \cos x(1-\sin^2 x)\,dx \overset{\sin x = t}{=} \int (1-t^2)\,dt$

$= t - t^3/3 + K = \sin x - \frac{\sin^3 x}{3} + K$

Passiamo ora al disegno di S_4 e al calcolo dell'area:

$2(1-x) = -1 \Rightarrow x = -3/2$;

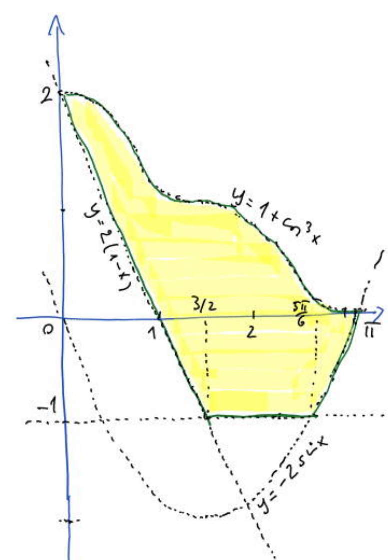
$-2\sin x = -1 \Rightarrow \sin x = +1/2 \Rightarrow$ (in $[0,\pi]$) $x=\frac{\pi}{6}$, $x=\frac{5\pi}{6}$

$$\text{Area}(S_4) = \int_0^{\pi}(1+\cos^3 x)\,dx + \int_{\pi}^{\frac{5\pi}{6}}(-2\sin x)\,dx + \int_{\frac{5\pi}{6}}^{3/2}(-1)\,dx + \int_{3/2}^{0} 2(1-x)\,dx =$$

$$= \left(x + \sin x - \frac{\sin^3 x}{3}\right|_0^{\pi} + \left(2\cos x\right|_{\pi}^{\frac{5\pi}{6}} + \left(-x\right|_{\frac{5\pi}{6}}^{3/2} + \left(2x - x^2\right|_{3/2}^{0}$$

$$= (\pi) - (0) + (-\sqrt{3}) - (-2) + (-3/2) - \left(-\frac{5\pi}{6}\right) + (0) - (3/4) =$$

$$= \frac{11\pi}{6} - \frac{1}{4} - \sqrt{3} \sim 3{,}5$$



RISOLUZIONE DEGLI ESERCIZI DI INTEGRAZIONE NON SVOLTI IN AULA

- $\displaystyle\int \frac{3x-1}{(x+2)^n}\,dx \quad (n\in\mathbb{N})$ $\qquad x+2=t \quad x=t-2 \quad dx=dt$

$$\int \frac{3t-6-1}{t^n}\,dt = 3\int t^{1-n}\,dt - 7\int t^{-n}\,dt$$

$$= \begin{cases} (\text{se } n=1) & 3t - 7\log|t| + K \\ (\text{se } n=2) & 3\log|t| + 7/t + K \\ (\text{se } n\geq 3) & \dfrac{3t^{2-n}}{2-n} - \dfrac{7t^{1-n}}{1-n} + K \end{cases} \quad \text{, poi rimettere } t=x+2.$$

- $\displaystyle\int \frac{2x^2+5x-2}{x^2-2x-3} = \int\left(2 + \frac{9x+4}{x^2-2x-3}\right)dx = 2x + \int \frac{9x+4}{x^2-2x-3}\,dx.$

$$\frac{9x+4}{x^2-2x-3} = \frac{A}{x+1} + \frac{B}{x-3} = \frac{(A+B)x + (B-3A)}{(x+1)(x-3)} \iff \begin{cases} A+B=9 \\ -3A+B=4 \end{cases} \iff \begin{cases} A = 5/4 \\ B = 31/4 \end{cases}$$

Dunque $\displaystyle\int = 2x + \frac{1}{4}\int\left(\frac{5}{x+1} + \frac{31}{x-3}\right)dx = 2x + \frac{1}{4}\left(5\log|x+1| + 31\log|x-3|\right) + K$

- $\displaystyle\int \frac{4x^2-3x+1}{4x^2+12x+9}\,dx = \int \frac{4x^2-3x+1}{(2x+3)^2}\,dx$ $\qquad$ Pongo $2x+3=t \Rightarrow x=\frac{t-3}{2} \Rightarrow dx=\frac{1}{2}dt$

Dunque $\displaystyle = \int \frac{\cancel{4}\cdot\frac{(t-3)^2}{\cancel{4}} - 3\,\frac{t-3}{2} + 1}{t^2}\cdot\frac{1}{2}\,dt = \frac{1}{4}\int\frac{2(t-3)^2 - 3(t-3) + 2}{t^2}\,dt$

$$= \frac{1}{4}\int\frac{2t^2 - 12t + 18 - 3t + 9 + 2}{t^2}\,dt = \frac{1}{4}\int\left(2 - \frac{15}{t} + \frac{29}{t^2}\right)dt =$$

$$= \frac{1}{4}\left(2t - 15\log|t| - 29/t\right) + K \quad \text{e poi risostituire } t = 2x+3.$$

