A.3 Multilinear algebra

Given a $\mathbb{R}$-vector space V, let us consider the dual space $V^* = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$, and costruct the tensor algebra $\otimes^\bullet V^*$ on V^* as follows. Set $\otimes^0 V^* = \mathbb{R}$ and, for $m \in \mathbb{N}$, consider

$$\otimes^m V^* = V^* \otimes_{\mathbb{R}} V^* \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} V^*$$

(with m factors): then $\otimes^m V^*$ is naturally identified with the space of *multilinear* (i.e., linear in all variables) *functions* of V^m with real values, or *m-forms* on V. Given a basis $\{\phi_i : i \in J\}$ of V^*, then $\{\phi_{i_1} \otimes \cdots \otimes \phi_{i_m} : (i_1, \dots, i_m) \in J^m\}$, where $(\phi_{i_1} \otimes \cdots \otimes \phi_{i_m})(v_1, \dots, v_m) = \phi_{i_1}(v_1) \cdots \phi_{i_m}(v_m)$, will be a basis of $\otimes^m V^*$. In particular, if $\dim V = n$ one has $\dim \otimes^m V^* = n^m$. The tensor product

$$\begin{aligned} \otimes : \quad & \otimes^m V^* \times \otimes^p V^* \longrightarrow \otimes^{m+p} V^*, \\ & (\alpha \otimes \beta)(v_, \dots, v_m, v_{m+1}, \dots, v_{m+p}) = \alpha(v, \dots, v_m)\beta(v_{m+1}, \dots, v_{m+p}), \end{aligned}$$

is associative and distributive with respect to the addition but not commutative, and gives to the tensor algebra $\otimes^\bullet V^* = \oplus_{m=0}^{+\infty}(\otimes^m V^*)$ the structure of graded $\mathbb{R}$-algebra[119] (the algebra of *forms* on V). To a morphism $f : V \to W$ is associated the morphism (*pull-back*) of algebras $f^* : \otimes^\bullet W^* \to \otimes^\bullet V^*$: given $\alpha \in \otimes^m W^*$, one defines $f^*\alpha \in \otimes^m V^*$ by $(f^*\alpha)(v_1, \dots, v_m) = \alpha(f(v_1), \dots, f(v_m))$; if $g : W \to W'$ is another morphism, it holds $(g \circ f)^* = f^* \circ g^*$. We then get a contravariant functor from $\mathfrak{Mod}(\mathbb{R})$ (the category of $\mathbb{R}$-vector spaces) to $\mathfrak{Alg}_{\deg}(\mathbb{R})$ (the category of graded $\mathbb{R}$-algebras).

We denote by $\wedge^m V^*$ the subspace of $\otimes^m V^*$ formed by the *alternating m-forms*, i.e. those forms which are sensitive to permutations of the arguments: it is the exterior algebra on V^*. Given $\alpha \in \otimes^m V^*$ and $\sigma \in \mathfrak{S}_m$, let $\alpha^\sigma(v_1, \dots, v_m) = \alpha(v_{\sigma(1)}, \dots, v_{\sigma(m)})$ (note that $(\alpha^\sigma)^\tau = \alpha^{(\sigma\circ\tau)}$): then $\alpha \in \wedge^m V^*$ if and only if $\alpha^\sigma = \mathrm{sign}(\sigma)\alpha$ for any $\sigma \in \mathfrak{S}_m$. This suggests the surjective linear application

$$\mathrm{Alt} : \otimes^m V^* \to \wedge^m V^*, \qquad \mathrm{Alt}(\alpha) = \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} \mathrm{sign}(\sigma)\alpha^\sigma$$

which expresses $\wedge^m V^*$ also as quotient of $\otimes^m V^*$. (Note that $\mathrm{Alt}(\alpha) = \alpha$ for any $\alpha \in \wedge^m V^*$.) The tensor product induces an external product

$$\wedge : \wedge^m V^* \times \wedge^p V^* \to \wedge^{m+p} V^*, \qquad \alpha \wedge \beta = \mathrm{Alt}(\alpha \otimes \beta)$$

which is easily verified to be associative (since $(\alpha\wedge\beta)\wedge\gamma = \mathrm{Alt}(\alpha\otimes\beta\otimes\gamma)$) and distributive with respect to addition. Given $I = (i_1, \dots, i_m) \in J^m$, thanks to the associativity of $\wedge$ it makes sense to set

$$\phi_I = \phi_{i_1} \wedge \cdots \wedge \phi_{i_m} := \mathrm{Alt}(\phi_{i_1} \otimes \cdots \otimes \phi_{i_m}).$$

Therefore, since Alt is surjective, the elements $\{\phi_I : I \in J^m\}$ generate $\wedge^m V^*$. On the other hand, if $\alpha, \beta \in V^*$ then one has the anticommutativity $\alpha \wedge \beta = \frac{1}{2}(\alpha\otimes\beta - \beta\otimes\alpha) = -\beta\wedge\alpha$.

[119] A $\mathbb{R}$-algebra $\mathcal{A}$ is *graded* if is a graded $\mathbb{R}$-vector space $\bigoplus_{n=0}^{+\infty} \mathcal{A}_n$ such that $\mathcal{A}_m\mathcal{A}_n \subset \mathcal{A}_{m+n}$.

Assuming to have a total order $\leq$ on J, this shows that (1) the elements $\{\phi_I : I = (i_1, \dots, i_m) \in J^m,\ i_1 < \cdots < i_m\}$ generate $\wedge^m V^*$, and since they are linearly indipendent (exercise), they form a basis for $\wedge^m V^*$; (2) one has also the general anticommutativity

$$\beta \wedge \alpha = (-1)^{mp} \alpha \wedge \beta, \qquad \alpha \in \wedge^m V^*,\ \beta \in \wedge^p V^*,$$

which gives also to the exterior algebra $\wedge^\bullet V^* = \oplus_{m=0}^{+\infty} (\wedge^m V^*)$ the structure of graded $\mathbb{R}$-algebra on V. Note that, if $\dim V = n$, one has $\wedge^m V^* = 0$ for $m > n$, $\dim \wedge^m V^* = \binom{n}{m}$ and hence $\dim \wedge^\bullet V^* = \sum_{m=0}^{n} \binom{n}{m} = 2^n$.

Example. In the case of $V = \mathbb{R}^n$, consider the canonical basis $\{e_1, \dots, e_n\}$ and let $\{u_1, \dots, u_n\}$ be the dual basis of $(\mathbb{R}^n)^*$ (the $u_i : \mathbb{R}^n \to \mathbb{R}$, given by $u_i(e_j) = \delta_{i,j}$, are the coordinate functions). An alternating m-form $\alpha \in \wedge^m (\mathbb{R}^n)^*$ can be uniquely written as $\alpha = \sum_{|I|=m} a_I u_I$, where $u_I = u_{i_1} \wedge \cdots \wedge u_{i_m}$.

