1 The fundamental group of a topological space

The first part of these notes deals with the (first) homotopy group —or "fundamental group"— of a topological space. We shall see that two topological spaces which are "homotopic" —i.e. continuously deformable one to the other, for example homeomorphic spaces— have isomorphic fundamental groups.
After defining the notion of homotopy (§1.1), we study the subsets which are "retract" (in various senses) of a given space (§1.2); then we come to the definition of the fundamental group of a topological space and investigate its invariance (§1.3), and we study the examples of the circle and of the other quotients of topological groups by discrete subgroups (§1.4). In computing a fundamental group, the theorem of Van Kampen allows one to decompose the problem on the subsets of a suitable open cover (§1.5). Deeply related to the fundamental group is the theory of covering spaces —i.e. local homeomorphisms with uniform fibers— of a topological space (§1.6), which enjoy the property of "lifting homotopies" (§1.7). The covering spaces have a simpler homotopy structure than the one of the original topological space, at the point that the graph of subgroups of the fundamental group of the latter describes the formers up to isomorphisms (§1.8). In particular, the "universal" covering space —whose characteristic subgroup is trivial— describes, by means of the covering automorphisms, the fundamental group itself (§1.9). We end by studying some examples, among them the fundamental group of manifolds and of real linear groups (§1.10).

The notes of this part are largely inspired by the lecture notes of a nice course held by Giuseppe De Marco in Padua while the author was still an undergraduate student.

Notation. In what follows, I denotes the closed interval $[0,1] \subset \mathbb{R}$, $\partial I = \{0,1\}$ its boundary points, $\{\mathrm{pt}\}$ the one-point set, $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$ the n-dimensional sphere and $\mathbb{B}^{n+1} = \{x \in \mathbb{R}^{n+1} : |x| \leq 1\}$ the $(n+1)$-dimensional closed ball (of which $\mathbb{S}^n$ is the boundary in $\mathbb{R}^{n+1}$). If not otherwise specified, on a subset of a topological space we shall always consider the induced topology.

1.1 Homotopy

Let us remind the notion of "path" in a topological space.

Definition 1.1.1. Let X be a topological space. A *path*, or *arc*, in X is a continuous function $\gamma : I \to X$. It is usual to denote by γ also the image $\gamma(I) \subset X$. The points $x_0 = \gamma(0)$ and $x_1 = \gamma(1)$ are called *extremities* (or *endpoints*) of the path. In the case where $x_1 = x_0$, one calls the path a *loop* based on x_0. (Analogously, a loop in X based on x_0 can be understood as a continuous function $\tilde{\gamma} : \mathbb{S}^1 \to X$ where, if we identify $\mathbb{S}^1 \subset \mathbb{C} \simeq \mathbb{R}^2$, we have $\tilde{\gamma}(1) = x_0$.) A *change of parameter* (or *reparametrization*) is a continuous function $p : I \to I$ such that $p(0) = 0$ and $p(1) = 1$ (note that the paths γ and $\gamma \circ p$ have the same image). The space X is called *arcwise connected* if for any pair of points $x_0, x_1 \in X$ there exists a path γ in X with extremities x_0 and x_1.

Remark 1.1.2. Recall that a topological space X is connected if it is not a disjoint union of two non empty open subsets or, equivalently, if all continuous functions of X with values in a discrete topological space are constant (hence, if $A \subset X$ is connected such is also $\overline{A}$). An arcwise connected space is also connected,[(1)] but not vice versa: for example, if $A = \{(x, y) \in \mathbb{R}^2 : x > 0,\ y = \sin(\frac{1}{x})\}$ then $B := \overline{A} = (\{0\} \times [-1, 1]) \cup A$ (commonly called *topologist's sinus*) is connected but not arcwise connected.[(2)] In any case one should be careful about which topology is being considered on the space: $X = \{p, q\}$ (the two-points space) is arcwise connected when endowed with the topology $\{\varnothing, \{p\}, X\}$, while the discrete topology makes it disconnected.[(3)]

Let X, Y be topological spaces, and denote by $C(X, Y)$ the space of continuous functions between them.[(4)] Given $f, g \in C(X, Y)$, we want to give a precise meaning to the idea of "deforming continuously the function f into the function g".

Definition 1.1.3. A *homotopy* between the functions f and g is a continuous function $h : X \times I \to Y$ such that $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$ for any $x \in X$. Two functions are called *homotopic* ($f \sim g$) if there exists a homotopy between them. More generally, given a subset $A \subset X$, the functions f and g are called *homotopic relatively to* A (also *homotopic rel* A for short) if there exists a homotopy h between them such that $h(x, t) = f(x) = g(x)$ for any $x \in A$ and $t \in I$. A function is called *nullhomotopic* if it is homotopic to a constant function.

[(1)] If X is arcwise connected, D is discrete, $f : X \to D$ is continuous, $x_0, x_1 \in X$ and γ is a path between them, then $f \circ \gamma$ is continuous and therefore (since I is connected) $f(\gamma(I)) \subset D$ is connected. But then $f(\gamma(I)) \subset D$ is a point, and in particular $f(x_0) = f(x_1)$.

[(2)] Let $\gamma : I \to B$ be a path joining a point of A and a point of $B \setminus A = \{0\} \times [-1, 1]$: then $\{t \in I : x(\gamma(t)) = 0\}$ is a non empty closed subset of I, and therefore there will exist its minimum $\delta > 0$. On the other hand, once more thanks to the continuity, there must exist a $\delta' \in]0, \delta[$ such that $|\gamma(t) - \gamma(\delta)| < \frac{1}{2}$ for any $t \in]\delta', \delta[$, but this is absurd: namely, in any left neighborhood of δ in I there are points where the value of $y \circ \gamma$ is 1 and other points where it is -1. Note that such argument does not apply (happily) to $B' = \{(x, y) \in \mathbb{R}^2 : x > 0,\ y = x \sin(\frac{1}{x})\} \cup \{(0, 0)\}$, which is indeed arcwise connected (it is a continuous image of I).

[(3)] If the topology is $\{\varnothing, \{p\}, X\}$, a path from p to q is $\gamma : I \to X$, $\gamma|_{[0,\frac{1}{2}[} \equiv p$ and $\gamma|_{[\frac{1}{2},1]} \equiv q$; if the topology is the discrete one, $X = \{p\} \cup \{q\}$ shows that X is disconnected.

[(4)] In other words, we have $C(X, Y) = \mathrm{Hom}_{\mathfrak{Top}}(X, Y)$ in the category $\mathfrak{Top}$ of topological spaces (see Appendix A.1).

Remark 1.1.4. Here are a couple of initial observations about the relation between the notions of homotopy and path.

(a) For $X = \{\mathrm{pt}\}$ and $Y = X$ one finds again the definition of path in X (as a homotopy between the functions of $\{\mathrm{pt}\}$ in X of values x_0 and x_1).

(b) For $X = I$ and $Y = X$ the functions f and g are paths in X. In the case of a homotopy of paths, it is frequent to require that the endpoints be fixed, i.e. that $h(0,t)$ and $h(1,t)$ do not depend on t (in particular $f(0) = g(0)$ and $f(1) = g(1)$):(5) at least this is the situation that we shall soon consider in the definition of fundamental groupoid/group.

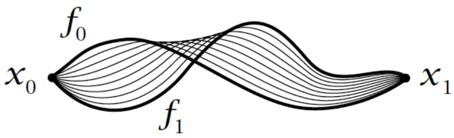

Figure 1 - Homotopy (with fixed endpoints) between two paths f_0 and f_1.(6)

For $t \in I$, one often uses the notation

$$h_t : X \to Y, \qquad h_t(x) := h(t,x) :$$

hence $f = h_0$ and $g = h_1$.

Examples. **(1)** If Y is a convex subset of a topological vector space, any two continuous functions $f, g : X \to Y$ are homotopic by means of the *affine homotopy* $h(x,t) = (1-t)f(x) + tg(x)$. Such homotopy is clearly rel $A = \{x \in X : f(x) = g(x)\}$. **(2)** Two constant functions with values in a topological space Y are homotopic if and only if such constants belong to the same arcwise connected component of Y. **(3)** As functions of $\mathbb{S}^1$ to itself, two rotations (i.e. multiplications by $e^{i\theta}$, with $\theta \in \mathbb{R}$) are homotopic. On the contrary, the identity is not nullhomotopic (in other words, as we shall say soon, $\mathbb{S}^1$ is not "contractible"), as it is well-known to those who have a basic knowledge of holomorphic functions (see 1.4.1).

Remark 1.1.5. (Compact-open topology) Let X be locally compact (i.e., any point has a compact neighborhood) and Hausdorff (hence it is possible to prove that any point has a basis of compact neighborhoods), and Y be Hausdorff. On the space of continuous functions $C(X,Y)$ one can consider the *compact-open* topology, generated by the subsets of type $M_{K,V} = \{f \in C(X,Y) : f(K) \subset V\}$ where K runs among the compact subsets of X and V among the open subsets of Y.(7) Now, $h : X \times I \to Y$ is a homotopy if and only

(5) In the terminology introduced just above, we could say that the homotopies between paths are frequently meant to be *rel* $\partial I = \{0,1\}$.

(6) This picture, as well as others in these notes, are taken from the book of Hatcher [8].

(7) Given a topology $\mathcal{T}$ on a set Z, one says that $S \subset \mathcal{T}$ is a *prebasis* of $\mathcal{T}$ if any element of $\mathcal{T}$ may be expressed as an arbitrary union of finite intersections of elements of S (in this case, the family of finite intersections of elements of S is said to be a *basis* of $\mathcal{T}$). Any $S \subset \mathcal{P}(Z)$ generates in such a way a topology $\mathcal{T}(S)$ on Z; if $S' \subset S$, it is clear that $\mathcal{T}(S') = \mathcal{T}(S)$ if and only if $S \subset \mathcal{T}(S')$. For example, if $Z = C(X,Y)$ and $S = \{M_{K,V} : K \text{ compact of } X,\ V \text{ open in } Y\}$, let us consider a family of compact subsets $\mathcal{K}$ of X containing a basis of neighborhoods of any point, and a prebasis $\mathcal{B}$ of the topology of Y, and set $S' = \{M_{K,V} : K \in \mathcal{K},\ V \in \mathcal{B}\}$: then also S' generates the compact-open topology on $C(X,Y)$. Namely, assumed that $\mathcal{B}$ is a basis (this is not restrictive, since $M_{K,V_1} \cap M_{K,V_2} = M_{K,V_1 \cap V_2}$), it is enough to prove

if (1) $h_t \in C(X,Y)$ for any $t \in I$, and (2) the function $\tilde{h} : I \to C(X,Y)$, $\tilde{h}(t) = h_t$, is a path in $C(X,Y)$.[(8)]

Lemma 1.1.6. (Gluing lemma) *Let X and Y be topological spaces, $X = \bigcup_{j=1}^{r} F_j$ a finite covering by closed subsets. Then $f : X \to Y$ is continuous if and only if so are the restrictions $f|_{F_j}$ $(j = 1, \dots, r)$.*

Proof. Follows immediately from the definition (f is continuous if and only if $f^{-1}(C)$ is closed in X for any closed $C \subset Y$...). □

Proposition 1.1.7. *Homotopy is an equivalence relation in $C(X,Y)$.*

Proof. Reflexivity and simmetry are obvious, while transitivity follows from Lemma 1.1.6 (exercise). □

Definition 1.1.8. (Homotopy category) The category $h\mathfrak{Top}$ has the topological spaces as objects, and the morphisms between two of them are the equivalence classes of continuous functions modulo homotopy. An isomorphism in $h\mathfrak{Top}$ is called *homotopic equivalence*; two spaces isomorphic in $h\mathfrak{Top}$ are said *homotopically equivalent* ($X \sim Y$). A space homotopically equivalent to a point is called *contractible*.