- $$\int \frac{x^3+3x^2-1}{x^2+6x+13}\,dx = \int\left(x-3+\frac{5x+38}{x^2+6x+13}\right)dx$$

$$\begin{array}{l|l} x^3+3x^2 \qquad -1 & x^2+6x+13 \\ \hline -x^3-6x^2-13x & x-3 \\ // \; -3x^2-13x-1 & \\ \quad +3x^2+18x+39 & \\ \quad // \quad 5x+38 & \end{array}$$

$$= \frac{x^2}{2}-3x+\int\frac{5x+38}{(x+3)^2+4}\,dx = \frac{x^2}{2}-3x+\frac{1}{4}\int\frac{5x+38}{\left(\frac{x+3}{2}\right)^2+1}\,dx$$

$$\left[\begin{array}{l} t=\frac{x+3}{2} \quad x=2t-3 \\ dx=2\,dt \end{array}\right] = \frac{x^2}{2}-3x+\frac{1}{4}\int\frac{10t-15+38}{t^2+1}\cdot 2\,dt =$$

$$= \frac{x^2}{2}-3x+\frac{1}{2}\int\frac{10t+23}{t^2+1}\,dt = \frac{x^2}{2}-3x+\frac{1}{2}\left(5\int\frac{2t}{t^2+1}\,dt+23\int\frac{1}{t^2+1}\,dt\right)$$

$$= \frac{x^2}{2}-3x+\frac{1}{2}\left(5\lg(t^2+1)+23\,\mathrm{arctg}\,t\right)+K$$, poi rimettere $t=\frac{x+3}{2}$.

- $$\int\frac{x}{4x^2-2x+1}\,dx = \int\frac{x}{\left(2x-\frac{1}{2}\right)^2+\frac{3}{4}}\,dx = \frac{4}{3}\int\frac{x}{\left(\frac{2x-1/2}{\sqrt{3}/2}\right)^2+1}\,dx$$

$$\left[\begin{array}{l} t=\frac{2x-1/2}{\sqrt{3}/2}=\frac{4x-1}{\sqrt{3}} \\ x=\frac{t\sqrt{3}+1}{4} \\ dx=\frac{\sqrt{3}}{4}\,dt \end{array}\right]$$

$$= \frac{4}{3}\int\frac{\frac{t\sqrt{3}+1}{4}}{t^2+1}\cdot\frac{\sqrt{3}}{4}\,dt = \frac{1}{4\sqrt{3}}\int\frac{t\sqrt{3}+1}{t^2+1}\,dt = \frac{1}{4\sqrt{3}}\left(\frac{1}{2}\lg(t^2+1)+\mathrm{arctg}\,t\right)+K$$, poi $t=\frac{4x-1}{\sqrt{3}}$.

- $$\int \overset{g}{x^2}\,\overset{F}{\lg|2x+1|}\,dx \qquad f=\frac{2}{2x+1} \quad G=\frac{x^3}{3}$$

$$= \frac{x^3}{3}\lg|2x+1|-\int\frac{x^3}{3}\cdot\frac{2}{2x+1}\,dx \qquad \left[\begin{array}{l} 2x+1=t \quad x=\frac{t-1}{2} \\ dx=\frac{1}{2}\,dt \end{array}\right]$$

$$= \frac{x^3}{3}\lg|2x+1|-\frac{\cancel{2}}{3}\int\frac{\left(\frac{t-1}{2}\right)^3}{t}\cdot\frac{1}{\cancel{2}}\,dt = \frac{x^3}{3}\lg|2x+1|-\frac{1}{24}\int\left(t^2-3t+3-\frac{1}{t}\right)dt$$

$$= \frac{x^3}{3}\lg|2x+1|-\frac{1}{24}\left(\frac{t^3}{3}-\frac{3t^2}{2}+3t-\lg|t|\right)+K$$, poi rimettere $t=2x+1$.

- $$\int\frac{x}{\sqrt{3-2x^2}}\,dx = \int x\,(3-2x^2)^{-1/2}\,dx = -\frac{1}{4}\int(-4x)(3-2x)^{-1/2}\,dx = -\frac{1}{4}\,\frac{(3-2x)^{1/2}}{1/2} = -\frac{\sqrt{3-2x}}{2}+K$$

- $$\int\frac{1}{\sqrt{3-2x^2}}\,dx = \frac{1}{\sqrt{3}}\int\frac{1}{\sqrt{1-\left(\sqrt{\frac{2}{3}}\,x\right)^2}}\,dx = \frac{1}{\cancel{\sqrt{3}}}\cdot\sqrt{\frac{\cancel{3}}{2}}\int\frac{\sqrt{2/3}}{\sqrt{1-\left(\sqrt{\frac{2}{3}}\,x\right)^2}}\,dx = \frac{1}{\sqrt{2}}\arcsin\left(\sqrt{\frac{2}{3}}\,x\right)+K$$