If $f : V \to W$ is a morphism, the pull-back $f^* : \otimes^\bullet W^* \to \otimes^\bullet V^*$ induces $f^* : \wedge^\bullet W^* \to \wedge^\bullet V^*$, and one has another contravariant functor from $\mathfrak{Mod}(\mathbb{R})$ to $\mathfrak{Alg}_{\text{deg}}(\mathbb{R})$. In particular, let $\dim V = n$, $f \in \text{End}(V)$ and consider $f^* : \wedge^n V^* \to \wedge^n V^*$: since this is a linear map between spaces of dimension 1, it must be indeed the multiplication by a constant, which is precisely $\det(f)$. In particular, one obtains

$$f^*\phi_1 \wedge \cdots \wedge f^*\phi_n = \det(f)\, \phi_1 \wedge \cdots \wedge \phi_n. \tag{A.7}$$

There is a similar costruction for $\odot^m V^*$, the subspace of $\otimes^m V^*$ formed by the *symmetric forms*, i.e. those forms which keep the sign unchanged for any permutation of variables (the symmetric algebra on V^*). Given $\alpha \in \otimes^m V^*$, one has $\alpha \in \odot^m V^*$ if and only if $\alpha^\sigma = \alpha$ for any $\sigma \in \mathfrak{S}_m$, and this gives the surjective linear application

$$\text{Sym} : \otimes^m V^* \to \odot^m V^*, \qquad \text{Sym}(\alpha) = \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} \alpha^\sigma$$

which expresses $\odot^m V^*$ also as quotient of $\otimes^m V^*$. The tensor product induces a symmetric product

$$\odot : \odot^m V^* \times \odot^p V^* \to \odot^{m+p} V^*, \qquad \alpha \odot \beta = \text{Sym}(\alpha \otimes \beta)$$

which is associative, distributive with respect to the addition and commutative and gives also to $\odot^\bullet V^* = \oplus_{m=0}^{+\infty} (\odot^m V^*)$ the structure of graded $\mathbb{R}$-algebra. A basis for $\odot^m V^*$ is given by $\{\phi_{i_1} \odot \cdots \odot \phi_{i_m} : i_1 \leq \cdots \leq i_m\}$; in particular, if $\dim V = n$ one has $\dim \odot^m V^* = \binom{n+m-1}{m}$.

A.4 Sheaves

We present some sketch of sheaf theory, referring for example to Kashiwara-Schapira [10] for a more complete exposition.

Presheaves and sheaves. Given a topological space X, consider the category $\mathcal{T}_X$ defined by the open subsets of X ordered by inclusion (i.e., $\mathrm{Hom}_{\mathcal{T}_X}(U, V) = \{\mathrm{pt}\}$ if $U \subset V$ and $\varnothing$ otherwise). Fix a unitary ring A and the category $\mathfrak{Mod}(A)$ of left A-modules. A *presheaf* of A-modules on X is a contravariant functor $F : \mathcal{T}_X \to \mathfrak{Mod}(A)$. In other words, to any open subset $U \subset X$ is associated a A-module $F(U)$ (denoted also by $\Gamma(U; F)$), and whenever $U \subset V$ there is a "restriction" morphism $\rho_{VU} : F(V) \to F(U)$ —often denoted by $\cdot|_U$, without mentioning V— such that $\rho_{UU} = \mathrm{id}_{F(U)}$ and $\rho_{VU} \circ \rho_{WV} = \rho_{WU}$ if $U \subset V \subset W$. A $s \in F(U)$ is called a *section* on U of the presheaf F. One defines the zero presheaf and the direct sum of presheaves in a natural way. A *morphism of presheaves* $\varphi : F \to G$ is a family $\varphi_U \in \mathrm{Hom}_A(F(U), G(U))$ compatible with the restrictions, i.e. such that $\rho_{VU,G} \circ \varphi_V = \varphi_U \circ \rho_{VU,F}$ (where $U \subset V$). A morphism $\varphi : F \to G$ defines in a natural way other presheaves on X by means if the associations $U \mapsto (P\ker(\varphi))(U) = \ker(\varphi_U)$ (*kernel* presheaf of φ), and similarly for the presheaves *cokernel* $P\,\mathrm{coker}(\varphi)$*, image* $P\,\mathrm{im}(\varphi)$ *and coimage* $P\,\mathrm{coim}(\varphi)$ of φ. We shall denote by with $P\mathfrak{Mod}(A_X)$ the category of presheaves (of left A-modules) on X; hence for any open subset $U \subset X$, one has a functor $\Gamma(U, \cdot\,) : P\mathfrak{Mod}(A_X) \to \mathfrak{Mod}(A)$.

If $U \subset X$ is open, the *restriction* $F|_U$ of a presheaf F to U is the presheaf on U given by $\Gamma(V; F|_U) = \Gamma(V; F)$ for any open $V \subset U$. The *fiber* of a presheaf F in $x \in X$ is the A-module $F_x = \varinjlim_{U \ni x} F(U) = \bigsqcup_{U \ni x} F(U)/\sim$, where $(s \in F(U)) \sim (t \in F(V))$ if s and t coincide in some neighborhood of x, i.e. there exists $W \subset U \cap V$ such that $s|_W = t|_W$. Therefore, if $x \in U$ there is a morphism $F(U) \to F_x$ in $\mathfrak{Mod}(A)$ sending a section s into its "germ" $[s]_x$. (In particular, note that $[s]_x = 0$ if and only if $s|_W = 0$ for some $W \subset U$.) If $\varphi : F \to G$ is a morphism of presheaves and $x \in X$, one defines a morphism of A-modules $\varphi_x : F_x \to G_x$ by setting, for $s \in \Gamma(U, F)$ with $U \ni x$, $\varphi_x([s]_x) = [\varphi_U(s)]_x$ (exercise). In this way one obtains a functor $(\,\cdot\,)_x : P\mathfrak{Mod}(A_X) \to \mathfrak{Mod}(A)$ which commutes with kernel and cokernel: for example, $(P\ker(\varphi))_x = \ker(\varphi_x)$.

A *sheaf* on X is a presheaf F such that, if U is open and $U = \bigcup_{i \in I} U_i$ is any open cover of U, the following local conditions hold:

(F1) (*Local vanishing*) Any section $s \in F(U)$ such that $s|_{U_i} = 0$ for any $i \in I$ is itself zero.

(F2) (*Gluing*) Given a family of sections $s_i \in F(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for any $i, j \in I$, there exists a section $s \in F(U)$ such that $s|_{U_i} = s_i$ for any $i \in I$.

One verifies that (F1)+(F2) is equivalent to the fact that for any open cover $U = \bigcup_{i \in I} U_i$ stable by finite intersections, the natural morphism $F(U) \to \varinjlim_{i \in I} F(U_i)$ is an isomorphism.

The *support* $\mathrm{supp}(F)$ of a sheaf F is the closed subset complementary of the largest open $U \subset X$ such that $F|_U = 0$. Given a section $s \in F(U)$, the support of s is the complementary (in U) of the largest open $V \subset U$ such that $s|_V = 0$. We shall denote by $\Gamma_c(U;F)$ the submodule of $\Gamma(U;F)$ of sections with compact support.

Let F and G be sheaves on X. A *morphism* $\varphi : F \to G$ is a morphism as presheaves. One proves that (exercise) $\varphi : F \to G$ is a monomorphism (resp. isomorphism) if and only if $\varphi_x : F_x \to G_x$ is a monomorphism (resp. isomorphism) for any $x \in X$.