Hence, by definition $f : X \to Y$ is a homotopic equivalence if there exists $g : Y \to X$ such that $g \circ f \sim \mathrm{id}_X$ and $f \circ g \sim \mathrm{id}_Y$. We leave to the student to check that $h\mathfrak{Top}$ is a category (for example, the compatibility of the homotopy with the composition).

Proposition 1.1.9. *A topological space X is contractible if and only if id_X is nullhomotopic. In particular, a contractible space is arcwise connected, and its identity is nullhomotopic to any constant.*

Proof. The first statement follows immediately from the definitions. Then, if $h : X \times I \to X$ is a homotopy between id_X and the constant x_0, another point x_1 of X is connected to x_0 by the arc $h(x_1, \cdot)$. □

Examples. **(0)** From what has just been seen, topological spaces non arcwise connected (for example, discrete spaces with more than one point) cannot be contractible. **(1)** Star-shaped subsets of topological vector spaces are immediate examples of contractible spaces. **(2)** The space $X = \{(x,y) \in I \times I : xy = 0\} \cup \{(\frac{1}{n}, y) : n \in \mathbb{N},\ 0 \le y \le 1\}$ (the so-called *comb space*, see Figure 2) is contractible but non "contractible

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that, given a function $f \in C(X,Y)$, a compact $K \subset X$ and an open $V \subset Y$ such that $f \in M_{K,V}$, there exist compact subsets $K_1, \dots, K_r \in \mathcal{K}$ and open subsets $V_1, \dots, V_r \in \mathcal{B}$ such that $f \in \bigcap_{i=1}^{r} M_{K_i,V_i} \subset M_{K,V}$. For any $x \in K$ let $V_x \in \mathcal{B}$ be such that $f(x) \in V_x \subset V$, and let $K_x \in \mathcal{K}$ be a neighborhood of x such that $f(K_x) \subset V_x$ (i.e. $f \in M_{K_x,V_x}$); by compactness there exist $x_1, \dots, x_r \in K$ such that $K \subset \bigcup_{i=1}^{r} K_{x_i}$: hence one can set $K_i = K_{x_i}$ and $V_i = V_{x_i}$.

[(8)] More generally, ley us show that if T is any Hausdorff topological space then $h : X \times T \to Y$ is continuous if and only if $h_t \in C(X,Y)$ for any $t \in T$, and $\tilde{h} : T \to C(X,Y)$, $\tilde{h}(t) = h_t$ is a path in $C(X,Y)$. If h is continuous, obviously also its restrictions h_t will be continuous; let us show that $\tilde{h}$ is continuous in $t_0 \in T$. Let K be a compact of X, and let V be an open subset of Y such that $\tilde{h}(t_0)(K) \subset V$: then it is enough to prove that there exists an open neighborhood $U \subset T$ of t_0 such that $\tilde{h}(U) \subset M_{K,V}$. Since h is continuous, for any $x \in K$ there exist open neighborhoods $W_x \subset X$ of x and $U_x \subset T$ of t_0 such that $h(W_x \times U_x) \subset V$; if $x_1, \dots, x_k$ are such that $K \subset \bigcup_{j=1}^{k} W_{x_j}$, it will be enough to take $U = \bigcap_{j=1}^{k} U_{x_j}$. Conversely, let us show that h is continuous in $(x_0, t_0) \in X \times T$: if $V \subset Y$ is a neighborhood of $y_0 = h(x_0, t_0)$, we must find open neighborhoods $W \subset X$ of x_0 and $U \subset T$ of t_0 such that $h(W \times U) \subset V$. Since h_{t_0} is continuous, there exists W such that $h(W \times \{t_0\}) \subset V$; then, if $K \subset X$ is a compact neighborhood of x_0 contained in W (remember the hypotheses on X), due to the continuity of $\tilde{h}$ in t_0 there exists U such that $\tilde{h}(U) \subset M_{K,V}$. We then conclude that $h(W \times U) \subset V$.

rel $\underline{q}$", i.e. there does not exist a homotopy rel $\{\underline{q}\}$ of id_X with the constant function $X \to \{\underline{q}\} \subset X$.[(9)] **(3)** The space $X = \{(\frac{1-t}{n}, \mathrm{sign}(n)t) : n \in \mathbb{Z}_\times,\ t \in I\} \cup \{(0, 2t-1) : t \in I\}$ (see Figure 3) is arcwise connected but not contractible.[(10)] **(4)** Later we shall see that the spheres $\mathbb{S}^n$ are not contractible.

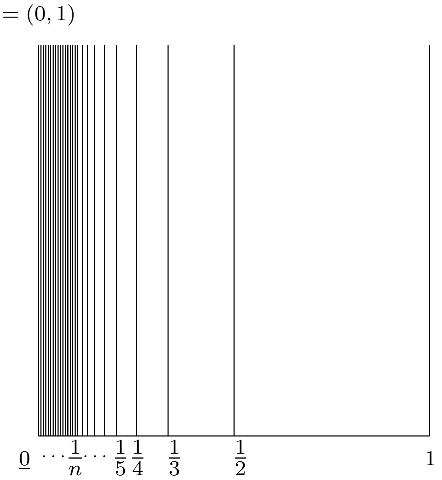

Figure 2 - The "comb space"

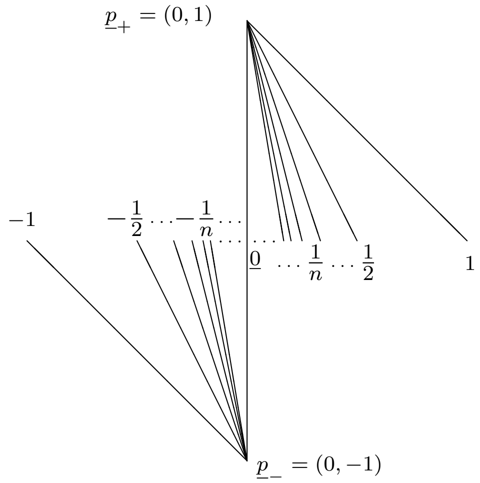

Figure 3 - A space arcwise connected but not contractible

Before studying the functions defined or having values in $\mathbb{S}^n$, let us recall the definition and some properties of quotient functions.

Definition 1.1.10. A surjective function $p : X \to Y$ is called *quotient* if the following condition holds: $V \subset Y$ is open in Y if and only if $p^{-1}(V)$ is open in X. (In particular, p is continuous.)

Of course, this is equivalent to saying that the topology of Y coincides with the quotient topology with rispect to p, i.e. the finest (=largest) topology on Y such that p is continuous. Now, on $Y \setminus p(X)$ such topology clearly coincides with the discrete topology, hence the non surjective case is not very interesting: therefore the problem is to check whether a

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(9) To write a homotopy (even rel $\{\underline{0}\}$) between id_X and the constant function $X \to \{\underline{0}\} \subset X$ is easy: for example, the function given by $h((x,y),t) = (x, (1-2t)y)$ for $t \in [0, \frac{1}{2}]$ and $h((x,y),t) = (2(1-t)x, 0)$ for $t \in][\frac{1}{2}, 1]$ (continuous by Lemma 1.1.6). To show the "non contractibility rel $\underline{q}$", it is enough to show that a hypothetical homotopy rel $\{\underline{q}\}$ between id_X and the constant function $X \to \{\underline{q}\} \subset X$ cannot be continuous: in fact such a homotopy should have constant value $\underline{q}$ on $\{\underline{q}\} \times I$, and to find a discontinuity point of one can argue as in the example that follows.

(10) Let us suppose that there exists a homotopy $h : X \times I \to X$ between id_X and the constant $\underline{0}$, and set $h = (h_1, h_2)$ in $\mathbb{R}^2$. For any $n \geq 1$ set $\underline{x}_{\pm n} = (\pm\frac{1}{n}, 0)$, and let $t_{\pm n} = \min\{t \in I : h(\underline{x}_{\pm n}, t) = \underline{p}_\pm\}$ (h is continuous, hence $t_{\pm n}$ exist > 0), and set $t_\pm = \liminf_{n \to +\infty} t_{\pm n}$. (Recall that, if $A \subset \mathbb{R}$, $x_0 \in \overline{A}$, $f : A \to \mathbb{R}$ and $\ell \in \mathbb{R}$ then $\ell = \liminf_{x \to x_0} f(x)$ means that (i) for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $x_n \geq x_0 - \epsilon$ for any $n \geq N$, and (ii) for any $\epsilon > 0$ and $N \in \mathbb{N}$ there exists $n \geq N$ such that $x_n \leq x_0 + \epsilon$.) If one out of the $t_\pm$'s would be zero (assume e.g. $t_+ = 0$) then h_2 (and hence h) cannot be continuous in $(\underline{0}, 0)$: otherwise for any $\epsilon > 0$ there would exist $\tau > 0$ and $N \in \mathbb{N}$ such that $|h_2(\underline{x}_n, t)| < \epsilon$ if $n \geq N$ and $t < \tau$, but this is false since for any $\epsilon > 0$ and any $\tau > 0$ there exists n such that $\frac{1}{n} < \epsilon$, $t_n < \tau$ and $h(\underline{x}_n, t_n) = \underline{p}_+$ (hence $h_2(\underline{x}_n, t_n) = 1$). On the other hand, if $t_\pm$ are both > 0, note that for any $0 \leq t < t_+$ it holds $h_2(\underline{0}, t) \geq 0$ (namely $\underline{x}_n \to 0$ and $h_2(\underline{x}_n, t) \geq 0$), and analogously for any $0 \leq t < t_-$ it holds $h_2(\underline{0}, t) \leq 0$. We conclude that $h_2(\underline{0}, t) \equiv 0$ for any $0 \leq t < \tilde{t} := \min\{t_+, t_-\}$. Arguing as above, we show that h_2 (and therefore h) cannot be continuous in $(\underline{0}, \tilde{t})$.

given continuous and surjective function $p : X \to Y$ is quotient or not. In general this is obviously false (it is enough to weaken the topology of Y: the continuity of p will be even improved). In any case, the following criterion holds.