- $$\int \overset{g}{e^{-x}}\,\overset{F}{\sin 2x}\,dx \qquad f=2\cos 2x \quad G=-e^{-x}$$

$$= -e^{-x}\sin 2x-\int(-e^{-x})\cdot 2\cos 2x\,dx = -e^{-x}\sin 2x+2\int \overset{g}{e^{-x}}\,\overset{F}{\cos 2x}\,dx \qquad f=-2\sin 2x \quad G=-e^{-x}$$

$$= -e^{-x}\sin 2x+2\left(-e^{-x}\cos 2x-\int(-2\sin 2x)(-e^{-x})\,dx\right)$$

$$= -e^{-x}\sin 2x-2e^{-x}\cos 2x-4\int e^{-x}\sin 2x\,dx$$ (porto al I° membro)

$$\Rightarrow \quad 5\int e^{-x}\sin 2x\,dx = -e^{-x}(\sin 2x+2\cos 2x)+K$$

$$\Rightarrow \quad \int e^{-x}\sin 2x\,dx = -\frac{1}{5}e^{-x}(\sin 2x+2\cos 2x)+K$$

- $$\int \frac{3x-2}{x^2+x+1}\,dx = \int \frac{3x-2}{\left(x+\frac12\right)^2+\frac34}\,dx = \frac43 \int \frac{3x-2}{\left(\frac{x+1/2}{\sqrt3/2}\right)^2+1}\,dx \qquad \left[\frac{x+1/2}{\sqrt3/2} = \frac{2x+1}{\sqrt3} = t,\quad x = \frac{t\sqrt3-1}{2},\quad dx = \frac{\sqrt3}{2}\,dt\right]$$
$$= \frac43 \int \frac{\frac32(t\sqrt3-1)-2}{t^2+1}\cdot\frac{\sqrt3}{2}\,dt = \frac{1}{\sqrt3}\int \frac{3(t\sqrt3-1)-4}{t^2+1}\,dt = \frac{1}{\sqrt3}\int \frac{3t\sqrt3-7}{t^2+1}\,dt$$
$$= \frac{1}{\sqrt3}\left(\frac{3\sqrt3}{2}\lg(t^2+1) - 7\operatorname{arctg} t\right) + K,$$ poi $t = \frac{2x+1}{\sqrt3}$.

- $F \quad g \quad f = 3x^2,\ G = \sin x$
$$\int\left(\frac{\sin x}{\sqrt{1-\cos x}} + 2x^3\cos x\right)dx = \int \sin x\,(1-\cos x)^{-1/2}\,dx + 2\int x^3\cos x\,dx$$
$F \quad g \quad f = 2x,\ G = -\cos x$
$$= \frac{(1-\cos x)^{1/2}}{1/2} + 2\left(x^3\sin x - \int 3x^2\sin x\,dx\right)$$
$$= 2\sqrt{1-\cos x} + 2\left(x^3\sin x - 3\left(-x^2\cos x - \int 2x(-\cos x)\,dx\right)\right)$$
$F \quad g \quad f = 1,\ G = \sin x$
$$= 2\sqrt{1-\cos x} + 2\left(x^3\sin x + 3x^2\cos x - 6\int x\cos x\,dx\right)$$
$$= 2\sqrt{1-\cos x} + 2\left(x^3\sin x + 3x^2\cos x - 6\left(x\sin x - \int 1\cdot\sin x\,dx\right)\right)$$
$$= 2\sqrt{1-\cos x} + 2\left((x^3-6x)\sin x + (3x^2-6x)\cos x\right) + K$$

- $F \quad g \quad f = 2,\ G = -e^{3-x}$
$$\int (2x+1)e^{3-x}\,dx = (2x+1)(-e^{3-x}) - \int 2\cdot(-e^{3-x})\,dx = -(2x+1)e^{3-x} + 2\int e^{3-x}\,dx$$
$$= -(2x+1)e^{3-x} - 2e^{3-x} = -(2x+3)e^{3-x} + K.$$

- $g \quad F \quad f = \frac1x,\ G = \frac{x^{-2}}{-2} = -\frac{1}{2x^2}$
$$\int \frac{\lg x}{x^3}\,dx = \int x^{-3}\lg x\,dx = -\frac{1}{2x^2}\lg x - \int\left(-\frac{1}{2x^2}\right)\cdot\frac1x\,dx$$
$$= -\frac{1}{2x^2}\lg x + \frac12\cdot\frac{x^{-2}}{-2} = -\frac{2\lg x + 1}{4x^2} + K$$

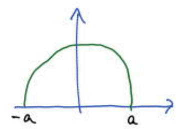
- $$\int \frac{2+\sqrt x}{x}\,dx = \int\left(\frac2x + \frac{1}{\sqrt x}\right)dx = 2\lg|x| + 2\sqrt x + K$$