Let $\mathfrak{Mod}(A_X)$ be the category of sheaves (of left A-modules) on X. Since not all presheaves are sheaves, $\mathfrak{Mod}(A_X)$ embeds as a full subcategory into $P\mathfrak{Mod}(A_X)$. Actually, to any presheaf F one can canonically associate a sheaf F^+, whose fibers coincide with those of F (and which, obviously, is isomorphic to F if F is already a sheaf) and a morphism of presheaves $\theta : F \to F^+$ such that any morphism of presheaves $\varphi : F \to G$, where G is a sheaf, can be uniquely factorized through θ, i.e. there exists a unique morphism of sheaves $\varphi^+ : F^+ \to G$ such that $\varphi = \varphi^+ \circ \theta$. The pair (F^+, θ) is unique up to isomorphisms. Moreover, for any $x \in X$, $\theta_x : F_x \to F^+_x$ is a isomorphism. The sheaf F^+ is costructed by considering, as sections on an open subset $U \subset X$, the functions of U with values in $\bigsqcup_{x\in U} F_x$ locally induced as germs of a single section t:

$$\begin{aligned} F^+(U) \;\; = \;\; \{s : U \to \bigsqcup_{x\in U} F_x : s(x) \in F_x, \text{ for any } x \in U \text{ exists an open } x \in V \subset U \\ \text{and a section } t \in F(V) \text{ with } [t]_y = s(y) \text{ for any } y \in V \ \}; \end{aligned}$$

the morphism θ is defined by setting $s \in F(U) \mapsto (x \mapsto [s]_x) \in F^+(U)$ (complete the verifications as an exercise).

Remark A.4.1. Passing to the category of sheaves fixes some strange situations: for example, it could happen (see the Examples below) that a presheaf F is nonzero even if $F_x = 0$ for any $x \in X$. In such a case, the associated sheaf is zero (namely in a sheaf, but not in a presheaf, to be zero is a local matter).

Let $\varphi : F \to G$ be a morphism of sheaves: it is easy to see that the presheaf $P\ker(\varphi)$ is a sheaf, that we denote by $\ker(\varphi)$. On the other hand, in general the presheaf $P\,\mathrm{coker}(\varphi)$ is not a sheaf, hence we shall set $\mathrm{coker}(\varphi) = (P\,\mathrm{coker}(\varphi))^+$. In any case it holds $(\ker(\varphi))_x = \ker(\varphi_x)$ and $(\mathrm{coker}(\varphi))_x = \mathrm{coker}(\varphi_x)$. One shows that $\mathfrak{Mod}(A_X)$ is an abelian category.

Examples. **(0)** One has $P\mathfrak{Mod}(A_{\{\mathrm{pt}\}}) = \mathfrak{Mod}(A_{\{\mathrm{pt}\}}) = \mathfrak{Mod}(A)$. **(1)** Let $M \in \mathfrak{Mod}(A)$: the *constant presheaf* with fiber M is defined by setting always $U \mapsto M$. In general this is not a a sheaf (namely (F1) is verified, but not (F2): if U_1 and U_2 are disjoint open subset of X, consider $s_j = m_j \in F(U_j)$ with $m_1 \neq m_2$). The associated sheaf, called *constant sheaf* of fiber M, is denoted by M_X and its sections on the open subset $U \subset X$ are the locally constant functions of U with values in M: note that $(M_X)_x = M$ for any $x \in X$. **(2)** More generally, it is very important to consider the *locally constant sheaves* of fiber M, i.e. the sheaves $F \in \mathfrak{Mod}(A_X)$ for which there exists an open cover $\{U_i : i \in I\}$ of X such that $F|_{U_i}$ is a constant sheaf on U_i. For example, let $\pi : Y \to X$ be a real vector bundle on a topological space X: the presheaf F_π on X given by $\Gamma(U; F_\pi) = \{s : U \to Y : \pi \circ s = \mathrm{id}_U\}$ (the sections of π on U) is a locally constant sheaf of $\mathbb{R}$-vector spaces. **(3)** The most direct examples of (pre)sheaves are provided by the functional spaces: the presheaf $U \mapsto \mathcal{C}^0_X(U)$ (real, or complex, continuous functions) is a sheaf of $\mathbb{C}$- or $\mathbb{C}$-vector spaces, denoted by $\mathcal{C}^0_X$; if X is a real analytic manifold one has the sheaves $\mathcal{A}_X$, $\mathcal{C}^\infty_X$, $\mathcal{D}b_X$ and $\mathcal{B}_X$ (analytic functions, smooth functions, distributions and hyperfunctions) or, more generally, the sheaves $\Omega^p(\mathcal{A}_X)$ of differential p-forms with coefficients in $\mathcal{A}_X$ etc.; if X is a complex analytic manifold, one has the sheaf $\mathcal{O}_X$ of holomorphic functions or, more generally, the sheaf $\Omega^p(\mathcal{O}_X)$ of holomorphic p-forms. **(4)** If X is a topological space endowed with a measure, the presheaf $U \mapsto L^\infty(U)$ satisfies (F1) but not (F2) (boundedness is not a local property), the same for the presheaf $U \mapsto L^1(U)$ (the associated sheaf is L^1_{loc}). **(5)** Consider the morphism of sheaves $\frac{\partial}{\partial z} : \mathcal{O}_\mathbb{C} \to \mathcal{O}_\mathbb{C}$, and the presheaf $F = P\,\mathrm{coker}(\partial/\partial z)$, i.e. $F(U) = \mathcal{O}_\mathbb{C}(U)/\frac{\partial}{\partial z}\mathcal{O}_\mathbb{C}(U)$: then F has fiber zero, because if U is an open disc the equation $\frac{\partial}{\partial z} f = g$ is solvable in U (any holomorphic function on a simply connected set admits a primitive there), but $F \neq 0$ since $F(\mathbb{C} \setminus \{0\}) \simeq \mathbb{C}$ (namely, $\frac{1}{z}$ is holomorphic on $\mathbb{C} \setminus \{0\}$ but it has no primitives defined on all of $\mathbb{C} \setminus \{0\}$: the branches of the complex logarithm are defined on angular wedges strictly smaller than 2π). Hence F is not a sheaf because (F1) is not valid, and the associated sheaf $(\mathrm{coker}(\partial/\partial z))$ is zero.

What we said for the complexes of A-modules and their cohomology extends naturally to the categories of presheaves and of sheaves, but paying attention to the fact that images and cokernels of morphisms in $P\mathfrak{Mod}(A_X)$ are not the same than in $\mathfrak{Mod}(A_X)$. In particular, a complex of sheaves which is an exact sequence in $P\mathfrak{Mod}(A_X)$ (i.e., on any open subset) is the same also in $\mathfrak{Mod}(A_X)$ (i.e., on any fiber), but the converse is not true: namely, the functor $\Gamma(U, \cdot\,)$ is obviously exact on $P\mathfrak{Mod}(A_X)$, but only left exact on $\mathfrak{Mod}(A_X)$ (exercise). An example is the one provided above, i.e. $\mathcal{O}_\mathbb{C} \xrightarrow{\partial/\partial z} \mathcal{O}_\mathbb{C} \to 0$: this is an exact sequence in $\mathfrak{Mod}(A_X)$ (namely $\mathrm{coker}(\partial/\partial z) = 0$) but, by applying $\Gamma(\mathbb{C} \setminus \{0\}, \cdot\,)$, as we saw one obtains a complex which is not exact in $\mathfrak{Mod}(A)$.

