2.3 Mayer-Vietoris principle

The Mayer-Vietoris principle allows one to reduce the problem of computing the cohomology of a space to the computations on two open subsets which cover the space: in this sense, Mayer-Vietoris (and its generalization due to Čech, see §2.9) is a sort of cohomological analogue of the Theorem of Van Kampen. However, not surprisingly in this abelian context, weaker hypotheses are required: neither the arcwise connectedness of the open subsets and of their intersections, nor the presence of a common base point.

Let $X$ be a $C^\infty$ manifold of dimension $n$; $U, V \subset X$ two open subsets with $X = U \cup V$; $\{\rho_U, \rho_V\}$ a partition of unity subordinate to $\{U, V\}$ (see Appendix A.5), and consider the open embeddings $\iota_U : U \cap V \to U$, $\iota_V : U \cap V \to V$, $j_U : U \to X$ and $j_V : V \to X$. Through the functor $\Omega^\ast$ one obtains a sequence of morphisms of complexes (called of Mayer-Vietoris):

\[
\begin{align*}
0 \to \Omega^\ast(X) \xrightarrow{j_U^\ast \oplus j_V^\ast} \Omega^\ast(U) \oplus \Omega^\ast(V) \xrightarrow{-\iota_U^\ast + \iota_V^\ast} \Omega^\ast(U \cap V) \to 0.
\end{align*}
\]

(Recall that the pull-back by the open embedding has the meaning of restricting the form to the open subset.)

**Proposition 2.3.1.** The sequence of Mayer-Vietoris (2.14) is exact.

**Proof.** The morphism $j_U^\ast \oplus j_V^\ast$ is injective, and it is clear that $\text{im}(j_U^\ast \oplus j_V^\ast) = \ker(\iota_U^\ast - \iota_V^\ast)$. Finally let $\omega \in \Omega^m(U \cap V)$; then $\rho_V\omega \in \Omega^m(U)$, $\rho_U\omega \in \Omega^m(V)$ and $\omega = (\rho_U\omega) + (\rho_V\omega)$ on $U \cap V$ (see Figure 16(a)). \hfill \Box

As we saw in Appendix A.2, one obtains a long exact sequence of cohomology

\[
\begin{align*}
0 \to H^0(X) \to H^0(U) \oplus H^0(V) \to H^0(U \cap V) \xrightarrow{j_U^\ast} H^1(X) & \to H^1(U) \oplus H^1(V) \to \cdots \to H^m(U \cap V) \xrightarrow{j_U^\ast} H^{m+1}(X) \to \cdots
\end{align*}
\]

**Figure 16:** (a-b) The Mayer-Vietoris sequence. (c) The Mayer-Vietoris sequence with compact support.

Let us recall the definition of coboundary morphism $\delta : H^{m-1}(U \cap V) \to H^m(X)$ (see Proposition A.2.3). Let $\omega \in Z^m(U \cap V) \subset \Omega^m(X)$. It is $\omega = (j_V^\ast - j_U^\ast)(\xi)$ with $\xi = (\rho_U\omega, \rho_V\omega)$. Since $(j_V^\ast - j_U^\ast)(d\xi) = d\omega = 0$ (i.e., $-d(\rho_U\omega)$ and $d(\rho_V\omega)$ coincide on
\[ U \cap V, \text{ see Figure 16(b)}, \text{ one has } d\xi \in \text{im}(j_U^c \oplus j_V^c): \text{ there exists } \tau \in \Omega_X^m(X) \text{ such that } \tau = d\xi \text{ (which in particular implies } \tau \in Z_X^m(X)). \text{ We then set}
\]
\begin{equation}
\delta([\omega]) = [\tau] = \begin{cases} 
-\frac{d(\rho \nu \omega)}{[d(\rho \nu \omega)]} & \text{(on } U) \\
[\text{on } V] 
\end{cases}
\end{equation}

For the cohomology with compact support, from Remark 2.2.10 one obtains a sequence of morphisms of complexes (called of Mayer–Vietoris with compact support)

\begin{equation}
\begin{array}{cccc}
0 & \to & \Omega_c^* (U \cap V) & \to & \Omega_c^* (U) \oplus \Omega_c^* (V) & \to & \Omega_c^* (X) & \to & 0.
\end{array}
\end{equation}