- $$\int \frac{2+\sqrt x}{x+1}\,dx \qquad \left[\begin{array}{l} x = t^2 \\ dx = 2t\,dt \end{array}\right] = \int \frac{2+t}{t^2+1}\cdot 2t\,dt = 2\int \frac{t^2+2t}{t^2+1}\,dt$$
$$= 2\int\left(1 + \frac{2t-1}{t^2+1}\right)dt = 2\left(t + \lg(t^2+1) - \operatorname{arctg} t\right) + K$$
$$= 2\left(\sqrt x + \lg(x+1) - \operatorname{arctg}\sqrt x\right) + K$$

• $\displaystyle\int \frac{3}{2x^2-1}\,dx = \frac{3}{2}\int \frac{1}{x^2-1/2}\,dx = \frac{3}{2}\int\left(\frac{1/\sqrt{2}}{x-\sqrt{2}/2} - \frac{1/\sqrt{2}}{x+\sqrt{2}/2}\right)dx$

$\displaystyle = \frac{3}{\sqrt{2}}\log\left|\frac{x-\sqrt{2}/2}{x+\sqrt{2}/2}\right| + K = \frac{3}{\sqrt{2}}\log\left|\frac{x\sqrt{2}-1}{x\sqrt{2}+1}\right| + K.$

• $\displaystyle\int \sqrt{a^2-x^2}\,dx$ (con $a>0$).



<u>1° modo</u>: PER PARTI. $\displaystyle\int\sqrt{a^2-x^2}\,dx = \int \underbrace{1}_{g}\cdot\underbrace{\sqrt{a^2-x^2}}_{F}\,dx = x\sqrt{a^2-x^2} - \int\left(-\frac{x}{\sqrt{a^2-x^2}}\right)x\,dx$

$f = -x/\sqrt{a^2-x^2}$
$G = x$

$\displaystyle = x\sqrt{a^2-x^2} - \int\frac{(a^2-x^2)-a^2}{\sqrt{a^2-x^2}}\,dx = x\sqrt{a^2-x^2} - \int\sqrt{a^2-x^2}\,dx + a^2\int\frac{dx}{\sqrt{a^2-x^2}}$

$\displaystyle = x\sqrt{a^2-x^2} - \int\sqrt{a^2-x^2}\,dx + a^2\int\frac{1/a\,dx}{\sqrt{1-(x/a)^2}} = x\sqrt{a^2-x^2} - \int\sqrt{a^2-x^2}\,dx + a^2\arcsin(x/a),$

dunque portando l'integrale al primo membro si ottiene

$$\int\sqrt{a^2-x^2}\,dx = \frac{1}{2}\left(x\sqrt{a^2-x^2} + a^2\arcsin\frac{x}{a}\right) + K.$$

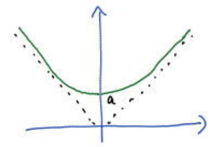
<u>2° modo</u>: SOSTITUZIONE. Poniamo $x = a\sin t$, e limitiamoci al caso $0\le x\le a$ (ciò non è restrittivo: poiché $\sqrt{a^2-x^2}$ è pari, la sua primitiva che si annulla per $x=0$ sarà dispari e dunque basta determinarla per $x>0$ [e poi estenderla a $x<0$]; allora $t = \arcsin x/a$, da cui $\cos t = \sqrt{1-x^2/a^2}$, e $dx = a\cos t\,dt$. Pertanto

$$\int\sqrt{a^2-x^2}\,dx = \int\sqrt{a^2-a^2\sin^2 t}\cdot a\cos t\,dt = a^2\int\cos^2 t\,dt =$$

$$= a^2\cdot\frac{t+\sin t\cos t}{2} + K = \frac{a^2}{2}\left(\arcsin\frac{x}{a} + \frac{x}{a}\sqrt{1-x^2/a^2}\right) + K$$

$\displaystyle = \frac{1}{2}\left(x\sqrt{a^2-x^2} + a^2\arcsin x/a\right) + K$. Notiamo che la primitiva che si annulla in $x=0$ è già in forma dispari, dunque la scrittura esibita (finora valida solo per $x>0$) è in realtà valida per ogni $-a<x<a$, e coincide con quella trovata prima.

- $\int \sqrt{a^2+x^2}\,dx$ (con $a>0$).



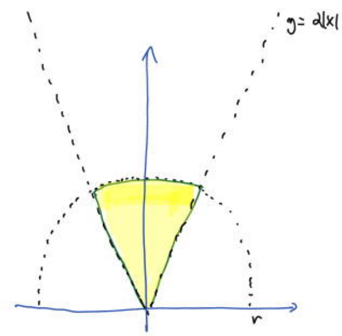
Si svolge in modo simile a quanto fatto per $\sqrt{a^2-x^2}$ qui sopra, tranne che nel 2° modo la sostituzione giusta è $x = a\sinh(t)$:

$$\int\sqrt{a^2+x^2}\,dx = \int a\cosh(t)\cdot a\cosh(t)\,dt = a^2\int\cosh^2(t)\,dt = \frac{a^2}{2}\int(1+\cosh(2t))\,dt$$

$$= \frac{a^2}{2}\left(t+\frac{1}{2}\sinh(2t)\right) = \frac{a^2}{2}\left(t+\sinh(t)\cosh(t)\right) = \frac{a^2}{2}t + (a\sinh(t))(a\cosh(t))$$

$$= \frac{a^2}{2}\operatorname{settsinh}\left(\frac{x}{a}\right) + x\cdot a\sqrt{1+\frac{x^2}{a^2}} + K = \frac{a^2}{2}\log\left(x+\sqrt{a^2+x^2}\right) + x\sqrt{a^2+x^2} + K.$$