Proposition 1.1.11. *Let $p : X \to Y$ be a continuous and surjective function. If p is open or closed, then p is quotient. Conversely, if p is quotient then p is open* (resp. *closed*) *if and only if for any subset open* (resp. *closed*) *$A \subset X$ the "p-saturated" $p^{-1}p(A) \subset X$ is open* (resp. *closed*) *in X.*

Proof. Exercise. $\square$

Corollary 1.1.12. *Let X be compact and Y be Hausdorff. Then a continuous and surjective function $p : X \to Y$ is quotient.*

Proof. Under these hypotheses p is closed, so we may apply Proposition 1.1.11. $\square$

Remark 1.1.13. As we have seen, in general a continuous and surjective function is far from being quotient. In particular, one should be careful when restricting quotient functions. For example, consider the exponential map $\pi : \mathbb{R} \to \mathbb{S}^1$ given by $\pi(t) = e^{2\pi i t}$: it is quotient, open and closed. The restriction $\pi|_{[0,1]}$ is closed (hence quotient) but non open, $\pi|_{]0,2[}$ is open (hence quotient) but non closed, and the bijection $q = \pi|_{[0,1[} : [0, 1[\to \mathbb{S}^1$ is even no longer quotient. This provides a confirmation that, as it is well-known, the continuous bijection q is not a homeomorphism between the interval $[0, 1[$ and the circle $\mathbb{S}^1$ with their usual topologies: in order to make it such, it would be necessary to refine the topology of $\mathbb{S}^1$ into the quotient one with respect to q.[11]

The quotient functions have the factorization property:

Proposition 1.1.14. *Let $p : X \to Y$ be a quotient function, $f : X \to Z$ a continuous function constant on the fibers of p. Then there exists a (unique) continuous function $\tilde{f} : Y \to Z$ such that $\tilde{f} \circ p = f$.*

Proof. Let us define $\tilde{f}(y) = f(x)$ for an arbitrary choice of $x \in p^{-1}(y)$: this function is well-defined by hypothesis, and uniqueness is obvious. If $W \subset Z$ is open, since p is quotient $\tilde{f}^{-1}(W)$ is open if and only if $p^{-1}\tilde{f}^{-1}(W) = f^{-1}(W)$ is open in X, and this is true since f is continuous. $\square$

A consequence:

Corollary 1.1.15. *Let $q : X \to Y$ be a quotient function, X_q the space of fibers of q* (i.e. the quotient of X by the equivalence relation $x_1 \sim x_2$ if and only if $q(x_1) = q(x_2)$) *endowed with the quotient topology. Then Y is canonically homeomorphic to X_q.*

Proof. The following is a standard argument which is often useful when one must prove the uniqueness (up to canonical identifications) of some structure starting from existence and uniqueness of morphisms. Let $\pi : X \to X_q$ be the canonical projection, and let us apply Proposition 1.1.14 repeatedly. For $p = q$ and $f = \pi$ we get that there exists a unique $\tilde{\pi} : Y \to X_q$ such that $\tilde{\pi} \circ q = \pi$; for $p = \pi$ and $f = q$ we get that there exists a unique $\tilde{q} : X_q \to Y$ such that $\tilde{q} \circ \pi = q$: this gives $(\tilde{q} \circ \tilde{\pi}) \circ q = q$. But for $p = f = q$ there is only one $\alpha : Y \to Y$ such that $\alpha \circ q = q$, and by uniqueness this α must be the identity: hence $\tilde{q} \circ \tilde{\pi} = \mathrm{id}_Y$. One shows similarly that $\tilde{\pi} \circ \tilde{q} = \mathrm{id}_{X_q}$, and the proof is over. $\square$

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[11] For $0 < \varepsilon < 1$, $[0, \varepsilon[$ (resp. $]0, \varepsilon]$) is a open subset of $[0, 1]$ (resp. a closed subset of $]0, 2[$) but its image is not open (resp. closed) in $\mathbb{S}^1$. Moreover, $V = \{e^{2\pi i t} : 0 \leq t < \varepsilon\}$ is not open in $\mathbb{S}^1$, while $\pi|_{[0,1[}^{-1}(V) = [0, \varepsilon[$ is open in $[0, 1[$. Therefore the quotient topology of $\mathbb{S}^1$ with respect to q must contain, among its open subset, also those of the form of V.

This shows that, up to canonical homeomorphisms, quotient functions are the maps of type $\pi : X \to \frac{X}{\sim}$, where $\sim$ is an equivalence relation in X and $\frac{X}{\sim}$ is the space of equivalence classes endowed with the quotient topology with respect to the canonical projection π.

Examples. **(1)** In $\mathbb{B}^{n+1}$ let us consider the equivalence relation $\sim$ which identifies all the points of the boundary $\mathbb{S}^n$: the quotient $\frac{\mathbb{B}^{n+1}}{\sim}$ is homeomorphic to $\mathbb{S}^{n+1}$.**(12)** **(2)** *(Suspensions and wedge sums)* Among the various classic topological constructions obtained by a quotient procedure, let us mention a couple of them. The *suspension* SX of a topological space X, is the quotient of $X \times I$ obtained by identifying $X \times \{0\}$ to a single point and $X \times \{1\}$ to another single point: so, for example, it is quite clear that $S\mathbb{S}^n$ is homeomorphic to $\mathbb{S}^{n+1}$.**(13)** The *wedge sum* $X \vee Y$ of two topological spaces X and Y with base points respectively $x_0 \in X$ and $y_0 \in Y$ is the quotient of the disjoint union $X \sqcup Y$ obtained by identifying the two points x_0 and y_0 to a single point: for example, $\mathbb{S}^1 \vee \mathbb{S}^1$ is homeomorphic to the "figure eight".

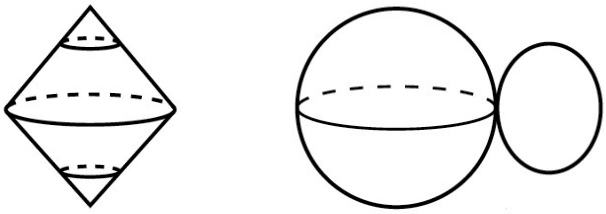


Figure 4 - The suspension $S\mathbb{S}^1$ and the wedge sum $\mathbb{S}^1 \vee \mathbb{S}^2$.

Remark 1.1.16. If $p : X \to Y$ is quotient, it is clear that $Y \simeq \frac{X}{\sim}$ is Hausdorff if and only any pair of different equivalence classes are contained in disjoint saturated open subsets. A necessary condition is that the classes are closed subsets of X (in other words, that Y is T_1). But this is not sufficient, as shows the following example of Jänich [9]. Let $X = [-\frac{\pi}{2}, \frac{\pi}{2}] \times \mathbb{R}$, $r_\pm = \{\pm\frac{\pi}{2}\} \times \mathbb{R}$, $s_1(c) = \{(x, \mathrm{tg}\,(x) + c) : x \in]-\frac{\pi}{2}, \frac{\pi}{2}[\}$, $s_2(c) = \{(x, \frac{1}{\cos(x)} + c) : x \in]-\frac{\pi}{2}, \frac{\pi}{2}[\}$ (where $c \in \mathbb{R}$) and consider the partitions $Y_1 = \{r_\pm,\ s_1(c)\ (c \in \mathbb{R})\}$ and $Y_2 = \{r_\pm,\ s_2(c)\ (c \in \mathbb{R})\}$. Both the classes of Y_1 and of Y_2 are closed subsets of X, but only Y_1 is of Hausdorff.**(14)**

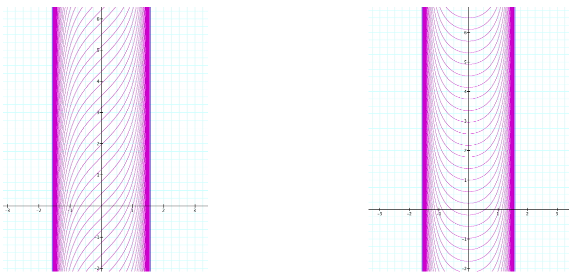


Figure 5 - The partitions Y_1 and Y_2 of the examples of Jänich.

(12) The idea of the proof: for example, for $n = 1$ identify, in $\mathbb{R}^3$, $\mathbb{B}^2 \simeq \{x^2 + y^2 + z^2 = 4,\ z \geq 0\}$ and $\mathbb{S}^2 \simeq \{x^2 + y^2 + (z-1)^2 = 1\}$, then consider the continuous map $\mathbb{B}^2 \to \mathbb{S}^2$ sending $(x, y, z) \in \mathbb{B}^2$ in the point of $\mathbb{S}^2$ given by the intersection with the horizontal half-line $\{(tx, ty, z) : t \geq 0\}$. Since $\mathbb{B}^2$ is compact and $\mathbb{S}^2$ is Hausdorff, by Corollary 1.1.12 the map is quotient.

(13) Try to visualize and write down the case $n = 1$ (see Figure 4), then argue in the general case.

(14) In both spaces the graphs $s_j(c)$ are separable both between them and from the two vertical lines $r_\pm$; on the other hand, $r_\pm$ can be separated only in Y_1.

So let us study the functions having $\mathbb{S}^n$ as domain or as codomain.

Proposition 1.1.17. *Let Y be a topological space and $f : \mathbb{S}^n \to Y$ be a continuous function. The following statements are equivalent:*

(i) *f is nullhomotopic;*

(ii) *f has a continuous extension $\tilde{f} : \mathbb{B}^{n+1} \to Y$;*

(iii) *f is nullhomotopic rel $\{x_0\}$ for any $x_0 \in \mathbb{S}^n$.*

Proof. (i) $\Rightarrow$ (ii): let $y_0 \in Y$ and $h : \mathbb{S}^n \times I \to Y$ be a homotopy such that $h(x, 0) = f(x)$ and $h(x, 1) = y_0$. If $u : I \to I$ is any continuous function with $u \equiv 0$ in a neighborhood of 0 and $u(1) = 1$, then $\tilde{f} : \mathbb{B}^{n+1} \to Y$ defined as $\tilde{f}(z) = h(\frac{z}{|z|}, 1 - u(|z|))$ (if $z \neq 0$) and $\tilde{f}(0) = y_0$ is continuous and extends f. Another proof can be obtained as follows. Let $g : \mathbb{S}^n \times I \to \mathbb{B}^{n+1}$ be given by $g(x, t) = (1 - t)x$: it is continuous, surjective and closed (both spaces are compact) and hence quotient by Proposition 1.1.11. The fiber of g above $z \in \mathbb{B}^{n+1}$ is the point $\{(z/|z|, 1 - |z|)\}$ (if $z \neq 0$) and $\mathbb{S}^n \times \{1\}$ (if $z = 0$): hence h is constant on the fibers of g. By Proposition 1.1.14 there exists a continuous function $\tilde{f} : \mathbb{B}^{n+1} \to Y$ such that $h = \tilde{f} \circ g$. The function $\tilde{f}$ is the desired extension. (ii) $\Rightarrow$ (iii): if we define $h : \mathbb{B}^{n+1} \times I \to Y$ by $h(x, t) = \tilde{f}(x_0 + (1 - t)(x - x_0))$, the function $h|_{\mathbb{S}^n \times I}$ is a homotopy rel x_0 between f and the constant $f(x_0)$. (iii) $\Rightarrow$ (i): obvious. □

Proposition 1.1.18. *Let X be a topological space, $f : X \to \mathbb{S}^n$ and $g : X \to \mathbb{S}^n$ continuous functions. If f and g are never antipodal* (i.e. if $f(x) \neq -g(x)$ for any $x \in X$) *then they are homotopic.*

Proof. The function $\alpha : X \times I \to \mathbb{R}^{n+1}$ given by $\alpha(x, t) = (1 - t)f(x) + tg(x)$ —i.e. the affine homotopy between f and g— never vanishes (if $(1 - t)f(x) = -tg(x)$, comparing the norms of both members one has $1 - t = t$, i.e. $t = 1/2$: hence $f(x) = -g(x)$, which contradicts the hypothesis): a homotopy is then $h = \alpha/|\alpha|$. □

Corollary 1.1.19. *Let X be a topological space. Any $f : X \to \mathbb{S}^n$ continuous and non surjective is nullhomotopic.*

Proof. If $y_0 \in \mathbb{S}^n \setminus f(X)$, then f and the constant map $g(x) \equiv -y_0$ are never antipodal. By Proposition 1.1.18, they are homotopic. □

Another consequence (if we accept to postpone the completion of the proof to the second part of the course, where we shall meet the notion of topological degree) is the following funny result. Recall that a *vector field* on a manifold is a continuous section of the tangent bundle, i.e. a continuous function $\varphi : \mathbb{S}^n \to \mathbb{R}^{n+1}$ such that, for any $x \in \mathbb{S}^n$, $\varphi(x)$ is a vector tangent to $\mathbb{S}^n$ in x (i.e., $\varphi(x) \in T_x\mathbb{S}^n$).