**Proposition 2.3.2.** The sequence of Mayer–Vietoris with compact support (2.17) is exact.

**Proof.** It is clear that the morphism \((-\nu \omega) \oplus (\nu \omega)_*) is injective and that \(\ker((\nu \omega)_* + (\nu \omega)_*) = \text{im}((-\nu \omega) \oplus (\nu \omega)_*)\). Finally, if \(\omega \in \Omega^m_c(X), \text{ then } \rho \nu \omega \in \Omega^m_c(U), \rho \nu \omega \in \Omega^m_c(V) \text{ and } \omega = (\nu \omega)_* \rho(\nu \omega) + (\nu \omega)_*(\nu \omega) \text{ (see Figure 16(c)).}\]

Hence we get another long exact sequence for the cohomology with compact support:

\begin{equation}
\begin{array}{cccc}
0 & \to & H^m_c(U \cap V) & \to & H^m_c(U) \oplus H^m_c(V) & \to & H^m_c(X) & \to & 0.
\end{array}
\end{equation}

Also in this case let us write explicitly the coboundary morphism \(\delta_c : H^{m-1}_c(X) \to H^m_c(U \cap V) \text{ (see Proposition A.2.3).} \text{ Let } \omega \in Z^{m-1}_c(X) \cap \Omega^{m-1}_c(U) \text{ and consider } \rho \nu \omega \in \Omega^{m-1}_c(U) \text{ and } \rho \nu \omega \in \Omega^{m-1}_c(V): \text{ from the exactness of } (2.17) \text{ it follows that } d(\rho \nu \omega) \text{ and } d(\rho \nu \omega) \text{ have compact support in } U \cap V \text{ and are opposite, and one has (neglecting the maps of extension by zero)}
\]
\begin{equation}
\delta_c([\omega]) = [-d(\rho \nu \omega)] = [d(\rho \nu \omega)].
\end{equation}

**Example.** (The circle) Let \(X = S^1\) and let \(U = \{(x, y) \in S^1 : y > -\frac{1}{2}\}\) and \(V = \{(x, y) \in S^1 : y < \frac{1}{2}\}\), open neighborhoods of the upper and lower emicyle whose intersection \(U \cap V\) has two connected components \((U \cap V)_+\) and \((U \cap V)_-\) (see Figure 17(a)). Since \(U\) and \(V\) are diffeomorphic to \(R\), and \(U \cap V\) to \(R \cup R\), one has \(H^0(U) = H^0(V) = \mathbb{R}, H^0(U \cap V) = \mathbb{R}^2\), and \(H^1(U) = H^1(V) = H^1(U \cap V) = 0\) for \(j \neq 0\). The map \(\gamma : H^0(U) \oplus H^0(V) \to H^0(U \cap V)\) is \(\gamma([\phi], [\psi]) = ([\psi - \phi]_+, [\psi - \phi]_-):\) hence from (2.15) it follows \(H^0(S^1) = \ker(\gamma) = \mathbb{R} \text{ and } H^1(S^1) = \text{coker}(\gamma) = \mathbb{R}\). A generator of \(H^0(S^1)\) is the constant function 1. For a generator of \(H^1(S^1)\), let us look for a closed 0-form (i.e. a locally constant function) on \(U \cap V\) which is not in the image of \(\gamma\); for example, the function \(\beta((U \cap V)_+) \equiv 1\) and \(\beta((U \cap V)_-) \equiv 0\). Then, \(\delta(\beta) = d(\rho \nu \beta)\) generates \(H^1(S^1)\): it is a closed 1-form on \(S^1\) supported by \((U \cap V)_+\) (see (2.16)).\(^8\) Obviously, since \(S^1\) is compact, its cohomology with compact support coincides with the one just found: this can be seen also with (2.18). Namely \(H^0_c(U) = H^0_c(V) = H^0_c(U \cap V) = 0, H^1_c(U) = H^1_c(V) = \mathbb{R}\) and \(H^1_c(U \cap V) = \mathbb{R}^2\). The morphism \(\epsilon : H^1_c(U \cap V) \to H^1_c(U) \oplus H^1_c(V)\) operates as follows: given \([\omega] = ([\omega_+], [\omega_-]) \in H^1_c(U \cap V)\),

\(^8\)Such generator has the form \(\alpha(x)dx\), where \(x\) is a local coordinate of \((U \cap V)_+\) and \(\alpha(x)\) is a function with compact support in \((U \cap V)_+\) with non zero integral. Roughly speaking, \(\alpha(x)dx\) seems like a little “bump” on \((U \cap V)_+\) (in fact the usual name for these generators is bump form). As we shall see, this will be the typical generator of cohomology with compact support of top degree of orientable manifolds.
we refer to yatcher more details about the Mayer-Vietoris sequence of singular homology and cohomology.