- $S_1 = \{(x,y) : x^2+y^2 \le r^2,\ y \ge \alpha|x|\}$ $(r,\alpha>0)$



$$\begin{cases} x^2+y^2=r^2 \\ y=\alpha|x| \end{cases} \Rightarrow (1+\alpha^2)x^2 = r^2 \Rightarrow x = \pm\frac{1}{\sqrt{1+\alpha^2}}\,r.$$

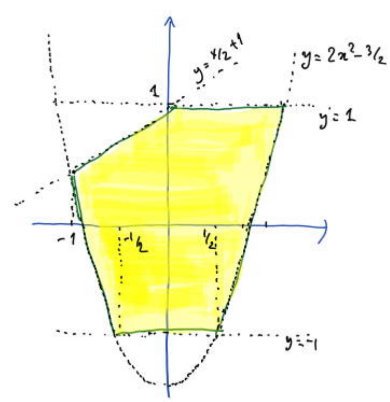
$$\text{Area}(S_1) = 2\left(\int_0^{\frac{1}{\sqrt{1+\alpha^2}}r}\sqrt{r^2-x^2}\,dx + \int_{\frac{1}{\sqrt{1+\alpha^2}}r}^{0}\alpha x\,dx\right)$$

[si era calcolato $\int\sqrt{a^2-x^2}\,dx = \frac{1}{2}\left(x\sqrt{a^2-x^2}+a^2\arcsin\frac{x}{a}\right)$]

$$= \cancel{2}\left(\frac{1}{\cancel{2}}\left(x\sqrt{r^2-x^2}+r^2\arcsin\frac{x}{r}\right) - \frac{1}{\cancel{2}}\alpha x^2\right)\Bigg|_0^{\frac{1}{\sqrt{1+\alpha^2}}r} = \frac{1}{\sqrt{1+\alpha^2}}r\cdot\frac{\alpha}{\sqrt{1+\alpha^2}}r + r^2\arcsin\frac{1}{\sqrt{1+\alpha^2}} -$$

$$-\alpha\frac{1}{1+\alpha^2}r^2 = r^2\arcsin\frac{1}{\sqrt{1+\alpha^2}} = r^2\theta,$$ ove $\theta = \arcsin\frac{1}{\sqrt{1+\alpha^2}}$ è l'angolo di semiapertura.

- $S_3 = \{(x,y) : 2x^2-3/2 \le y \le \frac{x}{2}+1,\ |y|\le 1\}$



$\begin{cases} y=2x^2-3/2 \\ y=x/2+1 \end{cases}$ $\quad 2x^2-3/2 = \frac{x}{2}+1 \quad 4x^2-x-5=0 \quad x=-1,\ 5/4$

$\begin{cases} y=2x^2-3/2 \\ y=1 \end{cases}$ $\quad x=\pm\frac{\sqrt5}{2}$; $\begin{cases} y=2x^2-3/2 \\ y=-1 \end{cases}$ $\quad x=\pm 1/2$

$$\text{Area}(S_3) = \int_{-1}^{0}\left(\frac{x}{2}+1\right)dx + \int_0^{\sqrt5/2}1\,dx + \int_{\sqrt5/2}^{1/2}(2x^2-3/2)\,dx$$

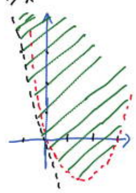
$$+\int_{1/2}^{-1/2}(-1)\,dx + \int_{-1/2}^{-1}(2x^2-3/2)\,dx =$$

$$\left(\frac{x^2}{4}+x\right|_{-1}^{0} + \left(x\right|_0^{\sqrt5/2} + \left(\frac{2x^3}{3}-\frac{3}{2}x\right|_{\sqrt5/2}^{1/2} + \left(-x\right|_{1/2}^{-1/2} + \left(\frac{2x^3}{3}-\frac{3}{2}x\right|_{-1/2}^{-1} = \ldots$$

- $S_5 = \{(x,y) : \sqrt{3x+y} \geq x \ , \ 2|y| \leq x+y+5\}$

$$\sqrt{3x+y} \geq x \iff \left(\begin{cases} 3x+y \geq 0 \\ x<0 \end{cases}\right) \vee \left(\begin{cases} 3x+y \geq 0 \\ x \geq 0 \\ 3x+y \geq x^2 \end{cases}\right)$$