Corollary 1.1.20. (Hairy Ball theorem) *There exist continuous and never vanishing vector fields on $\mathbb{S}^n$ if and only if n is odd.*

In particular, any continuous vector field on $\mathbb{S}^2$ has at least one zero (this provides a justification for the popular name "Hairy Ball theorem" of the result: it is impossible to "comb a sphere in a continuous way without creating some baldness").

Proof. To assume that there exists a vector field both continuous and never vanishing on $\mathbb{S}^n$ is obviously the same than assuming that there exists a continuous field of tangent versors $\varphi : \mathbb{S}^n \to \mathbb{S}^n$. Since φ can never be antipodal neither to $\mathrm{id}_{\mathbb{S}^n}$ nor to $-\mathrm{id}_{\mathbb{S}^n}$ (since neither $x \in \mathbb{S}^n$ nor $-x$ belong to $T_x\mathbb{S}^n$), by Proposition 1.1.18 it is homotopic to both and then, by transitivity, one has $\mathrm{id}_{\mathbb{S}^n} \sim -\mathrm{id}_{\mathbb{S}^n}$. Now, the topological degree of a proper continuous function is invariant under proper homotopies (the sphere $\mathbb{S}^n$ is compact, so of course everything is proper), and one proves that this degree is 1 for $\mathrm{id}_{\mathbb{S}^n}$ and $(-1)^{n+1}$ for $-\mathrm{id}_{\mathbb{S}^n}$ (this will be done in the second part of the course: see Examples 2.7): so we get a contradiction when n is even. On the other hand, when n is odd a field of tangent versors both continuous and never vanishing is easy to exhibite: just set $\varphi(x_1, x_2, \ldots, x_n, x_{n+1}) = (x_2, -x_1, \ldots, x_{n+1}, -x_n)$. □

1.2 Retractions

There are various ways to "shrink" a space into a subset of it: let us investigate them.

Definition 1.2.1. Let X be a topological space and $A \subset X$. A is said to be a *retract* of X if there exists a continuous function $r : X \to A$ (*retraction*) such that $r(x) = x$ for any $x \in A$.[15]

Lemma 1.2.2. *Let X and Y be topological spaces, $f, g : X \to Y$ two continuous functions. If Y is Hausdorff, then $A = \{x \in X : f(x) = g(x)\}$ is closed in X.*

Proof. Let $(f, g) : X \to Y \times Y$ the product map (continuous). Recalling (exercise) that Y is Hausdorff if and only if the diagonal Δ_Y is closed in $Y \times Y$, it is enough to observe that $A = (f, g)^{-1}(\Delta_Y)$. □

The following proposition provides a first, elementary bound to the fact of being a retract of a Hausdorff space.

Proposition 1.2.3. *If X is a Hausdorff topological space and $A \subset X$ is a retract of X, then A is closed.*

Proof. Note $A = \{x \in X : \mathrm{id}_X(x) = r(x)\}$, then apply Lemma 1.2.2. □

Examples. **(1)** For any $x_0 \in X$, $A = \{x_0\}$ is a retract of X. **(2)** $\mathbb{S}^n$ is a retract of $\mathbb{R}^{n+1}_\times$ (with $r(x) = x/|x|$), but not of $\mathbb{B}^{n+1}$, as we shall see soon (Theorem 1.2.5). **(3)** Let $X = I \times I$ and A be the "comb space" (see Figure 2 of §1.1): although these spaces are both contractible, A is not a retract of X.[16]

In the notion of "retract" the whole set X is sent into A with all points of A kept fixed, but this happens all at once, without any progressiveness: in other words, the idea of homotopy does non appear. Consequently it is not so surprising that even two homotopically elementary spaces as those of the last example could be not the retract of each other. In fact, the homotopy comes into play in the following notions of "deformation retract".

Definition 1.2.4. Let X be a topological space and $A \subset X$. A is a *weak deformation retract* of X if the canonical inclusion $i_A : A \to X$ is a homotopic equivalence. A is a *deformation retract* of X if there exists a retraction $r : X \to A$ such that $i_A \circ r \sim \mathrm{id}_X$. Finally, A is a *strong deformation retract* of X if exists a retraction $r : X \to A$ such that $i_A \circ r \sim \mathrm{id}_X$ rel A.

From the definitions it is clear that "strong deformation retract" implies "deformation retract", which in turn separately implies "retract" and "weak deformation retract". On the other hand the two latter notions are independent, as it is shown in the following examples.

[15] In other words, the map r is a left inverse of the canonical inclusion $i_A : A \to X$ in the category $\mathfrak{Top}$.

[16] Namely, let $\underline{q} = (0, 1) \in A$ and V a neighborhood of $\underline{q}$ in X: then $A \cap V$ is a neighborhood (which, up to shrinking V, can be assumed to be disconnected) of $\underline{q}$ in A. Would there exists a retraction $r : X \to A$, since $r(\underline{q}) = \underline{q}$ and r is continuous there would exist a connected neighborhood U of $\underline{q}$ in X such that $r(U) \subset A \cap V$, which implies $r(U) \subset \{(x, y) \in A \cap V : x = 0\}$. But this is absurd, since r induces the identity on A (and in particular on $A \cap U$).

Examples. **(1)** $\mathbb{S}^n$ (resp. $\mathbb{B}^{n+1}$) is a strong deformation retract of $X = \mathbb{R}^{n+1}_\times$ (resp. $X = \mathbb{R}^{n+1}$): the retractions are $r(x) = x/|x|$ and $r(x) = \begin{cases} x & (|x| \leq 1) \\ x/|x| & (|x| \geq 1) \end{cases}$, and the latter is continuous by the Gluing lemma. As an exercise, find the homotopies between $i_A \circ r$ and id_X. **(2)** Any contractible subset A of a contractible space X is a weak deformation retract of X: namely, if x_0 is any point of A, the constant function $g : X \to A$, $g \equiv x_0$ is a homotopic inverse of i_A (any contractible space is arcwise connected, hence its identity is homotopic to any constant: see Proposition 1.1.9). But it is false that A must necessarily be a retract of X: we already saw it with $X = I \times I$ and A the comb space, and a even more immediate example is given by $X = \mathbb{R}^{n+1}$ and A the open ball $\dot{\mathbb{B}}^{n+1}$ which, being non closed, can not be retract of X. **(3)** On the other hand, if X is a topological space, x_0 a point of X and $A = \{x_0\}$, we saw that A is always a retract of X, while it is clear that A is deformation retract of X if and only if X is contractible. **(4)** If X is contractible, any point of X is a deformation retract of X: namely, this amount to saying (see Proposition 1.1.9) that the identity is nullhomotopic to any constant. So, for example, both points $(0,0)$ and $\underline{q} = (0,1)$ are deformation retracts of the comb space X, but only the first point is a strong deformation retract: namely we saw (see p. 7) that there does not exist a homotopy rel $\underline{q}$ of the identity into the constant $\underline{q}$.

The following theorem shows a fact about $\mathbb{S}^n$ which has been already announced: however, once more we must wait for the second part of the course (when we shall use the notion of cohomology) to complete the proof.

Theorem 1.2.5. *The following equivalent statements hold:*

(i) $\mathbb{S}^n$ *is not contractible;*

(ii) $\mathbb{S}^n$ *is not a retract of* $\mathbb{B}^{n+1}$*;*

(iii) (Brouwer's Fixed Point Theorem) *Any continuous function of* $\mathbb{B}^{n+1}$ *into itself has at least one fixed point.*

Proof. First let us prove that the three statements are equivalent. (i) $\Leftrightarrow$ (ii): just apply Proposition 1.1.17 to $\mathrm{id}_{\mathbb{S}^n}$. (ii) $\Rightarrow$ (iii): let $f : \mathbb{B}^{n+1} \to \mathbb{B}^{n+1}$ be a continuous function without fixed points: then a retraction $r : \mathbb{B}^{n+1} \to \mathbb{S}^n$ is obtained by setting, for $z \in \mathbb{B}^{n+1}$, the image $r(z)$ as the intersection point with $\mathbb{S}^n$ of the half line starting from $f(z)$ and passing through z (note that $f(z) \neq z$ by hypothesis).(17) (iii) $\Rightarrow$ (ii): conversely, if $r : \mathbb{B}^{n+1} \to \mathbb{S}^n$ is a retraction, the function $f = -i_{\mathbb{S}^n} \circ r$ is continuous and without fixed points. • Now let us start the proof of (ii), which will be completed later. Using Weierstrass' approximation theorem, it is easy to prove that it is enough to show that there does not exist retractions *of class* $\mathcal{C}^\infty$ of $\mathbb{B}^{n+1}$ on $\mathbb{S}^n$.(18) It could be possible to provide a direct but some cumbersome proof of the latter fact (or even something more: that there does not exist retractions of class $\mathcal{C}^1$), based on arguments of differential topology, but we rather prefer to wait for the notion of cohomology in second part of the course (see Corollary 2.5.6). □

(17)Here are some details of the computation. For $z \in \mathbb{B}^{n+1}$ let $y = f(z)$: the half line is parametrized by $y + t(z - y)$ for $t > 0$. Let us look for the unique $t(z) > 0$ such that $|y + t(z)(z - y)| = 1$: the computations yield $t(z) = \frac{|y|^2 - (z\cdot y) + \sqrt{|z|^2 + |y|^2 - |z|^2|y|^2 - 2(z\cdot y) + (z\cdot y)^2}}{|z|^2 + |y|^2 - 2(z\cdot y)}$ (e.g. check that for $|z| = 1$, $z = 0$ and $y = 0$ the result is as expected), and $r(z) = f(z) + t(z)(z - f(z))$ is the desired retraction.