There is a similar sequence of Mayer-Vietoris for the singular cohomology, which is constructed analogously to the one for the cohomology of de Rham: one has an exact sequence of complexes in $C(\mathbb{Z})$

\begin{equation}
0 \to S^\bullet(X, \mathbb{Z}) \to S^\bullet(U, \mathbb{Z}) \oplus S^\bullet(V, \mathbb{Z}) \to S^\bullet(U \cap V, \mathbb{Z}) \to 0
\end{equation}

which gives rise to a long exact sequence in $\mathcal{M}\text{od}(\mathbb{Z})$

\begin{equation}
0 \to H^0(X, \mathbb{Z}) \to H^0(U, \mathbb{Z}) \oplus H^0(V, \mathbb{Z}) \to H^0(U \cap V, \mathbb{Z}) \to \cdots \to H^m(U \cap V, \mathbb{Z}) \to H^m(U, \mathbb{Z}) \oplus H^m(V, \mathbb{Z}) \to \cdots
\end{equation}

Also the singular homology satisfies a sequence of Mayer-Vietoris, with the arrows going in opposite direction with respect to (2.20) and (2.21): one has an exact sequence of complexes (of chains)

\begin{equation}
0 \to S_\bullet(U \cap V, \mathbb{Z}) \to S_\bullet(U, \mathbb{Z}) \oplus S_\bullet(V, \mathbb{Z}) \to S_\bullet(X, \mathbb{Z}) \to 0
\end{equation}

and a long exact sequence (of chains) in $\mathcal{M}\text{od}(\mathbb{Z})$

\begin{equation}
\cdots \to H_{m+1}(X, \mathbb{Z}) \to H_m(U \cap V, \mathbb{Z}) \to H_m(U, \mathbb{Z}) \oplus H_m(V, \mathbb{Z}) \to H_m(X, \mathbb{Z}) \to \cdots
\end{equation}

For more details about the Mayer-Vietoris sequence of singular homology and cohomology we refer e.g. to Hatcher [8].

**Figure 17:** Mayer-Vietoris calculus for: (a) the circle $S^1$; (b) the bouquet of two circles.

**Examples.** (1) *(Bouquet of k circles; chain of k circles)* The singular cohomology of the circle can be computed as above, and gives $H^0(S^1, \mathbb{Z}) = H^1(S^1, \mathbb{Z}) = \mathbb{Z}$. As for the bouquet (i.e. the wedge sum) $X$ of two circles, the singular cohomology is $H^0(X, \mathbb{Z}) = \mathbb{Z}$, $H^1(X, \mathbb{Z}) = \mathbb{Z}^2$ and zero in the other degrees. Namely take as $U$ and $V$ the two open subsets of Figure 17(b): since $U$ and $V$ are homotopically equivalent to $S^1$ and $U \cap V$ is contractible, the long exact sequence (2.21) becomes

\begin{equation}
0 \to H^0(X, \mathbb{Z}) \to \mathbb{Z}^2 \xrightarrow{\alpha} \mathbb{Z} \xrightarrow{\beta} H^1(X, \mathbb{Z}) \xrightarrow{\gamma} \mathbb{Z}^2 \to 0 \to H^2(X, \mathbb{Z}) \to 0 \to \cdots
\end{equation}
We know that $H^0(X, \mathbb{Z}) = \mathbb{Z}$ since $X$ is arcwise connected: this implies that $\alpha$ is surjective, $\beta = 0$ and $\gamma$ is an isomorphism, so that $H^1(X, \mathbb{Z}) = \mathbb{Z}^2$ and $H^j(X, \mathbb{Z}) = 0$ for $j \geq 2$. Arguing by induction on $k$, one can prove by similar arguments that the cohomology of the bouquet of $k$ circles and of the chain of $k$ circles (i.e. a sequence of $k$ circles where each circle has a common point with the previous one) is $\mathbb{Z}$ in degree zero, $\mathbb{Z}^k$ in degree one and 0 otherwise. (2) (Wedge sums) The argument used for the bouquet of two circles applies more generally for finite wedge sums of locally contractible spaces (see also p. 24 for an analogous situation for fundamental groups). Let $X = \bigsqcup_{j=1}^n X_j$ be the wedge sum of a family of arcwise connected pointed spaces $(X_j, x_j)$, and for each $j = 1, \ldots, n$ let $V_j$ be a contractible open neighborhood of $x_j$ in $X_j$: then $H^0(X, \mathbb{Z}) = \mathbb{Z}$, and $H^k(X, \mathbb{Z}) = \bigoplus_{j=1}^n H^k(X_j, \mathbb{Z})$ for any $k \geq 1$. Namely this is obvious for $n = 1$; arguing by induction, set $U = X_n \vee (\bigsqcup_{j=1}^{n-1} V_n)$ and $V = V_n \vee (\bigsqcup_{j=1}^{n-1} X_j)$. Then $U \cap V$ is contractible, $U$ and $V$ deformation-retract to $X_n$ and $\bigsqcup_{j=1}^{n-1} X_j$ respectively, so the induction proceeds by Mayer-Vietoris.
2.4 Orientation and integration

