A Appendices

A.1 Categories

We refer e.g. to Gelfand-Manin [4] or Kashiwara-Schapira [10] for further details.

Categories. A *category* $\mathcal{C}$ consists in the following data:

(a) a family of objects $\mathrm{Ob}(\mathcal{C})$,

(b) for any $X, Y \in \mathrm{Ob}(\mathcal{C})$ a set of morphisms $\mathrm{Hom}_{\mathcal{C}}(X, Y)$,

(c) for any triple $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ a composition law

$$\cdot \circ \cdot : \mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(Y,Z) \to \mathrm{Hom}_{\mathcal{C}}(X,Z), \qquad (f,g) \mapsto g \circ f$$

such that (1) $\cdot \circ \cdot$ is associative and (2) for any $X \in \mathrm{Ob}(\mathcal{C})$ there exists an "identity morphism" $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X,X)$ such that $f \circ \mathrm{id}_X = f$ and $\mathrm{id}_X \circ g = g$ for any $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(Y,X)$. A morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ is said to be a *monomorphism* (resp. *epimorphism*) if $f \circ g = f \circ g'$ (resp. $g \circ f = g' \circ f$) implies $g = g'$. f is said *isomorphism* if there exists a (unique) $g \in \mathrm{Hom}_{\mathcal{C}}(Y,X)$ (the inverse of f) such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. In such a case we shall write also $X \simeq Y$. Note that a isomorphism is both a monomorphism and an epimorphism. To denote $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ one often uses the functional notation $f : X \to Y$, without mentioning $\mathcal{C}$ (which should be clear from the context).

A category $\mathcal{C}$ is said to be *small* if $\mathrm{Ob}(\mathcal{C})$ is a set. Given a category $\mathcal{C}$, the *opposed* category $\mathcal{C}^{\mathrm{op}}$ is characterized by $\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C})$ and $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(Y,X)$. We shall say that a category $\mathcal{C}''$ is a *subcategory* of $\mathcal{C}$ if $\mathrm{Ob}(\mathcal{C}'') \subset \mathrm{Ob}(\mathcal{C})$ and, for any $X, Y \in \mathrm{Ob}(\mathcal{C}'')$, it holds $\mathrm{Hom}_{\mathcal{C}''}(X,Y) \subset \mathrm{Hom}_{\mathcal{C}}(X,Y)$. If $\mathrm{Hom}_{\mathcal{C}''}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(X,Y)$, one says that the subcategory is *full*.

Examples. **(1)** The category $\mathfrak{Set}$ of sets and map of sets. A monomorphism (resp. epimorphism, isomorphism) in $\mathfrak{Set}$ is a map which is injective (resp. surjective, bijective). **(2)** The category $\mathfrak{Top}$ of topological spaces and continuous maps (it is a a subcategory, not full, of $\mathfrak{Set}$). A isomorphism in $\mathfrak{Top}$ is called homeomorphism. **(3)** The category $\mathfrak{Groups}$ of groups and homomorphisms of group. **(4)** The category $\mathfrak{Mod}(A)$ of left A-modules on a unitary ring A and A-linear morphisms (see Appendix A.2). In particular, for $A = \mathbb{Z}$ (resp. $A = \mathbb{K}$ a field), one obtains the category of abelian groups (resp. $\mathbb{K}$-vector spaces). Note that $\mathfrak{Mod}(A)^{\mathrm{op}}$ is the category of right A-modules. One uses the notation Hom_A instead of $\mathrm{Hom}_{\mathfrak{Mod}(A)}$. **(5)** Given a poset $(I, \leq)$[96], one defines the category $\mathcal{I}$ with $\mathrm{Ob}(\mathcal{I}) = I$ and $\mathrm{Hom}_{\mathcal{I}}(i,j) = \{\mathrm{pt}\}$ if $i \leq j$ and $= \varnothing$ otherwise.

[96] A *poset* is a set endowed with a *preorder*, i.e. a relation which is reflexive and transitive; hence a symmetric preorder is a equivalence, while an antisymmetric preorder is a (partial) order.

Functors. Let $\mathcal{C}$ and $\mathcal{C}'$ be two categories. A *covariant functor* (resp. *contravariant functor*) $F : \mathcal{C} \to \mathcal{C}'$ is the data of:

(a) a map $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C}')$,

(b) for any $X, Y \in \mathrm{Ob}(\mathcal{C})$ a "map of morphisms"

$$F : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}'}(FX, FY)$$

(resp. $F : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}'}(FY, FX)$),

which respects the identity and the compositions, i.e. $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ and $F(f \circ g) = F(f) \circ F(g)$ (resp. $F(f \circ g) = F(g) \circ F(f)$). Note that a contravariant functor $F : \mathcal{C} \to \mathcal{C}'$ is just a covariant functor (we shall simply say "functor", "covariant" being understood) from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{C}'$. The composition of functors and the identity functor $\mathrm{id}_{\mathcal{C}}$ are defined in a natural way. Two functors $F : \mathcal{C} \to \mathcal{C}'$ and $G : \mathcal{C}' \to \mathcal{C}$ are called *adjoint*, and F (resp. G) a *left adjoint* (resp. *right adjoint*) of G (resp. F), if for any $X \in \mathrm{Ob}(\mathcal{C})$ and any $Y \in \mathrm{Ob}(\mathcal{C}')$ ona has a functorial isomorphism (in both variables)[97]

$$\mathrm{Hom}_{\mathcal{C}}(X, GY) \simeq \mathrm{Hom}_{\mathcal{C}'}(FX, Y).$$

One shows (exercise) that, if it exists, the left (resp. right) adjoint of F (resp. G) is unique up to isomorphism.

Examples. **(1)** Given a category $\mathcal{C}$ and an object $X \in \mathrm{Ob}(\mathcal{C})$, $\mathrm{Hom}_{\mathcal{C}}(X, \cdot\,)$ (resp. $\mathrm{Hom}_{\mathcal{C}}(\,\cdot\,, X)$) is a covariant (resp. contravariant) functor from $\mathcal{C}$ to $\mathfrak{Set}$. For example, for the functor $\mathrm{Hom}_{\mathcal{C}}(X, \cdot\,)$ one has $\mathrm{Ob}(\mathcal{C}) \ni Y \mapsto \mathrm{Hom}_{\mathcal{C}}(X, Y) \in \mathrm{Ob}(\mathfrak{Set})$ and, given a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$, it holds $\mathrm{Hom}_{\mathcal{C}}(X, f) : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$, $\alpha \mapsto f \circ \alpha$. **(2)** Let $(I, \leq)$ be a poset. Any covariant (resp. contravariant) functor $\mathcal{I} \to \mathcal{C}$ is naturally identified with the classic definition of inductive (resp. projective) system in $\mathcal{C}$ indexed by I (see below). **(3)** Let A be a unitary ring, B a subring with center $Z(A)$ (for example, if A is commutative one can take $B = A$; in any case it is possible to choose $B = \mathbb{Z}$). Some classical examples of functors in the category of modules on a ring (see Appendix A.2): (a) for $\mathcal{C} = \mathcal{C}' = \mathfrak{Mod}(A)$ and $K \in \mathfrak{Mod}(B)$, one has the functors $\cdot \otimes_B K$ and $\mathrm{Hom}_B(K, \cdot\,)$; (b) for $\mathcal{C} = \mathfrak{Mod}(A)$ and $\mathcal{C}' = \mathfrak{Mod}(B)$, one has *for* (the functor which associates to any A-module the underlying B-module)[98] and, for $Q \in \mathfrak{Mod}(A)$, the functor $\mathrm{Hom}_A(Q, \cdot\,)$; (c) for $\mathcal{C} = \mathfrak{Mod}(B)$ and $\mathcal{C}' = \mathfrak{Mod}(A)$, one has $A \otimes_B \cdot$ (the extension of coefficients) and, given $Q \in \mathfrak{Mod}(A)$, the functor $\cdot \otimes_B Q$. One shows that the functors *for* and $A \otimes_B \cdot$ are adjoint to each other (exercise), as well as the functors in (a) and the second ones in (b) and (c) (see (A.1)).