Operations. Let $f : X \to Y$ be a continuous function. If F is a sheaf on X, the presheaf f_*F on Y defined by $f_*F(V) = F(f^{-1}(V))$ is a sheaf (exercise), called the *direct image* of F; it has has a subsheaf $f_!F$ (the *proper direct image* of F) defined by $f_!F(V) = \{s \in F(f^{-1}(V)) : f \text{ is proper on } \mathrm{supp}(s)\}$[120]. One then obtains two functors $f_*, f_! : \mathfrak{Mod}(A_X) \to \mathfrak{Mod}(A_Y)$, with $(g \circ f)_* = g_* \circ f_*$ and $(g \circ f)_! = g_! \circ f_!$. If G is a sheaf on Y, the presheaf $Pf^{-1}G$ on X defined by $Pf^{-1}G(U) = \varinjlim_{V \supset f(U),\,\text{open}} G(V)$ in general is not a sheaf: the associated sheaf $f^{-1}G = \big(Pf^{-1}G\big)^+$ is called the *inverse image* of G. One obtains a functor $f^{-1} : \mathfrak{Mod}(A_Y) \to \mathfrak{Mod}(A_X)$ with $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. Note that $(f^{-1}G)_x = G_{f(x)}$ for any $x \in X$.

Examples. **(1)** Let $F \in \mathfrak{Mod}(A_X)$ and $M \in \mathfrak{Mod}(A)$. Denoted by $a_X : X \to \{\mathrm{pt}\}$ the constant map, one has $a_{X*}F = \Gamma(X; F)$, $a_{X!}F = \Gamma_c(X; F)$ and $a_X^{-1}M = M_X$. **(2)** If $Z \subset X$ and $\iota : Z \to X$ is the canonical inclusion, the sheaf $\iota^{-1}F$ on Z is denoted by $F|_Z$ (the *restriction* of F to Z). For example, if Z is open one recovers the restriction previously defined; if $Z = \{x\}$ one has $\iota^{-1}F = F_x$; if Z is a real analytic manifold and X is a complexification —just think to $Z = \mathbb{R}^n \subset X = \mathbb{C}^n$, one has $\mathcal{A}_Z = \mathcal{O}_X|_Z$. **(3)** There is a natural morphism $\mathcal{C}^0_Y \to f_*\mathcal{C}^0_X$ of sheaves on Y (exercise).

Remark A.4.2. In general, given a topological space X and a subset $S \subset X$, if F is a sheaf on X one defines $\Gamma(S; F) := \Gamma(S; F|_S)$. One proves that, if X is Hausdorff and S is compact, or if X is paracompact[121] and S closed, then $\Gamma(S; F) = \varinjlim_{U \supset S} \Gamma(U; F)$, i.e.

[120] A continuous function is said *proper* if the inverse image of any compact subset is compact.

[121] A topological space is said *paracompact* if for any open cover $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ of X there exists an open cover $\mathcal{V} = \{V_\mu : \mu \in M\}$ finer than $\mathcal{U}$ (i.e. for any $\lambda \in \Lambda$ there exists $\mu \in M$ such that $V_\mu \subset U_\lambda$) and locally finite (i.e., for any compact subset $K \subset X$ the set $\{\mu \in M : V_\mu \cap K \neq \varnothing\}$ is finite). Locally compact spaces which are countable at infinity (i.e. countable union of compact subsets: for example, the manifolds) and metric spaces are paracompact; closed subspaces of paracompact spaces are paracompact.

"the sections of F on S are the restrictions to S of sections of F on some open neighborhood U of S".

If $F, G \in \mathfrak{Mod}(A_X)$, the presheaf $U \mapsto \mathrm{Hom}_{A_U}(F|_U, G|_U)$ is a sheaf on X (exercise), denoted by $\mathcal{H}om_{A_X}(F, G)$: one obtains a functor (covariant in both variables)

$$\mathcal{H}om_{A_X}(\cdot\,, \cdot) : \mathfrak{Mod}(A_X)^{\mathrm{op}} \times \mathfrak{Mod}(A_X) \to \mathfrak{Mod}(B_X)$$

(where B is a subring contained in the center of A). One verifies that $\mathcal{H}om_{A_X}(A_X, F) \simeq F$, which implies $\mathrm{Hom}_{A_X}(A_X, F) \simeq \Gamma(X; F)$. If $H \in \mathfrak{Mod}(A_X^{\mathrm{op}})$, the presheaf $U \mapsto H(U) \otimes_A F(U)$ in general is not a sheaf on X; the associated sheaf is denoted by $H \otimes_{A_X} F$ (*tensor product*), and one obtains a functor (covariant in both variables)

$$\cdot \otimes_{A_X} \cdot : \mathfrak{Mod}(A_X) \times \mathfrak{Mod}(A_X) \to \mathfrak{Mod}(B_X).$$

One verifies that $A_X \otimes_{A_X} F \simeq F$, $H \otimes_{A_X} A_X \simeq H$ and $(H \otimes_{A_X} F)_x \simeq H_x \otimes_A F_x$ for any $x \in X$.

Example. Let $P = (a_{i,j})$ be a matrix $(m \times n)$ with coefficients in A, and consider the associated morphism of sheaves of left A-modules $A_X^m \xrightarrow{\cdot P} A_X^n$ (multiplication on the right by P of row vectors). Setting $M_P = \mathrm{coker}(\cdot P)$, one has the exact sequence of sheaves $A_X^m \xrightarrow{\cdot P} A_X^n \to M_P \to 0$. Now let $N \in \mathfrak{Mod}(A_X)$, and apply the functor $\mathcal{H}om_{A_X}(\cdot\,, N)$: recalling that the functor Hom, and hence $\mathcal{H}om$, is left exact, one gets the exact sequence of sheaves of B-modules $0 \to \mathcal{H}om_{A_X}(M_P, N) \to N^n \xrightarrow{P\cdot} N^m$ (namely $\mathcal{H}om_{A_X}(A_X^p, N) \simeq N^p$ and the morphism $\mathcal{H}om_{A_X}(\cdot P, N)$ is the multiplication on the left by P of column vectors). Finally, given an open subset $U \subset X$, one can apply the left exact functor $\Gamma(U, \cdot)$ obtaining a functorial isomorphism $\mathrm{Hom}_{A_U}(M_P|_U, N|_U) \simeq \ker[\Gamma(U; N^n) \xrightarrow{P\cdot} \Gamma(U; N^m)]$. Hence the sheaf $\mathcal{H}om_{A_X}(M_P, N)$ represents on any open subset the solutions of the linear system $Px = 0$ in the unknown $x \in N^n$: all informations relative to the homogeneous problems associated to the linear system P are contained in the sheaf M_P. (Actually, one proves that also the informations relatives to the non-homogeneous problems are contained in M_P, but this requires a deeper knowledge of homological algebra than the one provided in these brief notes.)

A.5 Manifolds

Let X be a countable Hausdorff topological space. In what follows, by $\mathcal{C}^k$ we mean $k \in \mathbb{N} \cup \{0, \infty, \omega\}$, where $\mathcal{C}^\omega$ denotes analytic regularity.