(18)Reading again the proof (ii) $\Leftrightarrow$ (iii), this is equivalent to show that any $\mathcal{C}^\infty$ function of $\mathbb{B}^{n+1}$ into itself has at least one fixed point. But, thanks to Weierstrass' approximation theorem, this is equivalent to (iii) (hence to (ii)). Namely, assume by absurd that $f : \mathbb{B}^{n+1} \to \mathbb{B}^{n+1}$ is continuous and without fixed points, and let $\mu = \min_{x \in \mathbb{B}^{n+1}} |f(x) - x| > 0$. Take a function $h : \mathbb{B}^{n+1} \to \mathbb{R}^{n+1}$ of class $\mathcal{C}^\infty$ such that $\sup_{x\in\mathbb{B}^{n+1}} |h(x) - f(x)| < \frac{\mu}{2}$, and set $g = h/(1 + \mu/2)$: then $g : \mathbb{B}^{n+1} \to \mathbb{R}^{n+1}$ is obviously of class $\mathcal{C}^\infty$, $g(\mathbb{B}^{n+1}) \subset \mathbb{B}^{n+1}$ (since $|h(x)| < |f(x)| + \mu/2 \leq 1 + \mu/2$) and $\sup_{x\in\mathbb{B}^{n+1}} |g(x) - f(x)| < \mu$ (since $|g(x) - f(x)| = (1 + \mu/2)^{-1}|f(x) - h(x) + (\mu/2)f(x)| \leq (1 + \mu/2)^{-1}(|f(x) - h(x)| + (\mu/2)|f(x)|) < 2\mu/(2 + \mu) < \mu$), hence if x_0 is a fixed point of g one gets $|x_0 - f(x_0)| \leq |x_0 - g(x_0)| + |g(x_0) - f(x_0)| = |g(x_0) - f(x_0)| < \mu$, a contradiction.

1.3 The fundamental group of a topological space

We start with the notion of "groupoid".

Definition 1.3.1. A *groupoid* is a small (see §A.1) category $\mathcal{C}$ in which all morphisms are isomorphisms.

If $\mathcal{C}$ is a groupoid, it is clear that $\mathrm{Hom}_{\mathcal{C}}(a,a)$ is a group for any $a \in \mathrm{Ob}(\mathcal{C})$; moreover, if $a, b \in \mathrm{Ob}(\mathcal{C})$ are such that $\mathrm{Hom}_{\mathcal{C}}(a,b) \neq \varnothing$, any isomorphism $\varphi : a \to b$ provides a (non canonical) isomorphism of groups $\mathrm{Hom}_{\mathcal{C}}(a,a) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(b,b)$ where $f \mapsto \varphi \circ f \circ \varphi^{-1}$. The groupoid is said to be *connected* if $\mathrm{Hom}_{\mathcal{C}}(a,b) \neq \varnothing$ for any $a, b \in \mathrm{Ob}(\mathcal{C})$: in this case, the groupoid is associated to a precise class of isomorphism of groups.

Now we define the "fundamental (or Poincare's) groupoid" of a topological space X.

Let $\gamma : I \to X$ and $\phi : I \to X$ be two paths in X, and assume that $x_1 = \gamma(1) = \phi(0)$. In this case one can "join" the paths by setting

$$(\gamma \cdot \phi)(t) = \begin{cases} \gamma(2t) & (0 \leq t \leq \frac{1}{2}) \\ \phi(2t-1) & (\frac{1}{2} < t \leq 1) \end{cases},$$

and $\gamma \cdot \phi$ is also a path (exercise: just use the Gluing lemma).

Remark 1.3.2. This junction is not associative: however, if $\psi : I \to X$ is a path with $x_2 = \phi(1) = \psi(0)$, then $(\gamma \cdot \phi) \cdot \psi$ and $\gamma \cdot (\phi \cdot \psi)$ differ only by a change of parameter (exercise: determine this change explicitly).

Let us introduce into the space $C(I, X)$ of paths in X the already mentioned relation of homotopy rel $\partial I = \{0, 1\}$ (i.e., homotopy with fixed extremities, see Remark 1.1.4), and denote by $[\gamma]$ the equivalence class of the path γ. In the sequel, for $x \in X$ we denote by c_x the constant path on x (i.e., $c_x : I \to X$, $c_x(t) \equiv x$).

Proposition 1.3.3. *The following statements hold.*

(i) *If γ is a path with extremities $x_0 = \gamma(0)$ and $x_1 = \gamma(1)$, one has $[\gamma \cdot c_{x_1}] = [\gamma] = [c_{x_0} \cdot \gamma]$.*

(ii) *If γ is a path and p is a change of parameter, then $[\gamma \circ p] = [\gamma]$.*

(iii) *Let γ_j $(j = 1, 2)$ two paths with extremities x_0 and x_1, and let ψ_j $(j = 1, 2)$ two paths with extremities x_1 and x_2; then, assuming that $[\gamma_1] = [\gamma_2]$ and $[\psi_1] = [\psi_2]$, one has $[\gamma_1 \cdot \psi_1] = [\gamma_2 \cdot \psi_2]$.*

Proof. Exercise. □

Thanks to Proposition 1.3.3(iii), it makes sense to define the junction also at the level of equivalence classes of paths: $[\gamma] \cdot [\phi] = [\gamma \cdot \phi]$. Due to (ii) and to Remark 1.3.2, such junction of classes is associative: in the previous notation it holds $[(\gamma \cdot \phi) \cdot \psi] = [\gamma \cdot (\phi \cdot \psi)]$. This gives a sense to the following definition.

Definition 1.3.4. The *fundamental groupoid* (or *Poincaré groupoid*) of X is the groupoid whose objects are the points of X, the morphisms between them are the classes of homotopy rel ∂I of paths joining them, and the composition law is — given two paths γ and ψ with $\gamma(1) = \psi(0)$ — the law $[\psi] \circ [\gamma] := [\gamma \cdot \psi]$. If $x_0 \in X$, the *fundamental group* (or (first) *homotopy group*) *of* X *in* x_0 is the corresponding group of endomorphisms of x_0, i.e. the set $\pi_1(X; x_0)$ of equivalence classes of loops based on x_0 endowed with the operation of junction.

In the group structure of $\pi_1(X; x_0)$, the identity element is obviously $[c_{x_0}]$ and, given a loop γ based on x_0, the inverse of $[\gamma]$ is $[\gamma^{-1}]$, where γ^{-1} is the loop defined by $\gamma^{-1}(t) = \gamma(1-t)$ for $t \in I$.

It is clear that the fundamental groupoid of X is connected if and only if X is arcwise connected. In this case, as it has been seen, if γ is a path in X with $\gamma(0) = x_0$ and $\gamma(1) = x_1$, the map

$$h_{[\gamma]} : \pi_1(X, x_0) \to \pi_1(X, x_1), \qquad [\phi] \mapsto [(\gamma^{-1} \cdot \phi) \cdot \gamma] \tag{1.1}$$

gives a non-canonical isomorphism between the two groups. This explains the abuse of notation (frequent in arcwise connected spaces) of neglecting the base point by simply writing $\pi_1(X)$ and talking about *the* fundamental group of X.

Remark 1.3.5. Note that $\pi_1(X; x_0)$ is also the family of homotopy classes of continuous functions $\gamma : (\mathbb{S}^1, 1) \to (X, x_0)$, endowed with the structure of group. More generally, one can define the higher homotopy groups $\pi_n(X; x_0)$ set-theoretically as the family of homotopy classes of continuous functions $\gamma : (\mathbb{S}^n, \mathbf{1}) \to (X, x_0)$, where $\mathbf{1} = (1, 0, \ldots, 0)$ denotes the first versor of the canonical basis of $\mathbb{R}^{n+1}$, then endowed with a structure of group given by a "junction" naturally generalizing the one for loops when $n = 1$. The study of higher homotopy groups is very interesting and rich of fundamental questions which are still open — for example, even the homotopy groups of the sphere $\mathbb{S}^n$ are not completely known. These groups are commutative for $n \geq 2$, a feature which does not hold when $n = 1$ as we shall understand very soon. In fact we shall deal only with the basic case $n = 1$, and when talking about "homotopy group" we shall always mean "first" homotopy group.

The class of spaces which are trivial with respect to the structure of fundamental group should be already familiar to non-freshmen students:

Definition 1.3.6. A topological space is said to be *simply connected* if it is arcwise connected and its fundamental group is trivial.

Proposition 1.3.7. *An arcwise connected topological space is simply connected if and only if there is a unique homotopy class of paths connecting any two points.*

Proof. The condition is obviously sufficient; conversely, given any two paths γ_1 and γ_2 we have $[\gamma_1] = [\gamma_1 \cdot (\gamma_2^{-1} \cdot \gamma_2)] = [(\gamma_1 \cdot \gamma_2^{-1}) \cdot \gamma_2] = [\gamma_2]$, since $\gamma_1 \cdot \gamma_2^{-1}$ is a loop and the space is simply connected. $\square$

Examples. **(1)** A contractible topological space is simply connected. **(2)** $\mathbb{S}^1$ is not simply connected, as we shall prove soon. **(3)** On the other hand, for any $n \geq 2$ the spheres $\mathbb{S}^n$ —although non contractible, recall Theorem 1.2.5— are simply connected. This will be an easy consequence of Van Kampen theorem (§1.5); however we provide another proof of this fact in a exercise here below, taken from [8, §1].

Let $\mathfrak{Top}_p$ be the category of "pointed" topological spaces, whose objects are the pairs (X, x_0) with X a topological space and x_0 a point of X, and the morphisms between (X, x_0) and (Y, y_0) are the continuous functions $f : X \to Y$ such that $f(x_0) = y_0$. Then one can define the following functor:

$$\pi_1 : \mathfrak{Top}_p \to \mathfrak{Groups},$$

which associates to (X, x_0) the group $\pi_1(X, x_0)$, and which sends the continuous function $f : X \to Y$ such that $f(x_0) = y_0$ in the morphism of groups

$$f_\# = \pi_1(f) : \pi_1(X, x_0) \to \pi_1(Y, y_0), \qquad f_\#([\gamma]) = [f \circ \gamma].$$

We leave to the student the necessary verifications (for example, that $f_\#$ is well-defined and that $(g \circ f)_\# = g_\# \circ f_\#$). It is important to note that maps homotopic rel $\{x_0\}$ give rise to the same functor:

Proposition 1.3.8. *Let X and Y be topological spaces, $f, g : X \to Y$ two homotopic continuous functions. Then for any $x_0 \in X$ there exists a path ω in Y between $y_0 = f(x_0)$ and $y_1 = g(x_0)$ such that $g_\# = h_{[\omega]} \circ f_\#$. If morever f and g are homotopic rel x_0* (hence in particular $y_0 = y_1$) *then $f_\# = g_\#$.*

Proof. Let $h : X \times I \to Y$ be a homotopy between f and g, and consider the path $\omega : I \to Y$ defined by $\omega(t) = h(x_0, t)$. We are left with showing that for any $[\sigma] \in \pi_1(X, x_0)$ it holds $[g \circ \sigma] = [\omega^{-1} \cdot ((f \circ \sigma) \cdot \omega)]$, i.e. that $[\omega] \cdot [g \circ \sigma] \cdot [\omega^{-1}] \cdot [f \circ \sigma^{-1}] = [c_{y_0}]$. To this aim let us define the following continuous function on the boundary of the square $I \times I$ with values in Y:

$$H(t, \tau) = \begin{cases} f(\sigma(t)) & (\tau = 0) \\ g(\sigma(t)) & (\tau = 1) \\ \omega(\tau) & (t = 0, 1). \end{cases}$$

The function H admits a natural extension to all of $I \times I$ by setting $H(t, \tau) = h(\sigma(t), \tau)$: by a result completely analogous to Proposition 1.1.17 for $n = 1$, where $\mathbb{B}^2$ becomes $I \times I$ and $\mathbb{S}^1$ the boundary of $I \times I$ (we leave the adaptation of the proof as an exercise to the student), H is nullhomotopic rel $(0, 0)$. But this is precisely what we are looking for (exercise). Finally, if moreover f and g are homotopic rel x_0, then $\omega = c_{y_0}$ and therefore $f_\# = g_\#$. $\square$

A consequence of this fact is the invariance of the fundamental group under homotopy:

Theorem 1.3.9. *Arcwise connected topological spaces which are homotopically equivalent have isomorphic fundamental groups.*

Proof. If $f : X \to Y$ and $g : Y \to X$ are such that $g \circ f \sim \mathrm{id}_X$ and $f \circ g \sim \mathrm{id}_Y$, from Proposition 1.3.8 one gets that $g_\# \circ f_\# = (g \circ f)_\#$ is a isomorphism (hence $f_\#$ is injective and $g_\#$ surjective) and the same for $f_\# \circ g_\# = (f \circ g)_\#$ (hence $g_\#$ is injective and $f_\#$ surjective). Therefore both $f_\#$ and $g_\#$ are isomorphisms. $\square$

Example. This happens in particular when $Y = A \subset X$ is a weak deformation retract of X: hence, for example, it holds $\pi_1(\mathbb{S}^n) \simeq \pi_1(\mathbb{R}^{n+1}_\times)$ and $\pi_1(\dot{\mathbb{B}}^{n+1}) \simeq \pi_1(\mathbb{B}^{n+1}) \simeq \pi_1(\mathbb{R}^{n+1})$.