Inductive and projective limits. Let $\mathcal{C}$ be a category, $(I, \leq)$ a poset. An *inductive system* in $\mathcal{C}$ indexed by I is the datum of (a) a family $\{X_i : i \in I\}$ in $\mathrm{Ob}(\mathcal{C})$, and (b) for any

[97] This means that, given $f \in \mathrm{Hom}_{\mathcal{C}}(X, X')$ and $g \in \mathrm{Hom}_{\mathcal{C}'}(Y, Y')$, the following diagrams are commutative:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(X', GY) \arrow[r, "\sim"] \arrow[d, "\mathrm{Hom}_{\mathcal{C}}(f, GY)"] & \mathrm{Hom}_{\mathcal{C}'}(FX', Y) \arrow[d, "\mathrm{Hom}_{\mathcal{C}'}(F(f), Y)"] \\
\mathrm{Hom}_{\mathcal{C}}(X, GY) \arrow[r, "\sim"] & \mathrm{Hom}_{\mathcal{C}'}(FX, Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(X, GY) \arrow[r, "\sim"] \arrow[d, "\mathrm{Hom}_{\mathcal{C}}(X, G(g))"] & \mathrm{Hom}_{\mathcal{C}'}(FX, Y) \arrow[d, "\mathrm{Hom}_{\mathcal{C}'}(FX, g)"] \\
\mathrm{Hom}_{\mathcal{C}}(X, GY') \arrow[r, "\sim"] & \mathrm{Hom}_{\mathcal{C}'}(FX, Y')
\end{tikzcd}$$

[98] "*for*" stands for "*forgetful*", because a part of the structure of M on A gets forgotten when reduced to the one on B.

$i \le j$, of morphisms $\phi_{i,j} \in \mathrm{Hom}_{\mathcal{C}}(X_i, X_j)$ such that $\phi_{j,k} \circ \phi_{i,j} = \phi_{i,k}$ if $i \le j \le k$: such a system can be seen as a functor from the category $\mathcal{I}$ associated to I (see above) to $\mathcal{C}$. The *inductive limit* of the system is a object $X \in \mathrm{Ob}(\mathcal{C})$ together with a family of morphisms $\phi_i \in \mathrm{Hom}_{\mathcal{C}}(X_i, X)$ for any $i \in I$ such that $\phi_i = \phi_j \circ \phi_{i,j}$ if $i \le j$, characterized by the following universal property: given any $Y \in \mathrm{Ob}(\mathcal{C})$ and morphisms $\psi_i \in \mathrm{Hom}_{\mathcal{C}}(X_i, Y)$, there exists a unique morphism $\psi \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ such that $\psi_i = \psi \circ \phi_i$ for any $i \in I$. By considering the category $\mathcal{I}^{\mathrm{op}}$ instead of $\mathcal{I}$, one obtains the dual notions of *projective system* and *projective limit* in $\mathcal{C}$ indexed by I: the former is the datum of (a) a family $\{X_i : i \in I\}$ in $\mathrm{Ob}(\mathcal{C})$, and (b) for any $i \le j$, of morphisms $\phi_{j,i} \in \mathrm{Hom}_{\mathcal{C}}(X_j, X_i)$ such that $\phi_{j,i} \circ \phi_{k,j} = \phi_{k,i}$ if $i \le j \le k$, while the latter is an object $X \in \mathrm{Ob}(\mathcal{C})$ with a family of morphisms $\phi_i \in \mathrm{Hom}_{\mathcal{C}}(X, X_i)$ for any $i \in I$ such that $\phi_i = \phi_{j,i} \circ \phi_j$ if $i \le j$ such that, given any $Y \in \mathrm{Ob}(\mathcal{C})$ with morphisms $\psi_i \in \mathrm{Hom}_{\mathcal{C}}(Y, X_i)$, there exists a unique morphism $\psi \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$ such that $\psi_i = \phi_i \circ \psi$ for any $i \in I$. The inductive and projective limits, which (it they exist) are unique up to canonical isomorphism, are denoted respectively by $\varinjlim X_i$ and $\varprojlim X_i$.

Products and coproducts. A particular case of inductive and projective limits is given by the *product* and *coproduct* of a family $\{X_\lambda : \lambda \in \Lambda\}$ of objects of $\mathcal{C}$. The former is an object P of $\mathcal{C}$ with a family of morphisms $(p_\lambda \in \mathrm{Hom}_{\mathcal{C}}(P, X_\lambda))_{\lambda\in\Lambda}$ such that, given any object Y of $\mathcal{C}$ and a family of morphisms $(f_\lambda \in \mathrm{Hom}_{\mathcal{C}}(Y, X_\lambda))_{\lambda\in\Lambda}$, there exists a unique morphism $f \in \mathrm{Hom}_{\mathcal{C}}(Y, P)$ such that $f_\lambda = p_\lambda \circ f$ for any $\lambda \in \Lambda$; dually, the latter is an object C of $\mathcal{C}$ with a family of morphisms $(i_\lambda \in \mathrm{Hom}_{\mathcal{C}}(X_\lambda, C))_{\lambda\in\Lambda}$ such that, for any object Z of $\mathcal{C}$ and morphisms $(g_\lambda \in \mathrm{Hom}_{\mathcal{C}}(X_\lambda, Z))_{\lambda\in\Lambda}$, there exists a unique $g \in \mathrm{Hom}_{\mathcal{C}}(C, Z)$ such that $g_\lambda = g \circ i_\lambda$ for any $\lambda \in \Lambda$. Such objects, which (if they exist) are unique up to canonical isomorphism, are denoted respectively by $\prod_{\lambda\in\Lambda} X_\lambda$ and $\coprod_{\lambda\in\Lambda} X_\lambda$, and they clearly coincide respectively with $\varprojlim X_\lambda$ and $\varinjlim X_\lambda$ (with the trivial preorder on Λ).

Examples. (1) In $\mathfrak{Set}$, products and coproducts exist and are respectively the cartesian product and the disjoint union of the sets X_λ, with the natural maps of projection and inclusion. In this way, the universal properties of product and coproduct in a category $\mathcal{C}$ could be expressed by the formulas (where product and coproduct in the l.h.s. members are in $\mathcal{C}$, and in r.h.s. members in $\mathfrak{Set}$):

$$\mathrm{Hom}_{\mathcal{C}}(Y, \prod_{\lambda\in\Lambda} X_\lambda) \simeq \prod_{\lambda\in\Lambda} \mathrm{Hom}_{\mathcal{C}}(Y, X_\lambda), \qquad \mathrm{Hom}_{\mathcal{C}}(\coprod_{\lambda\in\Lambda} X_\lambda, Z) \simeq \prod_{\lambda\in\Lambda} \mathrm{Hom}_{\mathcal{C}}(X_\lambda, Z).$$

(2) In $\mathfrak{Groups}$, products and coproducts exist and are respectively the direct product and the *free product* (see §1.5) of groups X_λ. **(3)** In $\mathfrak{Mod}(A)$ (where A is a unitary ring, see Appendix A.2), products and coproducts exist and are respectively the direct product and the *direct sum* of the A-modules X_λ:

$$\bigoplus_{\lambda\in\Lambda} X_\lambda = \{(x_\lambda)_{\lambda\in\Lambda} \in \prod_{\lambda\in\Lambda} X_\lambda : x_\lambda = 0 \text{ excepted a finite number of } \lambda \in \Lambda\}.$$

In these three cases, if all of X_λ are equal to a same X, it is clear that $\prod_{\lambda\in\Lambda} X_\lambda \simeq X^\Lambda$ (functions of Λ in X) and, in $\mathfrak{Mod}(A)$, one also has $\bigoplus_{\lambda\in\Lambda} X_\lambda \simeq X^{(\Lambda)}$ (functions of Λ in X vanishing except than on a finite number of $\lambda \in \Lambda$). **(4)** In $\mathfrak{Top}$, products and coproducts exist and are respectively the usual topological product $X \times Y$ and the disjoint union (or "sum") $X \sqcup Y$. There are products and coproducts also in $\mathfrak{Top}_p$ (the category of pointed topological spaces, see p. 14): for (X, x_0) and (Y, y_0), they are respectively the pointed product $(X \times Y, (x_0, y_0))$ and the wedge sum $(X \vee Y, z_0)$ (see p. 9; here z_0 denotes the unique point in which both x_0 and y_0 are identified)

A.2 Modules on a ring

We shall deal with the category $\mathfrak{Mod}(A)$ of modules on a unitary ring A and of A-linear morphisms; recall that, if A is a ring, a *left* (resp. *right*) *A-module* is an abelian group $(M,+)$ endowed with a multiplication $A \times M \to M$ ($(a,m) \mapsto am$) such that $(a+b)m = am+bm$, $a(m+m') = am+am'$ and $a(bm) = (ab)m$ for any $a, b \in A$ (resp. $a(bm) = (ba)m$ for any $a, b \in A$ and $m, m' \in M$: in this case one better writes the multiplication on the right, i.e. $(ma)b = m(ab)$). Clearly, if A is commutative the two notions coincide. In particular, as we already said, for $A = \mathbb{Z}$ (resp. $A = \mathbb{K}$ a field), one obtains the category of abelian groups (resp. $\mathbb{K}$-vector spaces).