On an oriented manifold of dimension $k$ it is possible to integrate differential $k$-forms with compact support.

2.4.1 Orientation

Consider a change of coordinates in $\mathbb{R}^n$, i.e. a diffeomorphism $\Phi : V \rightarrow U$ of class $C^\infty$ with $U$ and $V$ open connected subsets of $\mathbb{R}^n$. Let $y = (y_1, \ldots, y_n)$ (resp. $x = (x_1, \ldots, x_n)$) be coordinates in $V$ (resp. in $U$) with $x_i = \Phi_i(y)$. Let $J_\Phi = \left( \frac{\partial \Phi_i}{\partial y_j}(y) \right)_{i,j}$ be the jacobian matrix of $\Phi$. The function $\text{sign}(\det(J_\Phi)(y))$ is constant on $V$, and we denote its value by $\text{sign}(\Phi)$. We say that $\Phi$ preserves (resp. inverts) the orientation if $\text{sign}(\Phi) = 1$ (resp. if $\text{sign}(\Phi) = -1$). Such definition extends naturally to diffeomorphisms between open subsets of $\mathbb{R}^n$: namely, by definition they are induced by diffeomorphisms between open neighborhoods in $\mathbb{R}^n$ of such subsets.

**Remark 2.4.1.** By identifying the variables $y$ and $x$, we note that $\Phi$ preserves the orientation if and only if $\Phi^*(dx_1 \cdots dx_n) = \alpha(x)dx_1 \cdots dx_n$ with $\alpha \in C^\infty(V)$, $\alpha > 0$.

Now let us switch to manifolds.

**Definition 2.4.2.** A $C^\infty$ manifold $X$ of dimension $n$ is said to be orientable if there exists an “oriented” atlas $\{(U_\lambda, \varphi_\lambda) : \lambda \in \Lambda\}$ of $X$, i.e. such that all transition functions $\varphi_\mu \circ \varphi_\lambda^{-1}$ preserve the orientation of $\mathbb{R}^n$. $X$ is said to be oriented if it is explicitly associated to a oriented atlas. A volume form on $X$ is a global differential form on $X$ of top degree (i.e. an element of $\Omega^n(X)$) which does not vanish at any point of $X$.

![Figure 18: The Möbius band and the Klein bottle, two classical examples of non orientable manifolds.](image)

**Examples.** (0) For $n = 0$, an oriented atlas for $X = \{x_\lambda : \lambda \in \Lambda\}$ is the assignation, to any point, of a number $\pm 1$. (1) If $X$ is a vector space, an oriented atlas for $X$ is the assignation of an ordered basis. (2) To orient $X = S^1$ is equivalent to choose a sense of rotation (usually counterclockwise, see the example at p. 71). (2) The real projective plane, the Möbius band, the Klein bottle (see Figure 18) are three classical examples of non orientable manifolds: this will be proven later by using Poincaré duality (see Theorem 2.6.6 and the subsequent examples).
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**Proposition 2.4.3.** A manifold is oriented if and only if it is explicitly associated to a volume form.