On the other hand, for the retracts one has:

Proposition 1.3.10. *If $A \subset X$ is a retract of X, $i : A \to X$ is the canonical injection, $r : X \to A$ is a retraction and $x_0 \in A$, then $i_\# : \pi_1(A, x_0) \to \pi_1(X, x_0)$ is injective and $r_\# : \pi_1(X, x_0) \to \pi_1(A, x_0)$ is surjective.*

Proof. Since $r \circ i = \mathrm{id}_A$, one has $(r \circ i)_\# = r_\# \circ i_\# = \mathrm{id}_{\pi_1(A,x)}$: this says that $i_\#$ is injective and $r_\#$ is surjective. □

Corollary 1.3.11. *Any retract of a simply connected topological space is itself simply connected.*

The fundamental group of a topological product is the product of the fundamental groups:

Proposition 1.3.12. (Fundamental group of a topological product). *Let X and Y be arcwise connected topological spaces, $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ be the canonical projections, $(x_0, y_0) \in X \times Y$. Then the morphism $p_{1\#} \times p_{2\#}$ defines an isomorphism*

$$\pi_1(X \times Y, (x_0, y_0)) \simeq \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

Proof. Follows from the definition of the product topology on $X \times Y$. □

Corollary 1.3.13. *If Y is simply connected, then*

$$\pi_1(X \times Y, (x_0, y_0)) \simeq \pi_1(X, x_0);$$

in particular, if X and Y are simply connected so is $X \times Y$.

Exercise. (1) *Prove directly that $\mathbb{S}^n$ is simply connected for $n \geq 2$.* (2) *Deduce from that and from* $\pi_1(\mathbb{S}^1) \simeq \mathbb{Z}$ (which will be proven in §1.4) *that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for any $n \neq 2$.*

Solution. (1) [8, Proposition 1.1.14] Let γ be a loop in $\mathbb{S}^n$ at some basepoint x_0. If the image of γ is disjoint from some other point $x \in \mathbb{S}^n$ then γ is nullhomotopic since $\mathbb{S}^n \setminus \{x\}$ is homeomorphic to $\mathbb{R}^n$, which is simply connected. So it is enough to find a homotopy of γ with another non surjective loop. To do this we will look at a small open ball B in $\mathbb{S}^n$ about any point $x \neq x_0$ and see that the number of times that γ enters B, passes through x, and leaves B is finite, and each of these portions of γ can be pushed off x without changing the rest of γ. At first glance this might appear to be a difficult task to achieve since the parts of γ in B could be quite complicated geometrically, for example space-filling curves. But in fact it turns out to be rather easy. The set $\gamma^{-1}(B)$ is open in $]0,1[$, hence is the union of a possibly infinite collection of disjoint open intervals $]a_i, b_i[$. The compact set $\gamma^{-1}(x)$ is contained in the union of these intervals, so it must be contained in the union of finitely many of them. Consider one of the intervals $]a_i, b_i[$ meeting $\gamma^{-1}(x)$. The path γ_i obtained by restricting γ to the closed interval $[a_i, b_i]$ lies in the closure of B, and its endpoints $\gamma(a_i)$ and $\gamma(b_i)$ lie in the boundary of B. If $n \geq 2$, we can choose a path ψ_i from $\gamma(a_i)$ to $\gamma(b_i)$ in the closure of B but disjoint from x. For example, we could choose ψ_i to lie in the boundary of B, which is a sphere of dimension $n-1$, hence path-connected if $n \geq 2$. Since the closure of B is homeomorphic to a convex set in $\mathbb{R}^n$ and hence simply connected, the path γ_i is homotopic to ψ_i by Proposition 1.3.7, so we may homotope γ by deforming γ_i to ψ_i. After repeating this process for each of the intervals $]a_i, b_i[$ that meet $\gamma^{-1}(x)$, we obtain a loop ψ homotopic to the original γ and with $\psi(I)$ disjoint from x. (2) [8, Proposition 1.1.16] First note that $\mathbb{R}^n \setminus \{x\}$ is homeomorphic to $\mathbb{S}^{n-1} \times \mathbb{R}$ (or also that $\mathbb{S}^{n-1}$ is a strong deformation retract of $\mathbb{R}^n \setminus \{x\}$), hence $\pi_1(\mathbb{R}^n \setminus \{x\}) \simeq \pi_1(\mathbb{S}^{n-1})$, i.e. $\pi_1(\mathbb{R}^2 \setminus \{x\}) \simeq \mathbb{Z}$ and $\pi_1(\mathbb{R}^n \setminus \{x\})$ is trivial for $n \geq 3$. Now, any homeomorphism $\varphi : \mathbb{R}^2 \to \mathbb{R}^n$ would induce, for any point $x \in \mathbb{R}^2$, a homeomorphism $\mathbb{R}^2 \setminus \{x\} \to \mathbb{R}^n \setminus \{\varphi(x)\}$: but this is a contradiction both when $n = 1$ (namely $\mathbb{R}^2 \setminus \{x\}$ is arcwise connected while $\mathbb{R} \setminus \{\varphi(x)\}$ is not) and when $n \geq 3$ (homeomorphic spaces should have isomorphic fundamental groups).

1.4 An example: the circle $\mathbb{S}^1$ and the discrete quotients of topological groups

Let us compute the fundamental group of the circle $\mathbb{S}^1 \subset \mathbb{C}$: as we have seen, this group shall be isomorphic to the fundamental group of $\mathbb{C}_\times = \mathbb{C} \setminus \{0\}$.

Remark 1.4.1. We briefly recall some elementary properties of holomorphic functions of one complex variable. Let $z = x + iy \in \mathbb{C}$, $\overline{z} = x - iy$, and consider the corresponding differential operators $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$ (holomorphic derivative) and $\partial_{\overline{z}} = \frac{1}{2}(\partial_x + i\partial_y)$ (antiholomorphic derivative, or *Cauchy-Riemann operator*) and differential forms $dz = dx + idy$, $d\overline{z} = dx - idy$ and $dzd\overline{z} = -2idxdy$: if $U \subset \mathbb{C}$ is an open subset and $f = f_1 + if_2 : U \to \mathbb{C}$ a function with real and imaginary parts $f_1, f_2 : U \to \mathbb{R}$ of class $\mathcal{C}^1$, one has $df = (\partial_x f)dx + (\partial_y f)dy = (\partial_z f)dz + (\partial_{\overline{z}} f)d\overline{z}$. Given a path $\gamma = \gamma_1 + i\gamma_2 : I \to U$ piecewise of class $\mathcal{C}^1$ one can define the *integral of f along γ* as the complex number[19]

$$\begin{aligned}\int_\gamma f(z)dz \quad &:= \quad \int_0^1 f(\gamma(t))\gamma'(t)dt \\ &= \quad \int_0^1 \big(f_1(\gamma)\gamma_1' - f_2(\gamma)\gamma_2'\big)(t)dt + i\int_0^1 \big(f_1(\gamma)\gamma_2' + f_2(\gamma)\gamma_1'\big)(t)dt.\end{aligned}$$

The theorem of Stokes in the plane shows that, if γ is a loop in U boundary of an open subset $V_\gamma \subset U$ and $z_0 \in V_\gamma$, it holds

$$f(z_0) = \frac{1}{2\pi i}\left(\int_\gamma \frac{f(z)}{z - z_0}dz + \int_{V_\gamma} \frac{\partial_{\overline{z}} f(z)}{z - z_0}dzd\overline{z}\right). \tag{1.2}$$

The function f is called *holomorphic* in U if $\partial_{\overline{z}} f = 0$, i.e. if $\partial_x f_1 = \partial_y f_2$ and $\partial_y f_1 = -\partial_x f_2$ (*Cauchy-Riemann system*) at any point of U; denoting by $\mathcal{O}_{\mathbb{C}}(U)$ the complex vector space of holomorphic functions on U, this defines a sheaf $\mathcal{O}_{\mathbb{C}}$ of $\mathbb{C}$-vector spaces. The condition $f \in \mathcal{O}_{\mathbb{C}}(U)$ is equivalent both to the existence of the complex limit $f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ in any $z_0 \in U$ (note that, since $\partial_x = \partial_z + \partial_{\overline{z}}$, in this case one has $f'(z_0) = \partial_z f(z_0) = \partial_x f(z_0)$) and to the closure of real differential forms $f_1 dx - f_2 dy$ and $f_2 dx + f_1 dy$. Hence, for $f \in \mathcal{O}_{\mathbb{C}}(U)$ the integral $\int_\gamma f(z)dz$ is invariant under homotopy rel ∂I of the path γ (in particular, the integral of a holomorphic function on a nullhomotopic loop is zero); this shows that if U is a simply connected open subset then any $f \in \mathcal{O}_{\mathbb{C}}(U)$ admits a *primitive* $F = F_1 + iF_2 \in \mathcal{O}_{\mathbb{C}}(U)$ (i.e. $\partial_z F = f$), because this is equivalent to finding a primitive F_1 (resp. F_2) for the closed form $f_1 dx - f_2 dy$ (resp. $f_2 dx + f_1 dy$), and one then has $\int_\gamma f(z)dz = F(\gamma(1)) - F(\gamma(0))$. We also note that if $f \in \mathcal{O}_{\mathbb{C}}(U)$ the formula (1.2) riduces to the well-known *Cauchy integral formula*:

$$f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - z_0}dz. \tag{1.3}$$

Finally, observe that $f \in \mathcal{O}_{\mathbb{C}}(U)$ is locally invertibile (with holomorphic inverse) in z_0 if and only if $f'(z_0) \neq 0$: namely, denoted by $J_f(z_0)$ the jacobian determinant of f, due to Cauchy-Riemann system it holds $J_f(z_0) = |f'(z_0)|^2 > 0$ and, denoted by g the local inverse, one has $g'(f(z_0)) = \frac{1}{f'(z_0)}$.