Definition A.5.1. A *local chart* of dimension n is a pair (U, φ) formed by an open subset $U \subset X$ and a homeomorphism $\varphi : U \xrightarrow{\sim} \mathbb{R}^n$. Two local charts (U_1, φ_1) and (U_2, φ_2) of dimension n are *k-compatible* if (a) $U_1 \cap U_2 = \varnothing$ or (b) $U_1 \cap U_2 \neq \varnothing$ and the transition function $\varphi_{12} = \varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$ is a diffeomorphism $\mathcal{C}^k$ between open subsets of $\mathbb{R}^n$. A $\mathcal{C}^k$ *differential atlas* of dimension n is a family $\{(U_\lambda, \varphi_\lambda) : \lambda \in \Lambda\}$ of k-compatible local charts of dimension n, where the U_λ's form an open cover of X.

Note that two local charts of dimension n are always 0-compatible.

Definition A.5.2. X is a (real) $\mathcal{C}^k$ *manifold of dimension* n if it is endowed with an atlas $\mathcal{C}^k$ $\{(U_\lambda, \varphi_\lambda) : \lambda \in \Lambda\}$ of dimension n, assumed to be maximal with respect to the inclusion. Denoting by $u_i : \mathbb{R}^n \to \mathbb{R}$ the ith coordinate function (i.e. $u_i(a) = a_i$), setting $x_{\lambda,i} = u_i \circ \varphi_\lambda : U_\lambda \to \mathbb{R}$ it holds $\varphi_\lambda = (x_{\lambda,1}, \dots, x_{\lambda,n})$: the n-tuple of functions $(x_{\lambda,i})_{i=1,\dots,n}$ is called a *system of local coordinates* on U_λ.

The local coordinates allow one to operate in U_λ as in $\mathbb{R}^n$. Note that a manifold is always locally simply connected.

Example. The sphere $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$ is a $\mathcal{C}^\infty$ manifold of dimension n. An atlas is given by $\{(U_i^\pm, \varphi_i^\pm) : i = 1, \dots, n+1\}$ with $U_i^\pm = \{x \in \mathbb{S}^n : x_i \gtrless 0\}$ and $\varphi_i^\pm = \psi \circ \widetilde{\varphi}\pm_i$, where $\widetilde{\varphi}_i^\pm : U_i\pm \xrightarrow{\sim} \dot{\mathbb{B}}^n$ is given by $\widetilde{\varphi}_i^\pm(x) = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1})$, and $\psi : \dot{\mathbb{B}}^n \xrightarrow{\sim} \mathbb{R}^n$ (the inverse is $(\varphi_i^\pm)^{-1} = (\widetilde{\varphi}_i^\pm)^{-1} \circ \psi^{-1}$, with $(\widetilde{\varphi}_i^\pm)^{-1}(u) = (u_1, \dots u_{i-1}, \sqrt{1-|u|^2}, u_i, \dots, u_n)$. Another atlas is provided by the stereographic projections $\{(\mathbb{S}^n \setminus \{N\}, \varphi_N), (\mathbb{S}^n \setminus \{S\}, \varphi_S)\}$, where $N = e_{n-1} = -S$ and for example, given $x \in \mathbb{S}^n \setminus \{N\}$, one has $\varphi_N(x) = (\frac{2x_1}{1-x_{n-1}}, \dots, \frac{2x_n}{1-x_{n-1}})$ (intersection of the half line coming from N and passing through x with the plane $\{x \in \mathbb{R}^{n+1} : x_{n+1} = -1\} = T_S\mathbb{S}^n \simeq \mathbb{R}^n$), and hence $\varphi_N^{-1}(u) = (\frac{4u_1}{|u|^2+4}, \dots, \frac{4u_n}{|u|^2+4}, \frac{|u|^2-4}{|u|^2+4})$. Finally, another atlas is given by the polar coordinates on $\mathbb{S}^n$: one of these charts is $\vartheta \circ \alpha^{-1}$, where $\vartheta : U =]0, 2\pi[\times (]0, \pi[)^{n-1} \xrightarrow{\sim} \mathbb{R}^n$ and $\alpha : U \xrightarrow{\sim} \mathbb{S}^n \setminus \{x \in \mathbb{S}^n : x_1 > 0, x_2 = 0\}$ is defined by

$$\begin{aligned}\alpha(\theta, \phi_1, \dots, \phi_{n-1}) \quad &= \quad (\cos\theta \sin\phi_1 \cdots \sin\phi_{n-1}, \sin\theta \sin\phi_1 \cdots \sin\phi_{n-1}, \\ &\qquad \cos\phi_1 \sin\phi_2 \cdots \sin\phi_{n-1}, \dots, \cos\phi_{n-2} \sin\phi_{n-1}, \cos\phi_{n-1}).\end{aligned}$$

Definition A.5.3. Let X (resp. Y) be a $\mathcal{C}^k$ manifold of dimension n (resp. m), $\{(U_\lambda, \varphi_\lambda) : \lambda \in \Lambda\}$ (resp. $\{(V_\mu, \psi_\mu) : \mu \in M\}$) a maximal differentiable atlas in X (resp. in Y). Given $h \leq k$, a continuous function $f : X \to Y$ is said to be *(of class)* $\mathcal{C}^h$ if such are all the functions $\psi_\mu \circ f \circ \varphi_\lambda^{-1}$.[122]

In particular, given an open subset $U \subset X$, a function $f : U \to \mathbb{R}$ is $\mathcal{C}^k$ if such are the maps $f \circ \varphi_\lambda^{-1} : \varphi_\lambda(U \cap U_\lambda) \to \mathbb{R}$ for any $\lambda \in \Lambda$ such that $U \cap U_\lambda \neq \varnothing$. The set $\mathcal{C}_X^k(U)$ of $\mathcal{C}^k$ functions on U has a natural structure of $\mathbb{R}$-algebra.
For $x \in X$, let $\mathcal{C}_{X,x}^k$ be the $\mathbb{R}$-algebra of germs of $\mathcal{C}^k$ functions in x, i.e.

$$\mathcal{C}_{X,x}^k = \{(U, f) : U \text{ open neighborhood of } x,\ f : U \to \mathbb{R} \text{ of class } \mathcal{C}^k\} / \sim\ ,$$

(122) More precisely, if such are $\psi_\mu \circ f|_{f^{-1}(V_\mu)} \circ \left(\varphi_\lambda|_{U_\lambda \cap f^{-1}(V_\mu)}\right)^{-1}$ for any $\lambda \in \Lambda$ and $\mu \in M$ such that $U_\lambda \cap f^{-1}(V_\mu) \neq \varnothing$).

where $(U,f) \sim (V,g)$ if there exists a open neighborhood $W \subset U \cap V$ of x such that $f|_W = g|_W$. In the terminology of sheaves, $\mathcal{C}^k_X(U)$ are the sections $\Gamma(U;\mathcal{C}^k_X)$ of the sheaf $\mathcal{C}^k_X$ on U, and $\mathcal{C}^k_{X,x} = \varinjlim_{U\ni x} \mathcal{C}^k_X(U)$ is the fiber of $\mathcal{C}^k_X$ in x.

From now on we shall assume that $k \geq 1$.