A.2.1 Tensor product

Let B be a subring of center $Z(A)$, and let M (resp. N) be a left (resp. right) A-module, K a B-module. We say that a map $\alpha : N \times M \to K$ is *$\otimes$-bilinear* if: (1) α is B-bilinear, i.e. $\alpha(nb+n'b', m) = b\alpha(n,m)+b'\alpha(n',m)$ and $\alpha(n, bm+b'm') = b\alpha(n,m)+b'\alpha(n,m')$ for any $b, b' \in B$, $n, n' \in N$ and $m, m' \in M$; (2) $\alpha(na, m) = \alpha(n, am)$ for any $a \in A$, $n \in N$ and $m \in M$.

Definition A.2.1. The *tensor product* of N and M is the B-module $N \otimes_A M$, characterized by the following universal property: (1) there exists a $\otimes$-bilinear map $t : N \times M \to N \otimes_A M$, and (2) given any B-module K, for any $\otimes$-bilinear map $\alpha : N \times M \to K$ there exists a unique map B-linear $\widetilde{\alpha} : N \otimes_A M \to K$ such that $\alpha = \widetilde{\alpha} \circ t$.

The uniqueness of $N \otimes_A M$, up to B-linear isomorphisms, is easy to prove (exercise). One is left with proving the existence. Let X the free B-module generated by the set $\{n \otimes m : (n,m) \in N \times M\}$, Y the submodule generated by the elements $(na) \otimes m - n \otimes (am)$, $(bn+b'n') \otimes m - b(n \otimes m) - b'(n' \otimes m)$ and $n \otimes (bm+b'm') - b(n \otimes m) - b'(n \otimes m')$ (where $a \in A$, $b, b' \in B$, $n, n' \in N$, $m, m' \in M$), and consider the B-module X/Y (i.e. the B-module with generators $n \otimes m$ and relations $(na) \otimes m = n \otimes (am)$, $(bn+b'n') \otimes m = b(n \otimes m) + b'(n' \otimes m)$ and $n \otimes (bm + b'm') = b(n \otimes m) + b'(n \otimes m')$). The obvious map $t : N \times M \to X/Y$, $t(n,m) = n \otimes m$ is $\otimes$-bilinear; given then α as above, define $\widetilde{\alpha}(n \otimes m) = \alpha(n,m)$. The well-posedness of $\widetilde{\alpha}$ comes from the $\otimes$-bilinearity of α, the fact that $\alpha = \widetilde{\alpha} \circ t$ is evident and the uniqueness of $\widetilde{\alpha}$ is ensured by the fact that the elements $n \otimes m$ generate $N \otimes_A M$.

Examples. **(1)** One can always perform the tensor product of abelian groups, in the sense of $\mathbb{Z}$-modules (hence $A = B = \mathbb{Z}$). For example, if k is a field with $\chi(k) \neq 2$ it holds $k \otimes_{\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z}) = 0$ (namely, $r \otimes 0 = 0$, and $r \otimes 1 = 2(r/2) \otimes 1 = (r/2) \otimes 2(1) = 0$). **(2)** *Extension of the scalars.* Let R be a ring, $\phi : A \to R$ a ring monomorphism, M a left A-module: since R is naturally a right A-module, it makes sense to considerar $\widetilde{M} = R \otimes_A M$. The B-module $\widetilde{M}$ has a structure of left R-module (hence, in particular, of left A-module), by defining $r(s \otimes m) = (rs) \otimes m$.

Now let us list some properties (that the student could verify by exercise).

(1) If A is commutative, then $N \otimes_A M \simeq M \otimes_A N$ and $P \otimes_A (N \otimes_A M) \simeq (P \otimes_A N) \otimes_A M$ (as A-modules).

(2) The B-module $A \otimes_A M$ has a structure of A-module (see Example A.2.1), and there is a natural isomorphism $A \otimes_A M \simeq M$.

(3) Let K be a B-module and Q be a left A-module. The B-module $K \otimes_B Q$ (resp. $\mathrm{Hom}_B(K,Q)$, $\mathrm{Hom}_B(Q,K)$) has a structure of left (resp. left, right) A-module by defining $a(k \otimes q) = k \otimes (aq)$ (resp. $(a\varphi)(k) = a\varphi(k)$, $(\psi a)(q) = \psi(aq)$). The same statements (exchanging right with left) hold if Q is a right A-module.

(4) *Adjunction* Hom-$\otimes$. Given a B-module K, two left A-modules M and N and a right A-module Q, one has the following functorial isomorphisms of B-modules:

$$\begin{array}{llll} \text{(A.1)} & \mathrm{Hom}_A(K \otimes_B N, M) \simeq & \mathrm{Hom}_A(N, \mathrm{Hom}_B(K,M)) \simeq & \mathrm{Hom}_B(K, \mathrm{Hom}_A(N,M)), \\ & \mathrm{Hom}_B(Q \otimes_A M, K) \simeq & \mathrm{Hom}_A(Q, \mathrm{Hom}_B(M,K)) \simeq & \mathrm{Hom}_A(M, \mathrm{Hom}_B(Q,K)). \end{array}$$

(5) *Bifunctoriality.* Given a morphism $f : N \to N'$ (resp. $g : M \to M'$) of right (resp. left) A-modules, it is possible to define the morphism of B-modules $f \otimes g : N \otimes_A M \to N' \otimes_A M'$ by extending for B-linearity the map $n \otimes m \mapsto f(n) \otimes g(m)$, i.e. $(f \otimes g)(\sum_i b_i(n_i \otimes m_i)) = \sum_i b_i(f(n_i) \otimes g(m_i))$;

(6) *Commutations.* With the direct sum: $N \otimes_A (M \oplus M') \simeq (N \otimes_A M) \oplus (N \otimes_A M')$. With the cokernel: given a morphism of A-modules $f : M \to M'$, there is an isomorphism of B-modules $N \otimes_A \mathrm{coker}(f) \simeq \mathrm{coker}(\mathrm{id}_N \otimes f)$.

More generally, one can define a tensor product for graded A-modules:[(99)] given a right graded A-module N and a left graded A-module M, their tensor product $N \otimes_A M$ is a B-module which is graded by

$$(N \otimes_A M)_n = \bigoplus_{i+j=n} N_i \otimes_A M_j. \tag{A.2}$$

A.2.2 Complexes and cohomology

We introduce the notion of cohomology of complexes (of cochains) in the category $\mathfrak{Mod}(A)$; however, the notions introduced here will be valid in the natural generalization of abelian categories[(100)].

[(99)] An A-module M is said to be *graded* if it is endowed with a family of submodules $\{M_n : n = 0, 1, 2, \dots\}$ such that $M = \bigoplus_{n=0}^{+\infty} M_n$. A *morphism* of graded A-modules $f : M \to N$ is a morphism of A-modules such that $f(M_n) \subset N_n$ for any $n \in \mathbb{N} \cup \{0\}$. We denote by $\mathfrak{Mod}_{\mathrm{deg}}(A)$ the subcategory of $\mathfrak{Mod}(A)$ of graded modules.