Let $\gamma : I \to \mathbb{S}^1$ be a loop based at 1. When γ is piecewise of class $\mathcal{C}^1$, one defines the *index of γ in* 0 as

$$\mathrm{Ind}_\gamma(0) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z}dz = \frac{1}{2\pi i}\int_0^1 \frac{\gamma'(t)}{\gamma(t)}dt.$$

[19] One easily sees that $\int_\gamma f(z)dz$ does not depend on the chosen parametrization: if $p : I \to I$ is a change of parameter piecewise of class $\mathcal{C}^1$, one has $\int_0^1 f(\gamma(p(t)))\gamma'(p(t))p'(t)dt = \int_0^1 f(\gamma(\tau))\gamma'(\tau)d\tau$, with $\tau = p(t)$.

More generally, if γ is only continuous, note that in $[\gamma]$ there are anyway representatives which are piecewise of class $\mathcal{C}^1$.[20] It is well-known that the index has integer values and that it is invariant under homotopy rel ∂I,[21] and hence one gets a function

$$\mathrm{Ind}(0) : \pi_1(\mathbb{S}^1, 1) \to \mathbb{Z}.$$

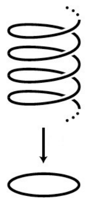
Proposition 1.4.2. Ind(0) *induces a group isomorphism*

$$\pi_1(\mathbb{S}^1, 1) \simeq \mathbb{Z}.$$

Proof. The function Ind(0) is a surjective homomorphism of groups: namely

$$\begin{aligned}\mathrm{Ind}(0)([\gamma_1 \cdot \gamma_2]) &= \frac{1}{2\pi i}\left(\int_0^{\frac{1}{2}} \frac{2\gamma_1'(2t)}{\gamma_1(2t)} dt + \int_{\frac{1}{2}}^1 \frac{2\gamma_2'(2t-1)}{\gamma_2(2t-1)} dt\right) \\ &= \frac{1}{2\pi i}\left(\int_0^1 \frac{\gamma_1'(\tau)}{\gamma_1(\tau)} d\tau + \int_0^1 \frac{\gamma_2'(\tau)}{\gamma_2(\tau)} d\tau\right) = \mathrm{Ind}(0)([\gamma_1]) + \mathrm{Ind}(0)([\gamma_2]),\end{aligned}$$

and for $n \in \mathbb{Z}$ it holds $\mathrm{Ind}(0)([e^{2\pi i n t}]) = n$. We are left with showing that Ind(0) is injective. Let γ be piecewise of class $\mathcal{C}^1$ and such that $\mathrm{Ind}_\gamma(0) = 0$, and let us show that γ is nullhomotopic. If we consider the exponential function $\epsilon : \mathbb{R} \to \mathbb{S}^1$, $\epsilon(t) = e^{2\pi i t}$ we note that there exists a unique loop $\psi : I \to \mathbb{R}$ based in 0 such that $\gamma = \epsilon \circ \psi$: namely, from $\psi' = \frac{\gamma'}{2\pi i \gamma}$ one gets $\psi(t) = \frac{1}{2\pi i} \int_0^t \frac{\gamma'(\tau)}{\gamma(\tau)} d\tau \ + c$ for some $c \in \mathbb{R}$, and the conditions $0 = \psi(0) = c$ and $0 = \psi(1) = \mathrm{Ind}_\gamma(0) + c = c$ determine univoquely $c = 0$.[22] At this point, since $\mathbb{R}$ is contractible and hence simply connected, just choose a homotopy h rel ∂I of ψ with the constant loop c_0 (for example, the affine homotopy $h(\tau, t) = (1-t)\psi(\tau)$): then, $\epsilon \circ h$ is a homotopy rel ∂I between γ and c_1. □



We have proved that $\pi_1(\mathbb{S}^1)$ is isomorphic to $\mathbb{Z}$, with a generator given by the class of the loop $I \ni t \mapsto e^{2\pi i t} \in \mathbb{S}^1$; in the meanwhile, we have met the exponential function $\epsilon : \mathbb{R} \to \mathbb{S}^1$, $\epsilon(t) = e^{2\pi i t}$ (represented in the picture, where $\mathbb{R}$ appears as a helix over $\mathbb{S}^1$), which is the first important example of a "covering space", a notion that we are going to study soon and which will be useful in the computation of fundamental groups.

As a first consequence of the fact that $\pi_1(\mathbb{S}^1) \simeq \mathbb{Z}$, let us provide a proof of the Fundamental Theorem of Algebra.

Corollary 1.4.3. *Any nonconstant polynomial with coefficients in* $\mathbb{C}$ *has a complex root.*

Proof. Suppose that $p(z) = z^n + a_1 z^{n-1} + \cdots + a_n \in \mathbb{C}[z]$ has no roots in $\mathbb{C}$. Then, for $r \geq 0$ one gets a family of loops in $\mathbb{S}^1$ based at 1 mutually homotopic by setting $\gamma_r(t) = \frac{p(re^{2\pi i t})/p(r)}{|p(re^{2\pi i t})/p(r)|}$. Since $\gamma_0 \equiv 1$, one has $[\gamma_r] = 0$ for any $r \geq 0$. Now let $r >> 1 + |a_1| + \cdots + |a_n|$: if $|z| = r$ one has $|z^n| = r^n = r \cdot r^{n-1} > (|a_1| + \cdots + |a_n|)|z^{n-1}| \geq |a_1 z^{n-1} + \cdots + a_n|$: hence, for any $\tau \in I$ the polynomial

[20] Just consider a partition $0 = t_0 < t_1 < \cdots < t_{m-1} < t_m = 1$ such that $\sup\{|\gamma(t) - \gamma(t_j)| : t \in [t_j, t_{j+1}]\} < 2$ for any $j = 0, \ldots, m-1$, construct the loop ϕ in $\mathbb{C}$ whose image is the poligonal joining the points $\gamma(0) = 1,\ \gamma(t_1), \ldots,\ \gamma(t_{m-1}),\ \gamma(1) = 1$ (note that 0 does not belong to the image of ϕ) and then consider the loop $\varphi = \phi/|\phi|$ in $\mathbb{S}^1$, which will be homotopic to γ (exercise: for any piece $[t_j, t_{j+1}]$ consider the affine homotopy in $\mathbb{C}$ between γ and φ...).

[21] Let $\varphi(t) = \exp\left(\int_0^t (\gamma'(\tau)/\gamma(\tau))\, d\tau\right)$: then $\varphi' = (\gamma'/\gamma)\varphi$, therefore $(\varphi/\gamma)' = (\varphi'\gamma - \varphi\gamma')/\gamma^2 = 0$, and so, since $\gamma(0) = \gamma(1)$, one has $1 = \varphi(0) = \varphi(1) = \exp(2\pi i\, \mathrm{Ind}_\gamma(0))$: in other words, $\mathrm{Ind}_\gamma(0) \in \mathbb{Z}$. As for the invariance under homotopy, observe that the function $\frac{1}{z}$ is holomorphic in $\mathbb{C}_\times$.

[22] In a future terminology, the loop ψ will be called a *lifting* of γ by the map ϵ; and we will see that the property of ϵ of lifting the paths in $\mathbb{S}^1$ with uniqueness will be a consequence of the fact that ϵ is a so-called "covering space".

$p_\tau(z) = z^n + \tau(a_1 z^{n-1} + \cdots + a_n)$ has no roots on $|z| = r$. It follows that, by replacing $p(z)$ with $p_\tau(z)$ in the definition of γ_r and letting τ run in I, if $r >> 1 + |a_1| + \cdots + |a_n|$ one gets a homotopy between $e^{2\pi int}$ and $\gamma_r(t)$: this implies that $n = [e^{2\pi int}] = [\gamma_r(t)] = 0$, in other words p is constant. □

Now we aim to generalize Proposition 1.4.2. Recall that a topological group G is a group endowed with a topology which makes continuous both the operation $(x, y) \mapsto xy$ and the inversion $x \mapsto x^{-1}$ (or, analogously, the "division" map $(x, y) \mapsto xy^{-1}$). Given a subgroup H of G denote by $p : G \to G/H$ the canonical projection, where the *homogeneous space* G/H is meant to be endowed with the quotient topology with respect to p: note that p is an open map.**(23)**

Let G be a simply connected topological group, $Z(G)$ its center, e the identity element, and let H be a discrete normal subgroup of G.

Remark 1.4.4. In the preceding hypotheses, it holds $H \subset Z(G)$. Namely, for $g \in H$ consider the (continuous) map $k_g : G \to G$, $x \mapsto xgx^{-1}g^{-1}$. Since H is normal in G one has $k_g(G) \subset H$; since G is connected, also $k_g(G)$ will be the same; but then, H being discrete, it holds $k_g(G) = \{e\}$. Hence $g \in Z(G)$.

We need the following result of general topology, due to Lebesgue. Recall that in a metric space (X, d) the *diameter* of a subset $A \subset X$ is $\operatorname{diam}(A) = \sup\{d(x, y) : x, y \in A\}$.

Lemma 1.4.5. (Lebesgue). *Let (X, d) be a compact metric space, $\mathcal{U}$ an open cover of X. Then there exists $\delta > 0$ such that for any $A \subset X$ with* $\operatorname{diam}(A) < \delta$ *there exists $U \in \mathcal{U}$ such that $A \subset U$.*

Let $L_{\mathcal{U}} \subset]0, +\infty[$ be the segment of δ which satisfy the conditions of Lemma. The supremum (which is also the maximum)**(24)** of $L_{\mathcal{U}}$ is called *Lebesgue number* of the open cover $\mathcal{U}$ in (X, d).

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X such that $B_n = \{x : d(x, x_n) \leq 1/n\}$ is not contained in any element of the cover, $x_0 \in X$ a closure point for the sequence, $U \in \mathcal{U}$ such that $x_0 \in U$ and $M \in \mathbb{N}$ such that $B = \{x : d(x, x_0) \leq 1/M\} \subset U$. If x_0 would not be an accumulation point (i.e., if x_0 would be an isolated point of $\{x_n : n \in \mathbb{N}\}$) there would exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ which would be definitely *equal* to x_0, i.e. such that there would exists $k_0 \in \mathbb{N}$ such that $x_{n_k} \equiv x_0$ for any $k \geq k_0$: but then, for $k \geq \max\{M, k_0\}$ one would have $B_{n_k} \subset B \subset U$, which is absurd. Assume then that x_0 is an accumulation point for the sequence, and let $N \in \mathbb{N}$ such that $d(x_N, x_0) \leq 1/2M$; since such N can be chosen to be arbitrarily large, we may assume that $N > 2M$. But then, once more we get $B_N \subset B \subset U$, absurd. □

Lemma 1.4.6. *Let $K \subset \mathbb{R}^n$ be a compact subset star-shaped with respect to* 0, *$f : K \to G/H$ a continuous function, $g_0 \in G$ such that $p(g_0) = f(0)$. Then there exists a unique continuous function* (lifting) *$\tilde{f} : K \to G$ such that $\tilde{f}(0) = g_0$ and $f = p \circ \tilde{f}$ in $\mathfrak{Top}_p$.*

Proof. We may assume that $f(0) = eH$ and $g_0 = e$.**(25)** Let W be a neighborhood of e in G such that $W \cap H = \{e\}$, and let U be an open and symmetric (i.e. $U^{-1} = U$) neighborhood of e such that

(23) By the Proposition 1.1.11, a quotient function $p : X \to Y$ is open if and only if for any subset open $A \subset X$ the "p-saturated" $p^{-1}p(A) \subset X$ is open in X. Now, the p-saturated of an open subset $A \subset G$ is $AH = \bigcup_{h\in H} Ah$, which is indeed open as it is the union of right translates of A (in a topological group the right and left traslations by a fixed element are autohomeomorphisms).