Let X be a $\mathcal{C}^k$ manifold of dimension n, and $\{(U_\lambda, \varphi_\lambda) : \lambda \in \Lambda\}$ be a maximal $\mathcal{C}^k$ atlas. On U_λ it is defined the operator of ith *partial derivative*: if $1 \leq h \leq k$ is a integer,

$$\frac{\partial}{\partial x_{\lambda,i}} : \mathcal{C}^h_X(U_\lambda) \to \mathcal{C}^{h-1}_X(U_\lambda), \qquad \frac{\partial f}{\partial x_{\lambda,i}}(x) = \frac{\partial(f \circ \varphi_\lambda^{-1})}{\partial u_i}(\varphi_\lambda(x)).$$

Proposition A.5.4. *If $U_\lambda \cap U_\mu \neq \varnothing$, it holds*

$$\frac{\partial}{\partial x_{\mu,j}} = \sum_{i=1}^n \frac{\partial x_{\lambda,i}}{\partial x_{\mu,j}} \frac{\partial}{\partial x_{\lambda,i}}.$$

Proof. Just use the chain rule for maps between open subsets of affine spaces. Namely let $f \in \mathcal{C}^h_X(U_\lambda \cap U_\mu)$: it holds

$$\begin{aligned} \frac{\partial f}{\partial x_{\mu,j}}(x) &= \frac{\partial(f \circ \varphi_\mu^{-1})}{\partial u_j}(\varphi_\mu(x)) = \frac{\partial((f \circ \varphi_\lambda^{-1}) \circ (\varphi_\lambda \circ \varphi_\mu^{-1}))}{\partial u_j}(\varphi_\mu(x)) \\ &= \sum_{i=1}^n \frac{\partial(f \circ \varphi_\lambda^{-1})}{\partial u_i}((\varphi_\lambda \circ \varphi_\mu^{-1})(\varphi_\mu(x))) \frac{\partial(u_i \circ (\varphi_\lambda \circ \varphi_\mu^{-1}))}{\partial u_j}(\varphi_\mu(x)) \\ &= \sum_{i=1}^n \frac{\partial(f \circ \varphi_\lambda^{-1})}{\partial u_i}(\varphi_\lambda(x)) \frac{\partial((u_i \circ \varphi_\lambda) \circ \varphi_\mu^{-1})}{\partial u_j}(\varphi_\mu(x)) = \sum_{i=1}^n \frac{\partial f}{\partial x_{\lambda,i}}(x) \frac{\partial x_{\lambda,i}}{\partial x_{\mu,j}}(x). \end{aligned}$$

□

In other words, denoted by $J_{\lambda,\mu} = \left(\frac{\partial x_{\mu,j}}{\partial x_{\lambda,i}}\right)_{i,j}$ the jacobian transition matrix, and by $\frac{\partial}{\partial x_\lambda}$ and $\frac{\partial}{\partial x_\mu}$ the coordinate vectors, one has

$$\frac{\partial}{\partial x_\mu} = {}^t(J_{\lambda,\mu})^{-1} \frac{\partial}{\partial x_\lambda}. \tag{A.8}$$

For any $x \in U_\lambda$ it is naturally induced a operator $\frac{\partial}{\partial x_{\lambda,i}}(x) : \mathcal{C}^h_{X,x} \to \mathcal{C}^{h-1}_{X,x}$.

Definition A.5.5. The *tangent space* in x, denoted by T_xX, is the real vector space of dimension n generated by the operators $\frac{\partial}{\partial x_{\lambda,i}}(x)$ $(i = 1, \dots, n)$. The *tangent bundle* to X is defined as $TX = \{(x,v) : x \in X,\ v \in T_xX\}$. We denote by $\tau : TX \to X$ the natural projection on X.

Note that the definition of T_xX is well-posed thanks to (A.8).

Remark A.5.6. From the previous definitions we get the classical definition of embedded differential manifold, and the other two equivalent to it. Let $X \subset \mathbb{R}^N$, and let $\iota : X \to \mathbb{R}^N$ be the inclusion map. Then X is a $\mathcal{C}^k$ manifold of dimension n if one of following equivalent conditions holds:

(1) for any $x \in X$ there exists an open $V \subset \mathbb{R}^n$, an open neighborhood $U \subset \mathbb{R}^N$ of x and a homeomorphism $\phi : V \xrightarrow{\sim} U \cap X$ (*local parametrization* at x, the inverse of a local chart) such that the function $\iota \circ \phi : V \to \mathbb{R}^N$ is of class $\mathcal{C}^k$ with jacobian matrix of rank n at any point;

(2) for any $x \in X$ there exist $1 \le i_1 < \cdots < i_n \le N$, an open neighborhood $U' \subset \mathbb{R}^n$ of x' (where $x = (x', x'') \in \mathbb{R}^n \times \mathbb{R}^{N-n} \simeq \mathbb{R}^N$ with $x' = (x_{i_1}, \dots, x_{i_n})$) and a function $f : U' \to \mathbb{R}^{N-n}$ of class $\mathcal{C}^k$ such that, denoted by $q' : \mathbb{R}^N \to \mathbb{R}^n$ the projection $q'(x', x'') = x'$, one has $X \cap q'^{-1}(U') = \{x = (x', x'') \in U' \times \mathbb{R}^{N-n} : f(x') = x''\}$;

(3) for any $x \in X$ there exists an open neighborhood $U \subset \mathbb{R}^N$ of x and a function $g : U \to \mathbb{R}^{N-n}$ (*defining function* at x) of class $\mathcal{C}^k$ and submersive (i.e., with jacobian matrix of rank $N - n$) on $g^{-1}(0)$, such that $X \cap U = g^{-1}(0)$.

Moreover, given $x_0 \in X$ and denoted by $\phi : V \xrightarrow{\sim} U \cap X$ a local parametrization at x_0 (with $\phi(v_0) = x_0$) and by $g : U \to \mathbb{R}^{N-n}$ a defining function at x_0, one has $T_{x_0}X = \operatorname{im}\left[d\phi(v_0) : \mathbb{R}^n \hookrightarrow \mathbb{R}^N\right] = \ker\left[dg(x_0) : \mathbb{R}^N \twoheadrightarrow \mathbb{R}^{N-n}\right] \subset \mathbb{R}^N$.

Proposition A.5.7. *The tangent bundle TX is a vector bundle on X* (see Definition 1.6.1) *and has a structure of $\mathcal{C}^k$ manifold of dimension $2n$.*

Proof. Fixed $\lambda \in \Lambda$, a trivialization of TX over U_λ is the map $U_\lambda \times \mathbb{R}^n \to \tau^{-1}(U_\lambda)$ associating to (x, a) the pair $(x, \sum_{i=1}^n a_i \frac{\partial}{\partial x_{\lambda,i}}(x))$. Hence TX is a vector bundle on X. An atlas of TX is given by $\{(\tau^{-1}(U_\lambda), \Phi_\lambda) : \lambda \in \Lambda\}$ with $\Phi_\lambda(x, \sum_{i=1}^n a_i \frac{\partial}{\partial x_{\lambda,i}}(x)) = (\varphi_\lambda(x), a)$. For the transition function, if $\sum_{i=1}^n a_{\lambda,i} \frac{\partial}{\partial x_{\lambda,i}}(x) = \sum_{j=1}^n a_{\mu,j} \frac{\partial}{\partial x_{\mu,j}}(x)$, from (A.8) one immediately gets that $a_\mu = J_{\lambda,\mu}(x)\, a_\lambda$. □

Definition A.5.8. A section on an open $U \subset X$ of τ is called a *vector field* on U.