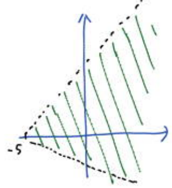
$$\iff \left(\begin{cases} x<0 \\ y \geq -3x \end{cases}\right) \vee \left(\begin{cases} x \geq 0 \\ y \geq -3x \\ y \geq x^2-3x \end{cases}\right)$$

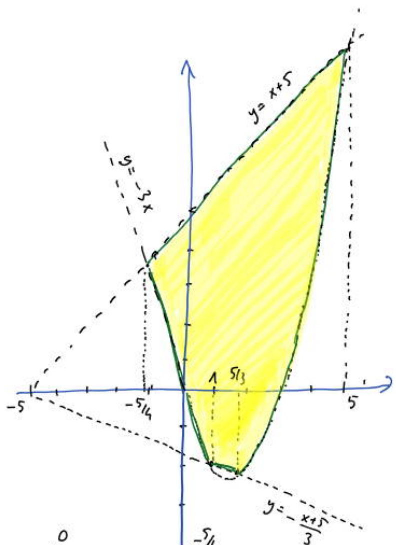


$2|y| \leq x+y+5$

(se $y \geq 0$) $2y \leq x+y+5$ cioè $y \leq x+5$

(se $y<0$) $-2y \leq x+y+5$ cioè $y \geq -\frac{x+5}{3}$





Intersezioni rette - parabole:

$\begin{cases} y = x^2-3x \\ y = x+5 \end{cases} \iff x=-1,\ x=5$; $\begin{cases} y = x^2-3x \\ y = -\frac{x+5}{3} \end{cases} \iff x=1,\ x=5/3$

$$\text{Area}(S_5) = \int_{-5/4}^{5} (x+5)\,dx + \int_{5}^{5/3} (x^2-3x)\,dx + \int_{5/3}^{1} \left(-\frac{x+5}{3}\right) dx + \int_{1}^{0} (x^2-3x)\,dx + \int_{0}^{-5/4} (-3x)\,dx$$

$$= \left(\frac{x^2}{2}+5x\right)\Big|_{-5/4}^{5} + \left(\frac{x^3}{3}-\frac{3x^2}{2}\right)\Big|_{5}^{5/3} + \left(-\frac{1}{3}\left(\frac{x^2}{2}+5x\right)\right)\Big|_{5/3}^{1} + \left(\frac{x^3}{3}-\frac{3x^2}{2}\right)\Big|_{1}^{0} + \left(-\frac{3x^2}{2}\right)\Big|_{0}^{-5/4}$$

$$= \ldots$$

- $\displaystyle\int \frac{dx}{\sqrt{e^{2x}-4}} \quad \left[\begin{array}{ll} e^{2x}-4 = t^2 & x = \frac{1}{2}\lg(t^2+4) \\ e^{2x} = t^2+4 & dx = \frac{t}{t^2+4}\,dt \\ 2x = \lg(t^2+4) & \end{array}\right] = \int \frac{1}{\cancel{t}} \cdot \frac{\cancel{t}}{t^2+4}\,dt = \frac{1}{4}\int \frac{dt}{1+(t/2)^2} =$

$$= \frac{1}{2}\int \frac{1/2\,dt}{1+(t/2)^2} = \frac{1}{2}\operatorname{arctg}(t/2) + K = \frac{1}{2}\operatorname{arctg}\left(\frac{\sqrt{e^{2x}-4}}{2}\right) + K$$

- $\displaystyle\int \frac{dx}{1-\sin x}$ Facciamo in due modi

(1) Con le formule parametriche: $t = \operatorname{tg} x/2$, $\sin x = \frac{2t}{1+t^2}$, $x = 2\operatorname{arctg} t \Rightarrow dx = \frac{2}{1+t^2}\,dt$

Dunque $\displaystyle\int \frac{dx}{1-\sin x} = \int \frac{1}{1-\frac{2t}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt = \int \frac{\cancel{1+t^2}}{(1-t)^2} \cdot \frac{2}{\cancel{1+t^2}}\,dt = \frac{2}{1-t} + K = \frac{2}{1-\operatorname{tg} x/2} + K$

(2) con le formule di bisezione $\sin^2\frac{u}{2} = \frac{1-\cos u}{2}$. Poniamo $u = \pi/2 - x$ (angoli complementari):

$$\int \frac{dx}{1-\sin x} = \int \frac{-du}{1-\sin(\pi/2-u)} = -\int \frac{du}{1-\cos u} = -\int \frac{1/2}{\sin^2(u/2)}\,du = \cot\left(\frac{u}{2}\right) =$$

$$= \cot\left(\pi/4 - x/2\right) = \left[\begin{array}{l}\text{ricordando che} \\ \cot(\alpha\pm\beta) = \frac{\cot\alpha\cot\beta\mp 1}{\cot\beta\pm\cot\alpha}\end{array}\right] \quad \frac{\cot\pi/4\,\cot x/2 + 1}{\cot x/2 - \cot\pi/4}$$

$$= \frac{\cot x/2 + 1}{\cot x/2 - 1} = \frac{1+\tan x/2}{1-\tan x/2} + h \quad \left(\text{notiamo che } \frac{2}{1-\tan x/2} = \frac{1+\tan x/2}{1-\tan x/2} + 1,\right.$$

dunque nei due casi abbiamo ottenuto la stessa famiglia di primitive).