(24) If $\overline{\delta}$ is such supremum and $A \subset X$ has $\operatorname{diam}(A) < \overline{\delta}$, by definition of supremum there exists a $\delta \in L_{\mathcal{U}}$ such that $\operatorname{diam}(A) < \delta \leq \overline{\delta}$: hence there exists $U \in \mathcal{U}$ such that $A \subset U$, and this proves that $\overline{\delta} \in L_{\mathcal{U}}$.

(25) Namely, let $\phi = f \cdot (g_0^{-1}H) : K \to G/H$ (remember that G/H is a group since H is normal in G): then $\phi(0) = eH$. If $\tilde{\phi} : K \to G$ is the unique continuous function such that $p \circ \tilde{\phi} = \phi$ and $\tilde{\phi}(0) = e$, then $\tilde{f} = \tilde{\phi} \cdot g_0$ solves the problem, because $\tilde{f}(0) = g_0$ and $(p \circ \tilde{f})(x) = p(\tilde{\phi}(x) \cdot g_0) = p(\tilde{\phi}(x)) \cdot p(g_0) = \phi(x) \cdot g_0 H = f(x)$.

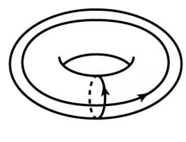
$U \cdot U \subset W$.[26] Then, setting $V = p(U)$, one has that $p|_U : U \to V$ is a homeomorphism,[27] and let $q : V \to U$ be the inverse map. Let U_0 be another open symmetric neighborhood of e such that $U_0 \cdot U_0 \subset U$. Since $\{p(gU_0) : g \in G\}$ form an open cover of G/H, $\{f^{-1}(p(gU_0)) : g \in G\}$ form an open cover of the compact K. Let $\delta > 0$ be the Lebesgue number of such cover (see Lemma 1.4.5), and let $N \in \mathbb{N}$ be such that $\mathrm{diam}(K)/N \leq \delta$. For $x \in K$, the segment $\{tx : t \in I\}$ is contained in K: let us write $f(x) = \prod_{j=1}^{N} f(\frac{j-1}{N}x)^{-1} f(\frac{j}{N}x)$ (recall that $f(0) = eH$). Since $|\frac{j}{N}x - \frac{j-1}{N}x| = \frac{|x|}{N} \leq \delta$ (where $j = 1, \dots, N$), the elements $f(\frac{j-1}{N}x)$ and $f(\frac{j}{N}x)$ belong to the same $p(gU_0)$: hence it holds $f(\frac{j-1}{N}x)^{-1} f(\frac{j}{N}x) \in p(gU_0)^{-1} p(gU_0) = p((gU_0)^{-1}(gU_0)) = p(U_0 U_0) \subset p(U)$. A function which satisfies the required hypotheses will then be $\tilde{f}(x) = \prod_{j=1}^{N} q\left(f(\frac{j-1}{N}x)^{-1} f(\frac{j}{N}x)\right)$. Finally let $\tilde{f}_1 : K \to G$ be another function with the same properties of $\tilde{f}$: if we denote for example by $\tilde{f}^{-1}$ the composition of $\tilde{f}$ with the inversion of G, it holds $p \circ (\tilde{f}\tilde{f}_1^{-1}) = (p \circ \tilde{f})(p \circ \tilde{f}_1)^{-1} = eH$, therefore $\tilde{f}\tilde{f}_1^{-1}(K) \subset \ker(p) = H$. But, K being connected, H discrete, $\tilde{f}\tilde{f}_1^{-1}$ continuous and $\tilde{f}\tilde{f}_1^{-1}(0) = e$, it holds $\tilde{f}\tilde{f}_1^{-1}(K) = \{e\}$: i.e., $\tilde{f}_1 = \tilde{f}$. $\square$

Proposition 1.4.7. $\pi_1(\frac{G}{H}, eH) \simeq H$.

Proof. Thanks to Lemma 1.4.6 (with $K = I \subset \mathbb{R}$) the loops $\gamma : I \to G/H$ based at eH lift uniquely to $\tilde{\gamma} : I \to G$ such that $\tilde{\gamma}(0) = e$. Since $\gamma = p \circ \tilde{\gamma}$, it holds $p(\tilde{\gamma}(1)) = eH$, and hence $\tilde{\gamma}(1) \in H$. We then define $\psi : \pi_1(\frac{G}{H}, eH) \to H$ by setting $\psi([\gamma]) = \tilde{\gamma}(1)$. For the well-posedness of ψ we observe, again by the Lemma 1.4.6 (with $K = I \times I \subset \mathbb{R}^2$), that the homotopies rel ∂I of loops in G/H based at eH lift uniquely to $\tilde{h} : I \times I \to G$ such that $\tilde{h}(0,0) = e$; now, since $h = p \circ \tilde{h}$, it holds $\tilde{h}(\{(0,\tau) : \tau \in I\}) \subset H$ and $\tilde{h}(\{(1,\tau) : \tau \in I\}) \subset H$. Since H is discrete and $\tilde{h}$ is continuous, we get $\tilde{h}(\{(0,\tau) : \tau \in I\}) = \{e\}$ (recall that $\tilde{h}(0,0) = e$) and $\tilde{h}(\{(1,\tau) : \tau \in I\}) = \{g\}$ for some $g \in H$: in other words, also $\tilde{h}$ is a homotopy rel ∂I of paths (no longer loops) in G from e to $g \in H$. Therefore, if $[\gamma_1] = [\gamma_2]$ we shall also have $\tilde{\gamma}_1(1) = \tilde{h}(1,0) = \tilde{h}(1,1) = \tilde{\gamma}_2(1)$.[28] Now let us show that ψ is a homomorphism. Let $[\gamma_1], [\gamma_2] \in \pi_1(\frac{G}{H}, eH)$: using Lemma 1.4.6 one gets $\widetilde{\gamma_1 \cdot \gamma_2} = \tilde{\gamma}_1 \cdot (\tilde{\gamma}_2)_g$, where $g = \tilde{\gamma}_1(1) \in H$ and $(\tilde{\gamma}_2)_g : I \to G$ is such that $\gamma_2 = p \circ (\tilde{\gamma}_2)_g$ and $(\tilde{\gamma}_2)_g(0) = g$. On the other hand, again by Lemma 1.4.6 one has $(\tilde{\gamma}_2)_g = g\tilde{\gamma}_2$, and hence $\psi([\gamma_1] \cdot [\gamma_2]) = \widetilde{\gamma_1 \cdot \gamma_2}(1) = g\tilde{\gamma}_2(1) = \psi([\gamma_1])\psi([\gamma_2])$. The surjectivity comes from the fact that G is arcwise connected: if $g \in H$, there exists a path $\alpha : I \to G$ with $\alpha(0) = e$ and $\alpha(1) = g$: then, since clearly $\widetilde{p \circ \alpha} = \alpha$, it holds $g = \psi([p \circ \alpha])$. As for the injectivity, note that $\ker(\psi) = \{[\gamma] : \tilde{\gamma} \text{ is a loop}\}$: since G is simply connected, there exists a homotopy $\tilde{h}$ rel ∂I from $\tilde{\gamma}$ to c_e, which gives a homotopy $h = p \circ \tilde{h}$ rel ∂I from γ to c_{eH}. $\square$

Examples. **(1)** Let $\mathbb{H} \simeq \mathbb{R} \oplus \mathbb{R}^3$ be the (noncommutative) fields of quaternions: recall that $(\lambda, u) + (\mu, v) = (\lambda + \mu, u + v)$, $(\lambda, u)(\mu, v) = (\lambda\mu - u \cdot v, \lambda v + \mu u + u \wedge v)$, where $\cdot$ (resp. $\wedge$) denotes the scalar (resp. vectorial) product; the conjugated of $q = (\lambda, u)$ is $\overline{q} = (\lambda, -u)$, the norm of q is $|q| = (q\overline{q})^{\frac{1}{2}} = (\lambda^2 + |u|^2)^{\frac{1}{2}}$, the inverse of q is $q^{-1} = \overline{q}/|q|^2$. Let $G = \mathbb{S}^3$ (the multiplicative group of quaternions of norm 1, which is simply connected as we shall see by the Theorem of Van Kampen), and $H = \{\pm 1\}$ (discrete normal subgroup): one has $G/H = \mathbb{P}^3(\mathbb{R})$ (real projective space of dimension 3) and hence $\pi_1(\mathbb{P}^3(\mathbb{R}); [1]) \simeq \mathbb{Z}/2\mathbb{Z}$. We shall see that this hold more generally for $\mathbb{P}^n(\mathbb{R})$ with $n \geq 2$. **(2)** Let $\mathbb{T}^n = (\mathbb{S}^1)^n$ the n-dimensional torus. From Propositions 1.3.12 and 1.4.2 one gets immediately $\pi_1(\mathbb{T}^n; x_0) \simeq \mathbb{Z}^n$ (where $x_0 = (1, \dots, 1) \in (\mathbb{S}^1)^n$): but this follows also from Proposition 1.4.7, noting that $\mathbb{T}^n \simeq G/H$ with $G = \mathbb{R}^n$ and $H = \mathbb{Z}^n$. Some generators (mutually commuting) of $\pi_1(\mathbb{T}^n; x_0)$ are the classes of loops $\gamma_i : I \to \mathbb{T}^n$ $(1 \leq i \leq n)$ whose ith component is $e^{2\pi i t}$ and whose others are the constant 1. In order to verify that they commute, let us examine the case $n = 2$ (the generalization will then be natural): the torus $\mathbb{T}^2$ can be seen as a filled square where one identifies the opposed sides pairwise, and hence the four vertices become a single point:

$$\mathbb{T}^2 = \begin{tikzcd} x_0 \arrow[r, "\gamma_1"] & x_0 \\ x_0 \arrow[u, "\gamma_2"] \arrow[r, "\gamma_1"'] & x_0 \arrow[u, "\gamma_2"'] \end{tikzcd} \quad \rightsquigarrow$$



[26] In a topological group G the symmetric neighborhoods form a basis of neighborhoods of e: namely if U is a neighborhood of e, such is also U^{-1} (the inversion is an autohomeomorphism), and hence also $U \cap U^{-1}$, which is symmetric.

[27] $p|_U$ is open (as a restriction of an open map to an open subset), continuous and surjective; if then $p(u_1) = p(u_2)$ one has $u_1 u_2^{-1} \in H$, but also $u_1 u_2^{-1} \in U \cdot U^{-1} = U \cdot U \subset W$, hence $u_1 u_2^{-1} \in W \cap H = \{e\}$: i.e., $u_1 = u_2$.

[28] We shall meet this fact also later (Monodromy lemma).

(if one visualizes the square as a sheet, this amounts to gluing the sides γ_1, then the sides —which in the meanwhile have become circles— γ_2 by overlapping the points x_0: in this way one obtains the usual picture of the torus as a doughnut in $\mathbb{R}^3$). The generators of $\pi_1(\mathbb{T}^2, x_0)$ are γ_1 and γ_2, and it is quite clear what should be a homotopy rel ∂I, passing through a diagonal of the square, of the loop $\gamma_1 \cdot \gamma_2$ into the loop $\gamma_2 \cdot \gamma_1$. (As an exercise, try to translate that homotopy on the doughnut.)