Hence, a $\mathcal{C}^k$ vector field on $U \subset U_\lambda$ can be uniquely written as $A = \sum_{i=1}^n A_i \frac{\partial}{\partial x_{\lambda,i}}$ with $A_i \in \mathcal{C}^k_X(U)$; in general, a vector field on $U \subset X$ is a family $A = (A_\lambda)_{\lambda \in \Lambda}$ where $A_\lambda = \sum_{i=1}^n A_{\lambda,i} \frac{\partial}{\partial x_{\lambda,i}}$ with $A_{\lambda,i} \in \mathcal{C}^k_X(U \cap U_\lambda)$ such that, whenever $U \cap U_\lambda \cap U_\mu \neq \varnothing$, one has

$$A_\mu = J_{\lambda,\mu}\, A_\lambda.$$

Definition A.5.9. Let X and Y be two $\mathcal{C}^k$ manifolds of dimension resp. n and m, $f : X \to Y$ a $\mathcal{C}^h$ map ($h \ge 1$). The *tangent map* $df : TX \to TY$ is defined by $df(x, v) = (f(x), df_x v)$ where $df_x v : \mathcal{C}^h_{Y,f(x)} \to \mathcal{C}^{h-1}_{Y,f(x)}$ is given by $df_x v([\alpha]) = v([\alpha \circ f])$.

In local coordinates, meaning $f : U_\lambda \to V_\mu$ (with $x \in U_\lambda$ and $f(x) \in V_\mu$) and $(x_1, \dots, x_n) \in U_\lambda$ and $(y_1, \dots, y_m) \in V_\mu$, setting $y_j = f_j(x)$ (for $j = 1, \dots, m$) the chain rule is still valid: $df(x, \frac{\partial}{\partial x_i}) = (f(x), \sum_{j=1}^m \frac{\partial f_j}{\partial x_i}(x) \frac{\partial}{\partial y_j})$. If $Y = \mathbb{R}$, df_x is the usual differential $T_x X \to \mathbb{R}$.

Definition A.5.10. The *cotangent bundle* of X is defined as $T^*X = \{(x, \omega) : x \in X,\ \omega \in T^*_x X\}$, where $T^*_x X$ is the dual vector space of $T_x X$. We denote by $\pi : T^*X \to X$ the natural projection on X.

Also the cotangent bundle T^*X is a vector bundle on X and has a structure of $\mathcal{C}^k$ manifold of dimension $2n$. Fixed $\lambda \in \Lambda$, introduce for any $x \in U_\lambda$ the dual basis

$$dx_{\lambda,i}(x) \in T^*_x X \ (i = 1, \dots, n), \qquad \left\langle \frac{\partial}{\partial x_{\lambda,i}}(x), dx_{\lambda,j}(x) \right\rangle = \delta_{i,j}.$$

Let (U_μ, φ_μ) with $U_\lambda \cap U_\mu \neq \varnothing$: the relation between the $dx_{\lambda,i}$ and the $dx_{\mu,j}$ is given by

$$dx_{\mu,j} = \sum_{i=1}^{n} \frac{\partial x_{\mu,j}}{\partial x_{\lambda,i}} dx_{\lambda,i}, \qquad \text{i.e.} \qquad dx_\mu = J_{\lambda,\mu}\, dx_\lambda. \tag{A.9}$$

Hence, if $\sum_{i=1}^n \alpha_{\lambda,i} dx_{\lambda,i}(x) = \sum_{j=1}^n \alpha_{\mu,j} dx_{\mu,j}(x)$, from (A.9) one computes (and this provides the transition functions) that

$$\alpha_\mu = {}^t J_{\lambda,\mu}^{-1}(x)\, \alpha_\lambda.$$

Definition A.5.11. A section on an open $U \subset X$ of π is called a *linear differential form* on U.

Analogously to vector fields, a $\mathcal{C}^k$ linear differential form one $U \subset U_\lambda$ can be uniquely written as $\omega = \sum_{i=1}^n \omega_i dx_{\lambda,i}$ with $\omega_i \in \mathcal{C}^k_X(U)$ and, in general, a linear differential form on $U \subset X$ is a family $\omega = (\omega_\lambda)_{\lambda\in\Lambda}$ where $\omega_\lambda = \sum_{i=1}^n \omega_{\lambda,i} dx_{\lambda,}i$ with $\omega_{\lambda,i} \in \mathcal{C}^k_X(U \cap U_\lambda)$ and

$$\omega_\mu = {}^t J_{\lambda,\mu}^{-1}\, \omega_\lambda.$$

This equality will be intrinsecally espressed by the equality of pull-back of linear differential forms on U_λ and U_μ with respect to the canonical inclusions of $U_\lambda \cap U_\mu$ (see §2.2).

Example. A natural example of linear differential forms is the differential of a function: as we have seen, if $f : X \to \mathbb{R}$, for any $x \in X$ is defined $df_x = df(x) \in T^*_x X$, and hence df is a linear differential form.

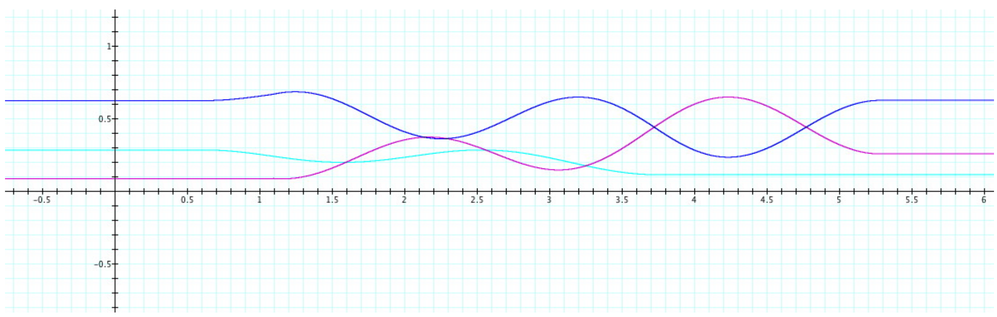

Figure 18: A partition of unity by three functions.

Definition A.5.12. Let X be a $\mathcal{C}^k$ manifold. A *partition of unity* is a family $\{\rho_\lambda : \lambda \in \Lambda\}$ of non negative $\mathcal{C}^k$ functions such that (a) $\{\rho_\lambda : \lambda \in \Lambda\}$ is locally finite, i.e. for any $x \in X$ there exists a neighborhood $U \subset X$ of x such that $\rho_\lambda|_U \neq 0$ only for a finite number of $\lambda \in \Lambda$, (b) $\sum_{\lambda\in\Lambda} \rho_\lambda = 1$.

In the case $k = \infty$, the following result is well-known (we refer e.g. to de Rham [11]). Recall that the *support* of $f : X \to \mathbb{R}$ is

$$\operatorname{supp}(f) = \overline{\{x \in X : f(x) \neq 0\}}.$$

Proposition A.5.13. *Let X be a $\mathcal{C}^\infty$ manifold, $\{U_\lambda : \lambda \in \Lambda\}$ a open cover of X. Then there exists a partition of unity $\{\rho_\lambda : \lambda \in \Lambda\}$ "subordinate to $\{U_\lambda : \lambda \in \Lambda\}$", i.e. such that* $\operatorname{supp}(\rho_\lambda) \subset U_\lambda$ *for any $\lambda \in \Lambda$. Moreover there exist partitions of unity $\{\rho_\mu : \mu \in M\}$ of functions with compact support, and a function $\gamma : M \to \Lambda$, such that supp$(\rho_\mu) \subset U_{\gamma(\mu)}$ for any $\mu \in M$.*

Let us conclude with the definition of manifold with boundary. Consider the half-space $\mathbb{H}^n = \{u \in \mathbb{R}^n : u_n \geq 0\}$ and its boundary $\partial\mathbb{H}^n = \{u \in \mathbb{R}^n : u_n = 0\} \simeq \mathbb{R}^{n-1}$.