[(100)] A category $\mathcal{C}$ is said to be *additive* if (a) for any $X, Y \in \mathrm{Ob}(\mathcal{C})$, $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is an abelian group and the composition law is bilinear; (b) there exists $0 \in \mathrm{Ob}(\mathcal{C})$ such that $\mathrm{Hom}\,(0,0) = \{0\}$; (c) for any $X, Y \in \mathrm{Ob}(\mathcal{C})$ there exists a unique $Z \in \mathrm{Ob}(\mathcal{C})$ such that for any $W \in \mathrm{Ob}(\mathcal{C})$ one has a isomorphism of abelian groups $\mathrm{Hom}_{\mathcal{C}}(X,W) \times \mathrm{Hom}_{\mathcal{C}}(Y,W) \simeq \mathrm{Hom}_{\mathcal{C}}(Z,W)$, functorial in W. (Such an object, denoted by $X \oplus Y$, is called the *direct sum* of X and Y.) If $\mathcal{C}$ and $\mathcal{C}'$ are additive, a functor $F : \mathcal{C} \to \mathcal{C}'$ is called *additive* if the maps of morphisms are morphisms of abelian groups. Given an additive category $\mathcal{C}$ and $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$, the *kernel* of f is $Z \in \mathrm{Ob}(\mathcal{C})$ with a monomorphism $\alpha_f \in \mathrm{Hom}_{\mathcal{C}}(Z,X)$ such that for any $\varphi \in \mathrm{Hom}_{\mathcal{C}}(W,X)$ with $f \circ \varphi = 0$ there exists a unique $\widetilde{\varphi} \in \mathrm{Hom}_{\mathcal{C}}(W,Z)$ such that $\varphi = \alpha_f \circ \widetilde{\varphi}$; such property individues (provided it exists) uniquely the object Z, which will be denoted by $\ker(f)$. Dually, the *cokernel* of f is $Z' \in \mathrm{Ob}(\mathcal{C})$ with a epimorphism $\beta_f \in \mathrm{Hom}_{\mathcal{C}}(Y,Z')$ such that for any $\psi \in \mathrm{Hom}_{\mathcal{C}}(Y,W)$ with $\psi \circ f = 0$ there exists a unique $\widetilde{\psi} \in \mathrm{Hom}_{\mathcal{C}}(Z',W)$ such that $\psi = \widetilde{\psi} \circ \beta_f$; such property individues (if it exists) uniquely the object Z', which will be denoted by $\mathrm{coker}(f)$. Moreover one defines $\mathrm{coim}(f) = \mathrm{coker}(\alpha_f)$ (*coimage* of f) and $\mathrm{im}(f) = \ker(\beta_f)$ (*image* of f); one then has a natural morphism $\tilde{f} \in \mathrm{Hom}_{\mathcal{C}}(\mathrm{coim}(f), \mathrm{im}(f))$. An additive category $\mathcal{C}$ si called *abelian* if (i) any morphism f admits kernel and cokernel; (ii) the canonical morphism $\tilde{f}$ is a isomorphism. For example, $\mathfrak{Mod}(A)$ —where A is a unitary ring— is abelian, while $\mathfrak{Ban}(\mathbb{C})$ —the category of Banach $\mathbb{C}$-vector spaces and continuous linear maps— is not (exercise).

A sequence of morphisms (with $n \in \mathbb{Z}$)

$$X^\bullet : \qquad \cdots \to X^{n-1} \xrightarrow{d_X^{n-1}} X^n \xrightarrow{d_X^n} X^{n+1} \to \cdots$$

where the module X^n is said "to be in degree n", is called *complex* if $d_X^n \circ d_X^{n-1} = 0$ for any $n \in \mathbb{Z}$. The family of morphisms d_X^n is called the *differential* of the complex $X^\bullet$. For any $k \in \mathbb{Z}$, the *shifted* complex of k is $(X[k]^n, d^n_{X[k]})_{n\in\mathbb{Z}}$, where $X[k]^n = X^{k+n}$ and $d^n_{X[k]} = (-1)^k d_X^{k+n}$.

In a complex one then has $\mathrm{im}(d_X^{n-1}) \subset \ker(d_X^n)$. The *cohomology* in degree n of the complex $X^\bullet$ is the A-module

$$H^n(X^\bullet) = \frac{\ker(d_X^n)}{\mathrm{im}(d_X^{n-1})}.$$

The *cohomology*[(101)] of the complex $X^\bullet$ is the graded A-module

$$H^\bullet(X^\bullet) = \bigoplus_{n\in\mathbb{Z}} H^n(X^\bullet). \tag{A.3}$$

If $\ker(d_X^{n_0}) = \mathrm{im}(d_X^{n_0-1})$ (i.e. $H^{n_0}(X^\bullet) = 0$), the complex is said to be *exact* in degree n_0; if this happens for any $n \in \mathbb{Z}$, the complex is said to be an *exact sequence*. In particular, an exact sequence

$$0 \to X' \xrightarrow{\alpha} X \xrightarrow{\beta} X'' \to 0$$

is called *short exact sequence*. (Here, of course, we mean that all the rest of the complex is formed by zeros and zero morphisms; moreover, we are not interesting in fixing the position of degree zero.) This means that α is injective, β is surjective and $\mathrm{im}(\alpha) = \ker(\beta)$.

If A' is another unitary ring, a covariant functor $F : \mathfrak{Mod}(A) \to \mathfrak{Mod}(A')$ is called *left exact* (resp. *right exact*) if, given any short exact sequence $0 \to X' \xrightarrow{\alpha} X \xrightarrow{\beta} X'' \to 0$, the sequence

$$0 \to F(X') \xrightarrow{F(\alpha)} F(X) \xrightarrow{F(\beta)} F(X'') \qquad (\text{resp. } F(X') \xrightarrow{F(\alpha)} F(X) \xrightarrow{F(\beta)} F(X'') \to 0)$$

is exact. A left and right exact functor is said to be *exact*. One shows (exercise) that the left adjoint of a left exact functor is right exact. Let us examine now two fundamental examples.

(101) We consider complexes of *cochains*: i.e., the morphisms go in the direction of increasing indexes. However, historically the theory started by using complexes of *chains*, i.e. sequences of morphisms $X_\bullet$: $\cdots \to X_{n+1} \xrightarrow{\partial^X_{n+1}} X_n \xrightarrow{\partial^X_n} X_{n-1} \to \cdots$ with $\partial^X_n \circ \partial^X_{n+1} = 0$ for any $n \in \mathbb{Z}$, whose morphisms go in the direction of decreasing indexes. It is usual to call *homology* in degree n of the complex $X_\bullet$ the A-module $H_n(X_\bullet) = \frac{\ker(\partial^X_n)}{\mathrm{im}(\partial^X_{n+1})}$. This does not cause serious problems: in a sense this is just a notational problem, since, given a complex of chains $(X_\bullet, \partial_\bullet)$ one can define a complex of cochains $(Y^\bullet, d^\bullet)$ by setting $Y^n := X_{-n}$ and $d^n := \partial_{-n}$, and it holds $H^n(Y^\bullet) = H_{-n}(X_\bullet)$ for any $n \in \mathbb{Z}$; on the other hand, the contravariance of the dual construction of cohomology is preferable for various reason, e.g. the presence of some extra structure as the "cup product", which makes the cohomology A-modules into A-algebras.

The functor Hom and its "derived" Ext. Let B be a subring contained in the center of A, and M b a A-module. One verifies easily (exercise) that the functors

$$\begin{aligned}
&\mathrm{Hom}_A(\,\cdot\,, M) : \mathfrak{Mod}(A)^{\mathrm{op}} \to \mathfrak{Mod}(B),\\
&\qquad (\varphi : N \to N') \mapsto (\mathrm{Hom}_A(\varphi, M) : \mathrm{Hom}_A(N', M) \to \mathrm{Hom}_A(N, M), \quad f \mapsto f \circ \varphi)\\
&\mathrm{Hom}_A(M, \,\cdot\,) : \mathfrak{Mod}(A) \to \mathfrak{Mod}(B),\\
&\qquad (\varphi : N \to N') \mapsto (\mathrm{Hom}_A(M, \varphi) : \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(M, N'), \quad g \mapsto \varphi \circ g)
\end{aligned}$$

are left exact: in other words, if $0 \to N' \to N \to N'' \to 0$ is a short exact sequence exact then one obtains "truncated" exact sequences

$$0 \to \mathrm{Hom}_A(N'', M) \to \mathrm{Hom}_A(N, M) \to \mathrm{Hom}_A(N', M) \tag{A.4}$$

$$0 \to \mathrm{Hom}_A(M, N') \to \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(M, N''). \tag{A.5}$$