Proof. Let $\omega \in \Omega^n(X)$ be a volume form and $\{(U_\lambda, \varphi_\lambda) : \lambda \in \Lambda\}$ an atlas for $X$. For any $\lambda \in \Lambda$ one has $\varphi_\lambda^*(dx_1 \cdots dx_n) = f_\lambda \omega$ with $f_\lambda(x) \neq 0$ for any $x \in U_\lambda$. Since $f_\lambda \in \mathbb{C}^\infty(U_\lambda)$, such $f_\lambda$ will have constant sign on $U_\lambda$; up to modifying the atlas by replacing sometimes $\varphi_\lambda$ with $-\varphi_\lambda$, we may assume that $f_\lambda > 0$ for any $\lambda \in \Lambda$. In such case, the pull-back of $dx_1 \cdots dx_n$ by any transition function $\varphi_\mu \circ \varphi_\lambda^{-1}$ will be a positive multiple of $dx_1 \cdots dx_n$ (namely one easily computes that $(\varphi_\mu \circ \varphi_\lambda^{-1})^*(dx_1 \cdots dx_n) = ((f_\mu/f_\lambda) \circ \varphi_\lambda^{-1}) dx_1 \cdots dx_n$), and therefore (by Remark 2.4.1) the atlas $\{(U_\lambda, \varphi_\lambda) : \lambda \in \Lambda\}$ is oriented. Conversely, let $\{(U_\lambda, \varphi_\lambda) : \lambda \in \Lambda\}$ be a oriented atlas of $X$. From $(\varphi_\mu \circ \varphi_\lambda^{-1})^*(dx_1 \cdots dx_n) = \alpha(x) dx_1 \cdots dx_n$ with $\alpha(x) > 0$, since the pull-back commutes with the external product one gets that $\varphi_\mu^*(dx_1 \cdots dx_n) = \varphi_\lambda^*(\alpha dx_1 \cdots dx_n) = (\alpha \circ \varphi_\lambda) \varphi_\lambda^*(dx_1 \cdots dx_n)$; hence, setting $\omega_\lambda = \varphi_\lambda^*(dx_1 \cdots dx_n)$, one has $\omega_\mu = (\alpha \circ \varphi_\lambda) \omega_\lambda$ on $U_\lambda \cap U_\mu$, with $\alpha \circ \varphi_\lambda > 0$. If $\{\rho_\lambda : \lambda \in \Lambda\}$ is a partition of unity subordinate to $\{U_\lambda : \lambda \in \Lambda\}$, then

$$\omega = \sum_{\lambda \in \Lambda} \rho_\lambda \omega_\lambda$$

is a volume form on $X$. □

By Proposition 2.4.3, we can give the following

**Definition 2.4.4.** Let $X$ be a connected and orientable $C^\infty$ manifold. In the class of volume forms on $X$ we introduce the equivalence relation $\omega \sim \theta$ if and only if $\theta = f \omega$ with $f > 0$ on $X$: any of the two equivalence classes is called an orientation on $X$. The **standard orientation** of $\mathbb{R}^n$ is the equivalence class of $dx_1 \cdots dx_n$.

In a manifold with boundary, an orientation induces an orientation on the boundary:

**Proposition 2.4.5.** If $X$ is a orientable manifold, such is also its boundary $\partial X$; moreover, an orientation of $X$ naturally induces an orientation of $\partial X$.

Proof. The boundary $\partial \mathbb{H}^n = \{u \in \mathbb{R}^n : u_n = 0\}$ has an orientation induced from the standard one of $\mathbb{H}^n$ (given by $dx_1 \cdots dx_n$): it is, by definition, the equivalence class of $(-1)^n dx_1 \cdots dx_{n-1} \in \Omega^{n-1}(\partial \mathbb{H}^n)$. Hence if $U_\lambda$ is an open subset of $X$ and $\varphi_\lambda : U_\lambda \to \mathbb{H}^n$ preserves the orientations, define the orientation of $\partial U_\lambda = \partial X \cap U_\lambda$ (the equality holds for what is said in Appendix A.5) as the class of the volume form $(\varphi_\lambda)^*(-1)^n dx_1 \cdots dx_{n-1}$). The transition functions preserve such orientation by the second part of Lemma A.5.15. □

Therefore, from now on, given a oriented manifold $X$, we shall mean that its boundary $\partial X$ is endowed with the orientation induced from the one of $X$.

**Example. (Orientation of a hypersurface)** Let $g : U \subset \mathbb{R}^{n+1} \to \mathbb{R}$ be a defining function, i.e. a $C^\infty$ function such that the system $\begin{cases} g(x) = 0 \\ dy(x) = 0 \end{cases}$ has no solutions: then $X = g^{-1}(\mathbb{R}_{\leq 0})$ is a manifold with boundary $\partial X = g^{-1}(0)$, the hypersurface of $U$ defined by $g$. The manifold $\bar{X} = X \setminus \partial X$ is an open subset of $\mathbb{R}^{n+1}$,

![Figure 19: The canonical orientation of a hypersurface $g(x) = 0$.](image)