Definition A.5.14. X is a $\mathcal{C}^k$ *manifold of dimension n with boundary* if it is endowed with an atlas ($\mathcal{C}^k$ of dimension n) $\{(U_\lambda, \varphi_\lambda) : \lambda \in \Lambda\}$ where φ_λ is a homeomorphism of U_λ on $\mathbb{R}^n$ or on $\mathbb{H}^n$ such that $\varphi_\mu \circ \varphi_\lambda^{-1} : \varphi_\lambda(U_\lambda \cap U_\mu) \xrightarrow{\sim} \varphi_\mu(U_\lambda \cap U_\mu)$ is a $\mathcal{C}^k$ diffeomorphism. The subset $\partial X = \{x \in X : x \in U_\lambda,\ \varphi_\lambda : U_\lambda \xrightarrow{\sim} \mathbb{H}^n,\ \varphi_\lambda(x) \in \partial\mathbb{H}^n\}$ is called *boundary* of X, and $\dot{X} = X \setminus \partial X$ the manifold (without boundary) associated X.

Example. **(1)** Let $g : \mathbb{R}^2 \to \mathbb{R}$, $g(x, y) = x^4 - 4(x^2 - y^2)$, and let $X = g^{-1}(\mathbb{R}_{\leq 0})$ (the "figure eight" filled inside). X is not a manifold with boundary: namely, no neighborhood V of $(0, 0) \in X$ is homeomorphic to $\mathbb{H}^2$ (let $\varphi : \mathbb{H}^2 \xrightarrow{\sim} V$ be a homeomorphism, $u_0, u_1 \in \mathbb{H}^2$ with $x(\varphi(u_0)) < 0$ and $x(\varphi(u_1)) > 0$, $\gamma : I \to \mathbb{H}^2$ with $\gamma(0) = u_0$, $\gamma(1) = u_1$ and $\varphi^{-1}(0, 0) \notin \gamma(I)$: then $\varphi \circ \gamma$ joins $\varphi(u_0)$ to $\varphi(u_1)$ without passing through $(0, 0)$, absurd). **(2)** Let $g : \mathbb{R}^2 \to \mathbb{R}$, $g(x, y) = y^2 - x^5$ and let $X = g^{-1}(\mathbb{R}_{\leq 0})$ (the cusp): it is manifold with boundary $\mathcal{C}^0$ (the boundary is $\partial X = g^{-1}(0)$). **(3)** In general, let $g : \mathbb{R}^{n+1} \to \mathbb{R}$ be a $\mathcal{C}^k$ function (with $k \geq 1$) such that the system $\begin{cases} g(x) = 0 \\ dg(x) = 0 \end{cases}$ has no solutions: then $X = g^{-1}(\mathbb{R}_{\leq 0})$ is a manifold with boundary $\partial X = g^{-1}(0)$, the hypersurface of $\mathbb{R}^{n+1}$ defined by g.

One sees immediately that $\dot{X}$ is a $\mathcal{C}^k$ manifold of dimension n (without boundary). Let us show that also ∂X is a manifold.

Lemma A.5.15. *A $\mathcal{C}^k$ autodiffeomorphism $F : \mathbb{H}^n \xrightarrow{\sim} \mathbb{H}^n$* (i.e., an autohomeomorphism which extends to a $\mathcal{C}^k$ diffeomorphism on some open neighborhood of $\mathbb{H}^n$) *induces a $\mathcal{C}^k$ autodiffeomorphism $f : \partial\mathbb{H}^n \xrightarrow{\sim} \partial\mathbb{H}^n$. Moreover, if F has Jacobian determinant everywhere positive, this holds also for f.*

Proof. As a consequence of the theorem of local inversion in $\mathbb{R}^n$ one obtains that $F(\partial\mathbb{H}^n) = \partial\mathbb{H}^n$ (it must be $F^{-1}(\dot{\mathbb{H}}^n) \subset \dot{\mathbb{H}}^n$, hence $F(\partial\mathbb{H}^n) \subset \partial\mathbb{H}^n$; then one can argue analogously with the inverse F^{-1}) and the first statement follows with $f = F|_{\partial\mathbb{H}^n}$. Now we show the second statement for $n = 2$, (the general case being similar). Let (y_1, y_2) (resp. (x_1, x_2)) be a coordinate system in the domain (resp. codomain), and let $F = (F_1, F_2)$: then $f(y_1) = F_1(y_1, 0)$. By hypothesis it holds $\det \begin{pmatrix} \frac{\partial F_1}{\partial y_1}(y_1, 0) & \frac{\partial F_1}{\partial y_2}(y_1, 0) \\ \frac{\partial F_2}{\partial y_1}(y_1, 0) & \frac{\partial F_2}{\partial y_2}(y_1, 0) \end{pmatrix} > 0$ for any y_1. Since $F_2(y_1, 0) \equiv 0$, one has $\frac{\partial F_2}{\partial y_1}(y_1, 0) \equiv 0$; moreover, since $F(\dot{\mathbb{H}}^n) \subset \dot{\mathbb{H}}^n$, one has $\frac{\partial F_2}{\partial y_2}(y_1, 0) > 0$. Hence $\frac{\partial f}{\partial y_1}(y_1) = \frac{\partial F_1}{\partial y_1}(y_1, 0) > 0$. □

Proposition A.5.16. *If X is a $\mathcal{C}^k$ manifold with boundary and dimension n, its boundary ∂X is a $\mathcal{C}^k$ manifold* without boundary *of dimension $n - 1$.*

Proof. Let $x \in \partial X$ and let $x \in U_\lambda$ with $\varphi_\lambda : U_\lambda \xrightarrow{\sim} \mathbb{H}^n$ and $\varphi_\lambda(x) \in \partial\mathbb{H}^n$. It is enough to show that $\varphi_\lambda^{-1}(\partial\mathbb{H}^n) = \partial X \cap U_\lambda$, because then $\varphi_\lambda|_{\partial X \cap U_\lambda} : \partial X \cap U_\lambda \to \partial\mathbb{H}^n \simeq \mathbb{R}^{n-1}$ would be a local chart of ∂X at the neighborhood of x. The inclusion $\varphi_\lambda^{-1}(\partial\mathbb{H}^n) \subset \partial X \cap U_\lambda$ is true by definition. Conversely, let

$U \subset U_\lambda$ and $\varphi : U \xrightarrow{\sim} \mathbb{H}^n$ be another local chart in U_λ compatible with φ_λ. Consider the diffeomorphism $\varphi_\lambda|_U \circ \varphi^{-1} : \mathbb{H}^n \xrightarrow{\sim} \varphi_\lambda(U) \subset \mathbb{H}^n$. Arguing as in the proof of Lemma A.5.15 one has $(\varphi_\lambda \circ \varphi^{-1})(\partial \mathbb{H}^n) \subset \partial \mathbb{H}^n$, i.e. $\varphi^{-1}(\partial \mathbb{H}^n) \subset \varphi_\lambda^{-1}(\partial \mathbb{H}^n)$, as desired. □