If $\mathrm{Hom}_A(\,\cdot\,, M)$ (resp. $\mathrm{Hom}_A(M, \,\cdot\,)$) is also exact, the A-module M is called *injective* (resp. *projective*).[102] Hence, for example, if M is injective the truncated short sequence (A.4) can be completed by zero. It is then natural to raise the question: *in general, if M is not injective, how could it be possible to continue* (A.4)*?* Now one can show that, even if M non is injective, there exists anyway an *injective resolution* of M, i.e. an exact sequence $0 \to M \to I^0 \to I^1 \to \cdots$ where the I^j's are injective A-modules; given another A-module K, one defines $\mathrm{Ext}^j_A(K, M)$ as the jth cohomology group of the complex of B-modules

$$0 \to \mathrm{Hom}_A(K, I^0) \to \mathrm{Hom}_A(K, I^1) \to \cdots$$

(where $\mathrm{Hom}_A(K, I^0)$ is in degree zero); one can show that such modules depend *only* on K and M and *not* on the particular injective resolution chosen for M; one easily notes that $\mathrm{Ext}^0_A(K, M) = \mathrm{Hom}_A(K, M)$, and that those groups Ext_A are exactly those who allow one to continue the sequence (A.4):

$$\begin{aligned}
0 \to \mathrm{Hom}_A(N'', M) &\to \mathrm{Hom}_A(N, M) \to \mathrm{Hom}_A(N', M) \to\\
&\to \mathrm{Ext}^1_A(N'', M) \to \mathrm{Ext}^1_A(N, M) \to \mathrm{Ext}^1_A(N', M) \to \mathrm{Ext}^2_A(N'', M) \to \cdots .
\end{aligned}$$

Of course, if M is injective then $\mathrm{Ext}^j_A(K, M) = 0$ for any K and any $j \geq 1$. Analogously, if M is not projective, there exists anyway a *projective resolution* of M, i.e. an exact sequence $\cdots \to P^1 \to P^0 \to M \to 0$ where the P^j's are projective A-modules; given another A-module K, one shows that $\mathrm{Ext}^j_A(M, K)$ is also equal to the jth cohomology group of the complex of B-modules

$$0 \to \mathrm{Hom}_A(P^0, K) \to \mathrm{Hom}_A(P^1, K) \to \cdots$$

(where $\mathrm{Hom}_A(P^0, K)$ is in degree zero)[103], and then one can make also (A.5) continue:

$$\begin{aligned}
0 \to \mathrm{Hom}_A(M, N') &\to \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(M, N'') \to\\
&\to \mathrm{Ext}^1_A(M, N') \to \mathrm{Ext}^1_A(M, N) \to \mathrm{Ext}^1_A(M, N'') \to \mathrm{Ext}^2_A(M, N') \to \cdots .
\end{aligned}$$

[102] The usual definition of injective (resp. projective) A-module M, is equivalent to the one that we have just given: for any monomorphism (resp. epimorphism) of A-modules $f : X \to Y$ and any morphism $\alpha : X \to M$ (resp. $\beta : M \to Y$) there exists a morphism $\tilde{\alpha} : Y \to M$ (resp. $\tilde{\beta} : M \to X$) such that $\alpha = \tilde{\alpha} \circ f$ (resp. $\beta = f \circ \tilde{\beta}$).

[103] Therefore we have understood that, in general, to compute $\mathrm{Ext}^j_A(N', N'')$ one may use either a projective resolution of N' or an injective one of N''.

If M is projective then $\mathrm{Ext}^j_A(M,K)=0$ for any K and any $j\geq 1$.[104]

The functor $\otimes$ and its "derived" Tor. Let N be a right A-module: by $N\otimes_A\cdot$ one can associate to any left A-module M a B-module $N\otimes_A M$ and, for $f\in\mathrm{Hom}_A(M,M')$ one can define $N\otimes_A f\in\mathrm{Hom}_B(N\otimes_A M, N\otimes_A M')$ in the most natural way, i.e. $N\otimes_A f=\mathrm{id}_N\otimes f$. In this way one obtains a covariant additive functor $N\otimes_A\,\cdot\,:\mathfrak{Mod}(A)\to\mathfrak{Mod}(B)$, which in general (being a left adjoint of the left exact functor $\mathrm{Hom}_B(N,\cdot))$ is only right exact: i.e., by applying $N\otimes_A\,\cdot$ to the short exact sequence $0\to X'\to X\to X''\to 0$ one obtains the truncated exact sequence of B-modules

$$N\otimes_A X'\to N\otimes_A X\to N\otimes_A X''\to 0. \tag{A.6}$$

A right A-module N such that $N\otimes_A\,\cdot$ is exact is called *flat*: for example, all projective A-modules are flat. Similarly, given a left A-module M the functor $\cdot\otimes_A M:\mathfrak{Mod}(A^{\mathrm{op}})\to\mathfrak{Mod}(B)$ is right exact.

Examples. (1) If $A=k$ (a field), all k-vector spaces are flat on k. **(2)** Consider the following example of non flat module: let $A=\mathbb{C}[x]$ and $N=A/Ax$. From the short exact sequence $0\to A\xrightarrow{x\cdot}A\to A/Ax\to 0$, by applying $N\otimes_A\,\cdot$ one gets the complex $0\to A/Ax\xrightarrow{x\cdot}A/Ax\to A/Ax\otimes_A A/Ax\to 0$. It is then enough to note that the morphism $x\cdot:A/Ax\to A/Ax$ is zero (hence non injective). As a consequence, one obtains the isomorphism $A/Ax\simeq A/Ax\otimes_A A/Ax$.

Also in this case, it is natural to raise the question: *in general, if N is not flat, how should* (A.6) *start?* Let $\cdots\to P^1\to P^0\to N\to 0$ be a resolution where the P^js are flat (possibly projective) right A-modules; given a left A-module K, one defines $\mathrm{Tor}^A_j(N,K)$ as the $(-j)$th cohomology group of the complex of B-modules

$$\cdots\to P^1\otimes_A K\to P^0\otimes_A K\to 0$$

(where $P^0\otimes_A K$ is in degree zero); one shows that such modules depend *only* on K and N, and *do not* depend on the particular flat resolution chosen for N; one easily notes that $\mathrm{Tor}^A_0(N,K)=N\otimes_A K$, and that these groups Tor^A allow one to continue (A.6) on the left hand side:

$$\begin{aligned}\cdots\to\mathrm{Tor}^A_2(N,X'')\to\mathrm{Tor}^A_1(N,X')\to\mathrm{Tor}^A_1(N,X)\to\\ \to\mathrm{Tor}^A_1(N,X'')\to N\otimes_A X'\to N\otimes_A X\to N\otimes_A X''\to 0.\end{aligned}$$

If N is flat then $\mathrm{Tor}^A_j(N,K)$ for any K and any $j\geq 1$.

Example. If $A=k$ with k commutative field, any k-vector space is free on k, hence there are no nonzero Ext^j_k and Tor^k_j for $j\neq 0$. As for the case of $A=\mathbb{Z}$ (abelian groups) we refer to the Appendix A.2.3.

Given two complexes $X^\bullet=(X^n,d^n_X)_{n\in\mathbb{Z}}$ and $Y^\bullet=(Y^n,d^n_Y)_{n\in\mathbb{Z}}$ in $\mathfrak{Mod}(A)$, a *morphism* $f:X^\bullet\to Y^\bullet$ is a family of morphisms $f^n\in\mathrm{Hom}_A(X^n,Y^n)$ such that $f^{n+1}\circ d^n_X=d^n_Y\circ f^n$,

[104] If in the place of $\mathrm{Hom}_A(M,\cdot)$ (resp. $\mathrm{Hom}_A(\cdot,M)$) one considers another covariant (resp. contravariant) left exact functor $F:\mathcal{C}\to\mathcal{C}'$ (with $\mathcal{C}$ and $\mathcal{C}'$ abelian categories: for example, categories of modules on a ring), one obtains the "derived functors" $R^jF:\mathcal{C}\to\mathcal{C}'$ (with $j\geq 0$) of the classical costruction of Cartan-Eilenberg [3]. Therefore, in this terminology one has for example $R^j\mathrm{Hom}_A(X,\cdot)=\mathrm{Ext}^j_A(X,\cdot)$.

i.e. the diagram

$$\begin{tikzcd}
X^\bullet : \arrow[d, "f"] & \cdots \arrow[r] & X^{n-1} \arrow[r, "d_X^{n-1}"] \arrow[d, "f^{n-1}"] & X^n \arrow[r, "d_X^n"] \arrow[d, "f^n"] & X^{n+1} \arrow[r] \arrow[d, "f^{n+1}"] & \cdots \\
Y^\bullet : & \cdots \arrow[r] & Y^{n-1} \arrow[r, "d_Y^{n-1}"] & Y^n \arrow[r, "d_Y^n"] & Y^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

commutes. It is immediate to verify that, in this way, one obtains a new category $\mathbf{C}(A)$ (of complexes of $\mathfrak{Mod}(A)$). From $f : X^\bullet \to Y^\bullet$ one defines a morphism between shifted complexes $f[k] : X[k]^\bullet \to Y[k]^\bullet$ by setting $f[k]^n = f^{k+n}$. Note also that a functor $F : \mathfrak{Mod}(A) \to \mathfrak{Mod}(B)$ induces in a natural way a functor $\mathbf{C}(F) : \mathbf{C}(A) \to \mathbf{C}(B)$, which will be also denoted by F.