- $$\int \frac{dx}{x^2+x+2} = \int \frac{dx}{(x+1/2)^2 + 7/4} = \frac{4}{7}\int \frac{dx}{\left(\frac{2x+1}{\sqrt7}\right)^2+1} = \frac{\cancel{4}^{\,2}}{\cancel{7}}\cdot\frac{\cancel{\sqrt7}}{\cancel{2}}\arctan\left(\frac{2x+1}{\sqrt7}\right) + K$$

- $\int \frac{dx}{\sqrt{x^2-5x+6}}$ — le radici di $x^2-5x+6=0$ sono 2 e 3. Converrà centrare la variabile per renderle simmetriche, poi useremo l'integrale immediato $\int \frac{du}{\sqrt{u^2-1}} = \operatorname{sett\,cosh} u$

$$\int \frac{dx}{\sqrt{x^2-5x+25/4-1/4}} = \int \frac{dx}{\sqrt{(x-5/2)^2-1/4}} = 4\int \frac{dx}{\sqrt{(2x-5)^2-1}} = \left(\begin{array}{l}\text{posto } 2x-5=u \\ x = \frac{u+5}{2} \\ dx = \frac12 du\end{array}\right) :$$

$$2\int \frac{du}{\sqrt{u^2-1}} = 2\operatorname{sett\,cosh} u + K = 2\operatorname{sett\,cosh}(2x-5) + K$$

- $$\int \frac{3x+1}{2x-3}\,dx \quad \left(\begin{array}{l}2x-3=t \\ x=\frac{t+3}{2} \\ dx = \frac12 dt\end{array}\right) = \int \frac{\frac32(t+3)+1}{t}\cdot\frac12\,dt = \frac14\int\frac{3t+11}{t}\,dt$$

$$= \frac14\int\left(3+\frac{11}{t}\right)dt = \frac14\left(3t + 11\log|t|\right) + K, \text{ poi } t = 2x-3.$$

- $\int e^{\arcsin x}\,dx$. Poniamo $x = \sin t \Rightarrow dx = \cos t\,dt$.

Dunque $\int e^{\arcsin x}\,dx = \int e^{\arcsin(\sin t)}\cos t\,dt = \int e^t\cos t\,dt$.

Ora $\int \overset{F}{e^t}\,\overset{g}{\cos t}\,dt \overset{\substack{f=e^t\\G=\sin t}}{=} e^t\sin t - \int \overset{F}{e^t}\,\overset{g}{\sin t} \overset{\substack{f=e^t\\G=-\cos t}}{=} e^t\sin t - \left((-\cos t)e^t - \int e^t(-\cos t)\,dt\right)$

$= e^t(\sin t + \cos t) - \int e^t\cos t \Rightarrow \int e^t\cos t = \frac12 e^t(\sin t + \cos t)$,

dunque $\int e^{\arcsin x}\,dx = \frac12 e^{\arcsin x}\left(\sin(\arcsin x) + \cos(\arcsin x)\right) + K$

$= \frac12 e^{\arcsin x}\left(x + \sqrt{1-x^2}\right) + K.$

• $\int \frac{x}{x^3+1}\,dx$. Come detto si può scrivere $\frac{x}{x^3+1} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1} =$

$$= \frac{(A+B)x^2+(-A+B+C)x+(A+C)}{(x+1)(x^2-x+1)} \iff \begin{cases} A+B=0 \\ -A+B+C=1 \\ A+C=0 \end{cases} \iff (A,B,C) = \left(-\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

Dunque $\int \frac{x}{x^3+1}\,dx = \frac{1}{3}\left(\int \frac{x+1}{x^2-x+1} - \frac{1}{x+1}\right)dx = \frac{1}{3}\left(\int \frac{x+1}{(x-1/2)^2+3/4}\,dx - \log|x+1|\right)$

$$= \frac{1}{3}\left(\frac{4}{3}\int \frac{x+1}{\left(\frac{2x-1}{\sqrt{3}}\right)^2+1}\,dx - \log|x+1|\right) \quad \left[\frac{2x-1}{\sqrt{3}} = t \quad x = \frac{t\sqrt{3}+1}{2} \quad dx = \frac{\sqrt{3}}{2}\,dt\right]$$

$$= \frac{1}{3}\left(\frac{4}{3}\int \frac{\frac{t\sqrt{3}+1}{2}+1}{t^2+1}\,\frac{\sqrt{3}}{2}\,dt - \log|x+1|\right) = \frac{1}{3}\left(\int \frac{t+\sqrt{3}}{t^2+1}\,dt - \log|x+1|\right)$$

$$= \frac{1}{3}\left(\frac{1}{2}\int \frac{2t}{t^2+1}\,dt + \sqrt{3}\int \frac{dt}{t^2+1} - \log|x+1|\right)$$

$$= \frac{1}{3}\left(\log\sqrt{t^2+1} + \sqrt{3}\,\mathrm{arctg}\, t - \log|x+1|\right) + K$$

$$= \frac{1}{3}\log\left(\frac{\sqrt{x^2-x+1}}{x+1} + \frac{1}{\sqrt{3}}\,\mathrm{arctg}\left(\frac{2x-1}{\sqrt{3}}\right)\right) + K$$