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hence it is naturally endowed with the orientation given by $\mathbb{R}^{n+1}$; on the boundary $\partial X$ is then induced a orientation, which coincides with the one described in Proposition 2.4.5, considering the volume form $\theta_{\partial X}$ obtained by restricting to $\partial X$ the n-form $\theta = \frac{1}{\sqrt{g}} \sum_{j=1}^{n+1} (-1)^{j+1} \frac{\partial g}{\partial x_j} dx_j \cdots dx_{n+1}$ on $\mathbb{R}^{n+1}$, where $\nabla g = \left( \frac{\partial g}{\partial x_1}, \ldots, \frac{\partial g}{\partial x_{n+1}} \right)$ is the gradient of $g$ (in other words, $dg = \nabla g \cdot dx = \frac{\partial g}{\partial x_1} dx_1 + \cdots + \frac{\partial g}{\partial x_{n+1}} dx_{n+1}$); namely note that $\frac{\partial g}{\partial x_j} \wedge \theta = dx_1 \cdots dx_{n+1}$. Analogously, denoting by $n_{x_0} = \frac{\nabla g(x_0)}{\sqrt{g(x_0)}}$ the normal versor to $\partial X$ issued from $x_0$ (orthogonal to the tangent space $T_{x_0}(\partial X) = \ker dg(x_0)$), a basis $\{v_1, \ldots, v_n\}$ of $T_{x_0}(\partial X)$ is positive if and only if the basis $\{x_0, v_1, \ldots, v_n\}$ is positive for the standard orientation of $\mathbb{R}^{n+1}$. For example, let $g(x) = \sum_{j=1}^{n+1} x_j^2 - 1$: one has $X = \mathbb{R}^{n+1}$, $\partial X = S^n$, $\nabla g(x_0) = 2x_0$, $\theta = \sum_{j=1}^{n+1} x_j dx_1 \cdots dx_{n+1}$, $n_{x_0} = \frac{x_0}{\sqrt{2}}$, denoting by $N = e_{n+1}$ (resp. $S = - e_{n+1}$) the North (resp. South) pole of $S^n$, one has $T_N S^n = T_0 S^n = \langle e_1, \ldots, e_n \rangle$; by what has been said in general, a positive basis for $T_N S^n$ (resp. for $T_0 S^n$) is $\langle -1 \rangle (e_1, \ldots, e_n)$.

2.4.2 Integration

Let $U \subset \mathbb{R}^n$ be open, and let $\omega = f dx_1 \cdots dx_n \in \Omega^n_c(U)$ with $f \in C^\infty_c(U)$.

**Definition 2.4.6.** The integral of the form $\omega$ is defined as $\int_U \omega = \int_U f(x) dx_1 \cdots dx_n$, where the right hand side is the Lebesgue integral of $f$.

If $\sigma \in \mathcal{S}_n$, then $\omega_{\sigma} = f(x) dx_{\sigma(1)} \cdots dx_{\sigma(n)} = \operatorname{sign}(\sigma) f(x) dx_1 \cdots dx_n = \operatorname{sign}(\sigma) \omega$ and hence $\int_U \omega_{\sigma} = \operatorname{sign}(\sigma) \int_U \omega$. More generally, let us see how the integral depends on a change of coordinates $\Phi : V \cong U$.

**Proposition 2.4.7.** It holds $\int_V \Phi^* \omega = \operatorname{sign}(\Phi) \int_U \omega$.

**Proof.** From (2.12), one has $\Phi^* \omega = (f \circ \Phi)^* d\Phi_1 \cdots d\Phi_n = (f \circ \Phi) \det(J_\Phi) dy_1 \cdots dy_n$, and hence $\int_V \Phi^* \omega = \int_U (f \circ \Phi) \det(J_\Phi) dy_1 \cdots dy_n$; on the other hand, the theorem of change of variables for the Lebesgue integral in $\mathbb{R}^n$ says that $\int_U \omega = \int_U (f \circ \Phi) |\det(J_\Phi)| dy_1 \cdots dy_n$, and the statement follows.