Proposition A.2.2. *A morphism $f : X^\bullet \to Y^\bullet$ induces a morphism of graded A-modules $H^\bullet f : H^\bullet(X^\bullet) \to H^\bullet(Y^\bullet)$.*

Proof. Given $x \in \ker(d_X^n)$, set $H^n f([x]) = [f^n(x)]$: since $d_Y^n(f^n(x)) = f^{n+1}(d_X^n(x)) = 0$, it is actually $f^n(x) \in \ker(d_Y^n)$; if then $x' \in \ker(d_X^n)$ with $[x] = [x']$ (i.e. $x - x' = d_X^{n-1}(x'')$ with $x'' \in X^{n-1}$), then $[f^n(x)] = [f^n(x')]$ since $f^n(x) - f^n(x') = f^n(d_X^{n-1}(x'')) = d_Y^{n-1}(f^{n-1}(x''))$. □

Such definition respects compositions, identity and the operations of morphisms: i.e., $H^\bullet$ is a additive functor from $\mathbf{C}(A)$ to $\mathfrak{Mod}_{\deg}(A)$. If $H^\bullet f$ is an isomorphism (i.e. if $H^n f : H^n(X^\bullet) \to H^n(Y^\bullet)$ is isomorphism for any n), one says that f is a *quasi-isomorphism*.

Proposition A.2.3. (Snake's Lemma) *Let $0 \to X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} Z^\bullet \to 0$ be a short exact sequence in $\mathbf{C}(A)$.*[105] *Then there exists a* (coboundary) *morphism $\delta : H^\bullet(Z^\bullet) \to H^{\bullet+1}(X^\bullet)$ such that the following complex of cohomologies in $\mathfrak{Mod}(A)$ is an exact sequence:*

$$\cdots \to H^{n-1}(Z^\bullet) \xrightarrow{\delta^{n-1}} H^n(X^\bullet) \xrightarrow{H^n f} H^n(Y^\bullet) \xrightarrow{H^n g} H^n(Z^\bullet) \xrightarrow{\delta^n} H^{n+1}(X^\bullet) \to \cdots.$$

Proof. Let us construct the morphism δ^n. Let $z \in \ker(d_Z^n)$, and let (g^n surjective) $y \in Y^n$ such that $g^n(y) = z$. Since $0 = d_Z^n(g^n(y)) = g^{n+1}(d_Y^n(y))$, one has $d_Y^n(y) \in \ker(g^{n+1}) = \mathrm{im}(f^{n+1})$, and so let $x \in X^{n+1}$ be such that $f^{n+1}(x) = y$: one then sets $\delta^n([z]) = [x]$ (note that $d_X^{n+1}(x) = 0$ because $f^{n+2}(d_X^{n+1}(x)) = d_Y^{n+1}(f^{n+1}(x)) = 0$ and f^{n+2} is injective).[106] The remaining verifications are left as an exercise. □

Note that $\mathfrak{Mod}(A)$ is a full subcategory of $\mathbf{C}(A)$ by identifying a A-module M with the complex $M^\bullet : \cdots \to 0 \to M \to 0 \to \cdots$ where M is in degree zero. If

$$0 \to M \xrightarrow{f} X^0 \xrightarrow{f^0} X^1 \xrightarrow{f^1} \cdots \to X^{m-1} \xrightarrow{f^n} \cdots$$

is an exact sequence where M is in degree zero, the complex $(X^\bullet, f^\bullet)$ with $X^m = 0$ for $m < 0$ is said a *resolution* of M. Note that the morphism $F : M^\bullet \to X^\bullet$ is defined by setting $F^0 = f$ and $F^j = 0$ for $j \neq 0$ is a quasi-isomorphism.

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[105] I.e., for any $n \in \mathbb{Z}$ the sequence $0 \to X^n \xrightarrow{f^n} Y^n \xrightarrow{g^n} Z^n \to 0$ is exact in $\mathfrak{Mod}(A)$.

[106] This is an example of what is usually called *"diagram chasing"*. In this case, the itinerary to find x starting from z reminds the shape of a snake, a fact which explains the popular name "Snake Lemma".

One says that a morphism $f : X^\bullet \to Y^\bullet$ is *homotopic to zero* if for any $n \in \mathbb{N}$ there exist morphisms $s^n : X^n \to Y^{n-1}$ such that $f^n = s^{n+1} \circ d_X^n + d_Y^{n-1} \circ s^n$:

$$\begin{tikzcd}
\cdots \arrow[r] & X^{n-1} \arrow[r, "d_X^{n-1}"] \arrow[d, "f^{n-1}"'] & X^n \arrow[r, "d_X^n"] \arrow[d, "f^n"] \arrow[dl, "s^n"'] & X^{n+1} \arrow[r] \arrow[d, "f^{n+1}"] \arrow[dl, "s^{n+1}"'] & \cdots . \\
\cdots \arrow[r] & Y^{n-1} \arrow[r, "d_Y^{n-1}"] & Y^n \arrow[r, "d_Y^n"] & Y^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

Two morphisms $f, g : X^\bullet \to Y^\bullet$ are said to be *homotopic* if $f - g$ is homotopic to zero. One shows (exercise) that if $f : X^\bullet \to Y^\bullet$ is homotopic to zero then $H^\bullet(f) = 0$: therefore, since $H^\bullet$ is an additive functor, two homotopic morphisms induce the same morphism in cohomology.[107] It is clear that the subset $\mathrm{Hom}^0_{\mathbf{C}(A)}(X^\bullet, Y^\bullet)$ of $\mathrm{Hom}_{\mathbf{C}(A)}(X^\bullet, Y^\bullet)$ formed by morphisms homotopic to zero is a subgroup; moreover, $g \circ f$ is homotopic to zero if so is at least one out of f and g (exercise). This allows to define a new category $\mathbf{K}(A)$ by setting $\mathrm{Ob}(\mathbf{K}(A)) = \mathrm{Ob}(\mathbf{C}(A))$ and $\mathrm{Hom}_{\mathbf{K}(A)}(X^\bullet, Y^\bullet) = \mathrm{Hom}_{\mathbf{C}(A)}(X^\bullet, Y^\bullet)/\mathrm{Hom}^0_{\mathbf{C}(A)}(X^\bullet, Y^\bullet)$. The functor of cohomology $H^\bullet : \mathbf{K}(A) \to \mathfrak{Mod}_{\mathrm{deg}}(A)$ is well-defined.

One sees immediately that, if $f : X^\bullet \to Y^\bullet$ is a isomorphism in $\mathbf{K}(A)$ (i.e., there exists un morphism $g : Y^\bullet \to X^\bullet$ such that $g \circ f$ is homotopic to $\mathrm{id}_{X^\bullet}$ and $f \circ g$ a $\mathrm{id}_{Y^\bullet}$) then f is a quasi-isomorphism. On the other hand, the converse is false: it shall be necessary to perform a procedure of "localization" of $\mathbf{K}(A)$ to come to the derived category $\mathbf{D}(A)$. (We refer to [4] or [10].)