• $\int \frac{\sqrt{x}}{x-1}\,dx \quad \left[\begin{array}{l} x=t^2 \\ dx = 2t\,dt \end{array}\right] = \int \frac{t}{t^2-1}\cdot 2t\,dt = 2\int \frac{t^2}{t^2-1}\,dt =$

$$= 2\int\left(1+\frac{1}{t^2-1}\right)dt = 2t + \int\left(\frac{1}{t-1} - \frac{1}{t+1}\right)dt = 2t + \log\left|\frac{t-1}{t+1}\right| + K$$, poi $t=\sqrt{x}$.

• $\int \frac{1+\sqrt{x}}{1+\sqrt[3]{x}}\,dx$. $\left[\begin{array}{l} x=t^6 \\ dx = 6t^5\,dt \end{array}\right] = \int \frac{1+t^3}{1+t^2}\cdot 6t^5\,dt = 6\int \frac{t^5(t^3+1)}{t^2+1}\,dt$

$$\begin{array}{l|l}
t^8 + \quad\quad t^5 & t^2+1 \\ \hline
-t^8 \quad -t^6 & t^6-t^4+t^3+t^2-t-1 \\ \hline
// \quad -t^6+t^5 & \\
\quad\ +t^6 \quad +t^4 & \\ \hline
\quad // \quad t^5+t^4 & \\
\quad\quad -t^5 \quad -t^3 & \\ \hline
\quad\quad // \quad t^4-t^3 & \\
\quad\quad\quad -t^4 \quad -t^2 & \\ \hline
\quad\quad\quad // \quad -t^3-t^2 & \\
\quad\quad\quad\quad +t^3 \quad +t & \\ \hline
\quad\quad\quad\quad // \quad -t^2+t & \\
\quad\quad\quad\quad\quad +t^2 \quad +1 & \\ \hline
\quad\quad\quad\quad\quad // \quad t+1 &
\end{array}$$

$$t^5(t^3+1) = (t^2+1)(t^6-t^4+t^3+t^2-t-1) + (t+1)$$

Dunque l'integrale vale $6\int\left(t^6-t^4+t^3+t^2-t-1+\frac{t+1}{t^2+1}\right)dt$

$$= 6\left(\frac{t^7}{7} - \frac{t^5}{5} + \frac{t^4}{4} + \frac{t^3}{3} - \frac{t^2}{2} - t + \frac{1}{2}\log(t^2+1) + \mathrm{arctg}\, t\right) + K$$, poi $t = \sqrt[6]{x}$

• $$\int \frac{1+\cos^2 x}{(1+\cos x)\sin x}\,dx = \int \frac{(1+\cos^2 x)\sin x}{(1+\cos x)\sin^2 x}\,dx = \int \frac{(1+\cos^2 x)\sin x}{(1+\cos x)(1-\cos^2 x)}\,dx.$$

$$\left[\begin{array}{l} \cos x = t \\ -\sin x\,dx = dt \end{array}\right] = \int \frac{t^2+1}{(t+1)(t^2-1)}\,dt = \int \frac{t^2+1}{(t+1)^2(t-1)}\,dt.$$

Seguendo gli esempi precedenti, cerchiamo una decomposizione della funzione razionale integranda:

$$\frac{t^2+1}{(t+1)^2(t-1)} = \frac{A}{t+1} + \frac{B}{(t+1)^2} + \frac{C}{t-1} = \frac{A(t^2-1)+B(t-1)+C(t+1)^2}{(t+1)^2(t-1)}$$

$$= \frac{(A+C)t^2+(B+2C)t+(-A-B+C)}{(t+1)^2(t-1)} \iff \begin{cases} A+C=1 \\ B+2C=0 \\ -A-B+C=1 \end{cases} \quad \begin{cases} C=1-A \\ B=-2C=-2+2A \\ -A+2-2A+1-A=1 \end{cases}$$

$$\begin{cases} A=1/2 \\ B=-1 \\ C=1/2 \end{cases}, \text{ dunque } \int \frac{t^2+1}{(t+1)^2(t-1)}\,dt = \frac{1}{2}\int \left(\frac{1}{t+1} - \frac{2}{(t+1)^2} + \frac{1}{t-1}\right)dt$$

$$= \frac{1}{2}\left(\log|t^2-1| + \frac{2}{t+1}\right) + K = \frac{1}{1+\cos x} + \log\sqrt{1-\cos^2 x} = \frac{1}{1+\cos x} + \log|\sin x| + K.$$