**Definition 2.4.8.** Let $X$ be a oriented $C^\infty$ manifold, $\{(U_\lambda, \varphi_\lambda) : \lambda \in \Lambda\}$ an oriented atlas of $X$, $\{\rho_\lambda : \lambda \in \Lambda\}$ a partition of unity subordinate to $\{U_\lambda : \lambda \in \Lambda\}$. Given $\omega \in \Omega^n_c(X)$, one defines the integral of $\omega$ on $X$ as

$$\int_X \omega = \sum_{\lambda \in \Lambda} \int_{U_\lambda} \rho_\lambda \omega \quad \text{where} \quad \int_{U_\lambda} \rho_\lambda \omega := \int_{\mathbb{R}^n \cap U_\lambda} (\varphi_\lambda^{-1})^* (\rho_\lambda \omega) .$$

We must verify that this is a good definition. So let $\{(V_\mu, \psi_\mu) : \mu \in M\}$ be another atlas of $X$ analogously oriented, and $\{\chi_\mu : \mu \in M\}$ a partition of unity subordinate to it: recalling that $\sum_{\lambda \in \Lambda} \rho_\lambda = \sum_{\mu \in M} \chi_\mu = 1$ and that $\rho_\lambda \chi_\mu \omega \in \Omega^n_c(U_\lambda \cap V_\mu)$ and sign$(\varphi_\lambda \circ \psi_\mu^{-1}) = 1$ whenever $U_\lambda \cap V_\mu \neq \emptyset$, using Proposition 2.4.7 one obtains $\int_{U_\lambda} \rho_\lambda \chi_\mu \omega = \int_{\varphi_\lambda(U_\lambda \cap V_\mu)} (\varphi_\lambda^{-1})^* (\rho_\lambda \chi_\mu \omega) = \int_{\psi_\mu(U_\lambda \cap V_\mu)} (\varphi_\lambda \circ \psi_\mu^{-1})^* (\rho_\lambda \chi_\mu \omega) = \int_{\psi_\mu(U_\lambda \cap V_\mu)} (\psi_\mu^{-1})^* (\rho_\lambda \chi_\mu \omega) = \int_{V_\mu} \rho_\mu \chi_\mu \omega$, and hence $\sum_{\lambda \in \Lambda} \int_{U_\lambda} \rho_\lambda \omega = \sum_{\mu \in M} \int_{U_\lambda} \rho_\lambda \chi_\mu \omega = \sum_{\mu \in M} \int_{V_\mu} \rho_\mu \chi_\mu \omega = \lambda \int_{V_\mu} \chi_\mu \omega$.

More generally, for any $0 \leq k \leq n$ the $k$-forms can be integrated one the oriented $k$-dimensional submanifolds of $X$.
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Definition 2.4.9. Let $Z$ be a oriented $k$-dimensional submanifold of $X$, $\iota : Z \to X$ the embedding map, and let $\omega \in \Omega^k(X)$ be such that $\iota^*\omega \in \Omega^k_c(Z)$.(84) One defines

$$\int_Z \omega = \int_Z \iota^*\omega.$$ 

Examples. (0) A $0$-form (i.e. a $C^\infty$ function) $f$ on $X$ can be integrated on a oriented $0$-dimensional submanifold of $X$, i.e. a discrete set $\Gamma = \{x_i : i \in I\}$ of points of $X$ endowed with signs $\text{sign}(x_i) \in \{\pm 1\}$, provided that $f(x_i) = 0$ except then on a finite set of $i \in I$: one then has $\int_\Gamma f = \sum_{i \in I} \text{sign}(x_i)f(x_i)$.

(1) Let $\gamma : \mathbb{R} \to \mathbb{R}^n$ be a parametrization of a simple curve $C$: hence $\gamma$ is a diffeomorphism of $\mathbb{R}$ on the curve $C = \gamma(I) \subset \mathbb{R}^n$, and provides (with $\gamma^{-1}$) an oriented atlas for $C$. Given $\omega = \sum_{i=1}^n f_i\,dx_i$ with $\text{supp}(\omega) \cap C$ compact in $C$ (it $\text{supp}(\omega) = \bigcap_{i=1}^n \text{supp}(f_i)$), one hence has $\int_C \omega = \int_C \iota^*\omega = \int_\gamma \gamma^*\omega = \sum_{i=1}^n f_i(\gamma(t))\gamma_i'(t)\,dt$ (namely $\gamma^*\omega = \frac{d\gamma_i}{dt} = \gamma_i'(dt)$).