Double complexes. A *double complex*

$$X^{\bullet,\bullet} = \{(X^{m,n}, \delta^{m,n}, d^{m,n}) : m, n \in \mathbb{Z}\}$$

is the data of a family of A-modules $\{X^{m,n} : m, n \in \mathbb{N} \cup \{0\}\}$ and of morphisms

$$\begin{cases} \delta^{m,n} : X^{m,n} \to X^{m,n+1} \\ d^{m,n} : X^{m,n} \to X^{m+1,n} \end{cases} \qquad \text{such that} \qquad \begin{cases} d^2 = 0 \\ \delta^2 = 0 \\ d \circ \delta = \delta \circ d \end{cases} .$$

In other words, one has a commutative diagram whose rows $(X^{m,\bullet}, \delta^{m,\bullet})$ and columns $(X^{\bullet,n}, d^{\bullet,n})$ are complexes in $\mathbf{C}(A)$:

$$\begin{tikzcd}
 & \cdots & \cdots & \cdots & \\
\cdots \arrow[r] & X^{m+1,n-1} \arrow[r, "\delta^{m+1,n-1}"] \arrow[u] & X^{m+1,n} \arrow[r, "\delta^{m+1,n}"] \arrow[u] & X^{m+1,n+1} \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & X^{m,n-1} \arrow[r, "\delta^{m,n-1}"] \arrow[u, "d^{m,n-1}"] & X^{m,n} \arrow[r, "\delta^{m,n}"] \arrow[u, "d^{m,n}"] & X^{m,n+1} \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & X^{m-1,n-1} \arrow[r, "\delta^{m-1,n-1}"] \arrow[u, "d^{m-1,n-1}"] & X^{m-1,n} \arrow[r, "\delta^{m-1,n}"] \arrow[u, "d^{m-1,n}"] & X^{m-1,n+1} \arrow[r] \arrow[u, "d^{m-1,n+1}"] & \cdots \\
 & \cdots \arrow[u] & \cdots \arrow[u] & \cdots \arrow[u] &
\end{tikzcd}$$

[107] Of course, there is a similar costruction for the complexes of chains and relative homology.

Given a double complex $X^{\bullet,\bullet}$ such that the set $\{(m,n) : m+n = k,\ X^{m,n} \neq 0\}$ is finite for any $k \in \mathbb{Z}$ (for example, such that $X^{m,n} = 0$ if $m < m_0$ or $n < n_0$ for some $m_0, n_0 \in \mathbb{Z}$), one can construct a simple complex $(s(X)^\bullet, D^\bullet)$ by "summing on the antidiagonals", i.e. by setting

$$s(X)^k = \bigoplus_{m+n=k} X^{m,n}, \qquad (D^k)^{i,j} = \delta^{i,j-1} + (-1)^j d^{i-1,j} : s(X)^k \to X^{i,j} \quad (i+j = k+1).$$

Note (and verify by exercise) that the presence of the alternating factor is indispensable in order that $s(X)$ become a complex.

Let $(Y^\bullet, d_Y^\bullet)$ be a complex with $Y^k = 0$ for $k < 0$, $X^{\bullet,\bullet}$ a double complex with $m_0 = n_0 = 0$, and let $f^p : Y^p \to X^{p,0}$ be morphisms such that $d^{p,0} \circ f^p = f^{p+1} \circ d_Y^p$ for any $p \in \mathbb{N}$. The double complex $X^{\bullet,\bullet}$ *augmented* with the column $Y^\bullet$ is

$$\begin{tikzcd}
 & \cdots & \cdots & \cdots & \cdots & \\
0 \arrow[r] & Y^2 \arrow[u] \arrow[r, "f^2"] & X^{2,0} \arrow[u] \arrow[r, "\delta^{2,0}"] & X^{2,1} \arrow[u] \arrow[r, "\delta^{2,1}"] & X^{2,2} \arrow[u] \arrow[r] & \cdots \\
0 \arrow[r] & Y^1 \arrow[u, "d_Y^1"] \arrow[r, "f^1"] & X^{1,0} \arrow[u, "d^{1,0}"] \arrow[r, "\delta^{1,0}"] & X^{1,1} \arrow[u, "d^{1,1}"] \arrow[r, "\delta^{1,1}"] & X^{1,2} \arrow[u, "d^{1,2}"] \arrow[r] & \cdots \\
0 \arrow[r] & Y^0 \arrow[u, "d_Y^0"] \arrow[r, "f^0"] & X^{0,0} \arrow[u, "d^{0,0}"] \arrow[r, "\delta^{0,0}"] & X^{0,1} \arrow[u, "d^{0,1}"] \arrow[r, "\delta^{0,1}"] & X^{0,2} \arrow[u, "d^{0,2}"] \arrow[r] & \cdots \\
 & 0 \arrow[u] & & & &
\end{tikzcd}$$

An analogous definition holds for a double complex augmented with the row $Y^\bullet$.

There is a natural morphism of complexes $\psi : Y^\bullet \to s(X)^\bullet$ sending $y \in Y^p$ in $(f^p(y), 0) \in s(X)^p = X^{p,0} \oplus \bigoplus_{m=1}^p X^{p-m,m}$ (verify that ψ commutes with the differentials d_Y and D): by Proposition A.2.2, it induces a morphism in cohomology $H^\bullet\psi : H^\bullet(Y^\bullet) \to H^\bullet(s(X))$.

Proposition A.2.4. *If all rows* (resp. *all columns*) *of the double complex $X^{\bullet,\bullet}$ augmented with the column* (resp. *with the row*) *$Y^\bullet$ are exact, then ψ is a quasi isomorphism: i.e., $H^\bullet\psi$ induces an isomorphism*

$$H^\bullet(Y^\bullet) \simeq H^\bullet(s(X)).$$

Proof. We shall bound to the case of a double complex augmented with the column $Y^\bullet$, the case of the row being similar. To show that $H^\bullet\psi$ is surjective, note that $x \in Z^p(s(X))$ is a $x = (x^{m,n})_{m+n=p} \in s(X)^p = \bigoplus_{m+n=p} X^{m,n}$ such that $d^{p,0}(x^{p,0}) = 0$, $\delta^{m,n}(x^{m,n}) + (-1)^{n+1} d^{m-1,n+1}(x^{m-1,n+1}) = 0$ (for $m+n = p$ and $m = 1, \dots, p$) and $\delta^{0,p}(x^{0,p}) = 0$. By the latter equality, one has $x^{0,p} = \delta^{0,p-1}(z)$ for a certain $z \in X^{0,p-1}$, hence, up to subtracting $D(z)$ from x, we may assume that $x^{0,p} = 0$ without changing the class of D-cohomology $[x]$ of x. By continuing this way, we come to a representative $\tilde{x}$ of $[x]$ with $\tilde{x}^{m,n} = 0$ for $n > 0$: therefore, further than $d^{p,0}(\tilde{x}^{p,0}) = 0$, we also have $\delta^{p,0}(\tilde{x}^{p,0}) = 0$, and hence there will exist $y \in Y_p$ such that $f^p(y) = \tilde{x}^{p,0}$; we even get $y \in Z^p(Y_\bullet)$, because $f^{p+1}(d_y^p(y)) = d^{p,0}(f^p(y)) = 0$ and f^{p+1} is injective. As for the injectivity of $H^\bullet\psi$, let $y \in Z^p(Y^\bullet)$ be such that $(f^p(y), 0) \in B^p(s(X))$. We shall then get a $z \in s(X)^{p-1} = X^{p-1,0} \oplus \bigoplus_{m=1}^{p-1} X^{p-1-m,m}$ such that $D(z) = (f^p(y), 0)$, i.e. $d^{p-1,0}(z^{p-1,0}) = f^p(y)$,

$\delta^{m,n}(z^{m,n})+(-1)^{n+1}d^{m-1,n+1}(z^{m-1,n+1}) = 0$ (for $m+n = p-1$ and $m = 1,\dots,p-1$) and $\delta^{0,p-1}(z^{0,p-1}) = 0$. Continuing as above, we may assume that $z^{p-1-m,m} = 0$ for $m > 0$: hence $d^{p-1,0}(z^{p-1,0}) = f^p(y)$ and $\delta^{p-1,0}(z^{p-1,0}) = 0$. Therefore we have $z^{p-1,0} = f^{p-1}(y')$ for a certain $y' \in Y^{p-1}$, which implies $f^p(d_Y^{p-1}(y')) = d^{p,0}(f^{p-1}(y')) = f^p(y)$, and therefore $y = d_Y^{p-1}(y')$ since f^p is injective. □

A.2.3 The case of abelian groups

As an example and for the particular interest of this case, let us say something more in detail about the case $A = \mathbb{Z}$ (abelian groups).

Proposition A.2.5. *Let*

$$X^\bullet : \qquad 0 \to X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-2}} X^{n-1} \xrightarrow{d^{n-1}} X^n \to 0$$

be a bounded complex of finitely generated abelian groups. Then

$$\sum_{j=0}^{n}(-1)^j \mathrm{rk}\, X^j = \sum_{j=0}^{n}(-1)^j \mathrm{rk}\, H^j(X),$$

where rk *denotes the rank.*[108]*, In particular,* $\sum_{j=0}^{n}(-1)^j \mathrm{rk}\, X^j = 0$ *if* $X^\bullet$ *is an exact sequence.*

Proof. We start by observing that if $0 \to H \to G \to K \to 0$ is a short exact sequence of abelian groups, then $\mathrm{rk}\,(G) = \mathrm{rk}\,(H)+\mathrm{rk}\,(K)$ (exercise). In our case, denoting $Z^j = \ker(d^j)$ and $B^j = \mathrm{im}(d^{j-1})$, for $j = 0,\dots,n$ we have short exact sequences $0 \to Z^j \to X^j \to B^{j+1} \to 0$ and $0 \to B^j \to Z^j \to H^j(X) \to 0$, which implies $\mathrm{rk}\, X^j = \mathrm{rk}\, Z^j + \mathrm{rk}\, B^{j+1} = \big(\mathrm{rk}\, B^j + \mathrm{rk}\, H^j(X)\big) + \mathrm{rk}\, B^{j+1}$, i.e. $\mathrm{rk}\, X^j = \mathrm{rk}\, H^j(X) + (\mathrm{rk}\, B^j + \mathrm{rk}\, B^{j+1})$; the result follows by moltiplying both members of the latter equality by $(-1)^j$ and summing on j. □

Remark A.2.6. Let k be a field of characteristic zero. Given a finitely generated abelian group G, $\mathrm{Hom}_{\mathbb{Z}}(G,k)$ and $k \otimes_{\mathbb{Z}} G$ are k-vector spaces of finite dimension $\mathrm{rk}\, G$, dual to each other.[109] Note that, by applying the functor $k \otimes_{\mathbb{Z}} \cdot$ to the complex $X^\bullet$, the Proposition A.2.5 remains valid by replacing "k-vector spaces of finite dimension" with "finitely generated abelian groups" and "$\dim_k$" with "rk".[110]

In the category $\mathfrak{Mod}(\mathbb{Z})$ the projective objects are the free abelian groups, the injective ones are the divisible groups and the flat ones the torsion-free groups.[111] Since any abelian

(108) Recall that the rank $\mathrm{rk}\,(G)$ of a finitely generated abelian group G is the (finite) number of components isomorphic to $\mathbb{Z}$ in any decomposition of G as direct sum of cyclic subgroups.

(109) It is clear that $\mathrm{Hom}_{\mathbb{Z}}(G,k)$ is a k-vector space of finite dimension $\mathrm{rk}\, G$. By applying the adjunction between $k \otimes_{\mathbb{Z}} \cdot$ and for to the group G and to the k-vector space k one obtains an isomorphism of k-vector spaces $(k \otimes_{\mathbb{Z}} G)^* = \mathrm{Hom}_k(k \otimes_{\mathbb{Z}} G, k) \simeq \mathrm{Hom}_{\mathbb{Z}}(G,k)$.

(110) Of course, one could have proved that fact directly, in the same way used for abelian groups.

(111) A free abelian group is a group of the form $\mathbb{Z}^{(\Lambda)}$ (recall, as we said in the Appendix A.1, that we denote $Y^\Lambda = \{\text{functions } \Lambda \to Y\} \simeq \prod_{\lambda\in\Lambda} Y_\lambda$ and $Y^{(\Lambda)} = \{\text{functions } \Lambda \to Y \text{ vanishing a. e.}\} \simeq \bigoplus_{\lambda\in\Lambda} Y_\lambda$). Any abelian group is the homomorphic image of a free group (this is clear: it is enough to take a family of generators A_0 of R and consider the natural surjective morphism $\mathbb{Z}^{(A_0)} \to R \to 0$ given by $(n_r)_{r\in A_0} \mapsto \sum_{r\in A_0} n_r r$), and subgroups of free groups are free. Dually, the abelian group G is divisible if $nG = G$ for any $n \in \mathbb{N}$. The structure theorem says that any divisible group is the direct sum of copies of $\mathbb{Q}$ (prototype of torsion-free divisible group) and of $\mathbb{Z}/p^\infty\mathbb{Z} := \bigcup_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z}$ where p is a prime number (prototype of divisible group of torsion: it holds $\mathbb{Q}/\mathbb{Z} \simeq \bigoplus_p \mathbb{Z}/p^\infty\mathbb{Z}$). The divisible groups enjoy properties dual to the ones of free groups: homomorphic images of divisible groups are divisible, and any abelian group is the subgroup (even essential, i.e. "dense" subgroup: a subgroup $H \subset G$ is called essential in G if the unique subgroup $K \subset G$ such that $H \cap K = \{0\}$ is $K = \{0\}$) of a divisibile group. Finally, an abelian group G is torsion-free if $nx = 0$ with $n \in \mathbb{Z}$ and $x \in G \setminus \{0\}$ implies that $n = 0$.

group is homeomorphic image of a free group and the subgroups of free groups free, any abelian group R has a projective resolution of type $0 \to \mathbb{Z}^{(A_1)} \to \mathbb{Z}^{(A_0)} \to R \to 0$; similarly, since any abelian group is the subgroup of a divisibile group, and any homeomorphic image of divisible groups is divisible, any abelian group G has an injective resolution of type $0 \to G \to D_0 \to D_1 \to 0$ with D_0 and D_1 divisible: it folows that, given two abelian groups R and G, the groups $\mathrm{Ext}^j_{\mathbb{Z}}(R,G)$ and $\mathrm{Tor}^{\mathbb{Z}}_j(R,G)$ vanish for $j \neq 0,1$. As we have seen, in general it holds $\mathrm{Ext}^0_{\mathbb{Z}}(R,G) = \mathrm{Hom}_{\mathbb{Z}}(R,G)$ and $\mathrm{Tor}^{\mathbb{Z}}_0(R,G) = R \otimes_{\mathbb{Z}} G$; for the computation of $\mathrm{Ext}^1_{\mathbb{Z}}(R,G)$ one could use either a free resolution of R or a divisible one of G (hence $\mathrm{Ext}^1_{\mathbb{Z}}(R,G) = 0$ if R is a free group, or if G is divisible), and for $\mathrm{Tor}^{\mathbb{Z}}_1(R,G)$ a free resolution of R or of G (hence $\mathrm{Tor}^{\mathbb{Z}}_1(R,G) = 0$ if at least one out of R and G is free or –more simply– flat, i.e. torsion-free).

Examples. **(1)** It holds $\mathrm{Ext}^1_{\mathbb{Z}}(\frac{\mathbb{Z}}{n\mathbb{Z}}, \mathbb{Z}) = \frac{\mathbb{Z}}{n\mathbb{Z}}$, $\mathrm{Ext}^1_{\mathbb{Z}}(\frac{\mathbb{Z}}{n\mathbb{Z}}, \frac{\mathbb{Z}}{m\mathbb{Z}}) = \frac{\mathbb{Z}}{(n,m)\mathbb{Z}}$ (where $(n,m) = \mathrm{MCD}\{n,m\}$) and $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) \neq 0$. **(2)** One has $\mathrm{Tor}^{\mathbb{Z}}_1(\frac{\mathbb{Z}}{m\mathbb{Z}}, \frac{\mathbb{Z}}{n\mathbb{Z}}) = \frac{\mathbb{Z}}{(m,n)\mathbb{Z}}$.[112]

[112] $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z})$ is usually computed by using the divisible resolution $0 \to \mathbb{Z} \to \mathbb{Q} \to \frac{\mathbb{Q}}{\mathbb{Z}} \to 0$ of $\mathbb{Z}$ but it turns out to be complicated (see for example [8, Section 3.G]). The other statements can be easily obtained using the free resolution $0 \to \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \to \frac{\mathbb{Z}}{n\mathbb{Z}} \to 0$ of $\frac{\mathbb{Z}}{n\mathbb{Z}}$.