(2) Let $F : \mathbb{R}^{n-1} \to \mathbb{R}$ be a $C^\infty$ function, and consider the hypersurface $S \subset \mathbb{R}^n$ parametrized by $h : \mathbb{R}^{n-1} \to \mathbb{R}^n$, $h(x) = (x, F(x))$ (in other words, $S$ is the graph of $F$); then $h^{-1}$ is an oriented atlas for $S$. Given $\omega \in \Omega^{n-1}(\mathbb{R}^n)$, with $\text{supp}(\omega) \cap S$ compact in $S$, it holds $\int_S \omega = \int_{\mathbb{R}^{n-1}} h^*\omega$. Written $\omega = \sum_{j=1}^n (-1)^{n-j}f_j\,dx_1\cdots\hat{dx_j}\cdots\,dx_n$, for (2.11) it holds $h^*\omega = \sum_{j=1}^n (-1)^{n-j}(f_j \circ h)\,dh_1\cdots\hat{dh_j}\cdots\,dh_n$, since $dh_j = dx_j$ for $1 \leq j \leq n-1$ and $dh_n = \sum_{j=1}^n (-\frac{\partial F}{\partial x_j})(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)\,dx_j$, one has $\int_S \omega = \int_{\mathbb{R}^{n-1}} \left((f \circ h)\cdot n \right)\,dx_1\cdots\hat{dx_j}\cdots\,dx_{n-1}$, where $n(x) = (-\frac{\partial F}{\partial x_1}(x), \ldots, -\frac{\partial F}{\partial x_n}(x))$ is orthogonal to $T_{h(x)}S = \{(a, \sum_{j=1}^n \frac{\partial F}{\partial x_j}(x) : a \in \mathbb{R}^{n-1}) \subset \mathbb{R}^n\}$. The quantity at the right hand side is called also “flow through $S$” of the vector field $f = (f_1, \ldots, f_n)$ associated to $\omega$.

Now we show the main result of the theory of integration on manifolds.

Theorem 2.4.10. (Stokes) Let $X$ be a oriented $C^\infty$ manifold with boundary, $U \subset X$ an open subset, $\partial U = U \cap \partial X$. Then for any $\omega \in \Omega^{n-1}(U)$ it holds

$$\int_U d\omega = \int_{\partial U} \omega.$$ 

Proof. Let $\iota : U \to X$ be the embedding: up to replacing $\omega$ with $\iota^*\omega$ we may assume that $U = X$. So let us prove the theorem in three steps. (1) Let $X = \mathbb{R}^n$. Since $\partial X = \emptyset$, it holds $\int_{\partial X} \omega = 0$. Hence, by linearity it is enough to prove that the left hand side vanishes when $\omega = f\,dx_1\cdots dx_n$ with $f \in C^\infty_c(\mathbb{R}^n)$: then $\omega = (-1)^{-1} f\left(\frac{\partial f}{\partial x_n}\right) dx_1\cdots dx_n$, and by Fubini one has $\int_{\mathbb{R}^n} \omega = (-1)^{-1} \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} \frac{\partial f}{\partial x_n}\,dx_n\right) dx_1\cdots dx_{n-1} = 0$ because $\int_{\mathbb{R}^{n}} \frac{\partial f}{\partial x_n}\,dx_n = \lim_{a \to +\infty} \lim_{b \to -\infty} [f(x_1, \ldots, x_{n-1}, b) - f(x_1, \ldots, x_{n-1}, a)] = 0$ since $f$ has compact support.

(2) Now let $X = \mathbb{H}^n$ and $\omega = \sum_{j=1}^n (-1)^{n-j} f_j\,dx_1\cdots\hat{dx_j}\cdots\,dx_n$ with $f_j \in C^\infty(\mathbb{H}^n)$, and hence $\omega = (-1)^{-1} \left(\sum_{j=1}^n \frac{\partial f_j}{\partial x_n}\right) dx_1\cdots dx_n$. Since the $f_j$ have compact support, if $j \neq n$ one has, again by Fubini, $\int_{\mathbb{H}^n} \frac{\partial f_j}{\partial x_n}\,dx_1\cdots dx_n = 0$ because $\int_{\mathbb{R}^{n}} \frac{\partial f_j}{\partial x_n}\,dx_n = 0$. Hence $\int_{\mathbb{H}^n} \omega = (-1)^{-1} \int_{\mathbb{H}^n} \frac{\partial f_j}{\partial x_n}\,dx_1\cdots dx_n = \int_{\mathbb{H}^n} \int_0^{+\infty} \frac{\partial f_j}{\partial x_n}\,dx_n\right) dx_1\cdots dx_{n-1} = (-1)^n \int_{\mathbb{R}^{n-1}} f_n(x_1, \ldots, x_{n-1}, 0)\,dx_1\cdots dx_{n-1}$. On the other hand, for $j \neq n$ the restrictions of the forms $f_j\,dx_1\cdots\hat{dx_j}\cdots\,dx_n$ to $\partial \mathbb{H}^n$ are zero, while the restriction of $f_j\,dx_1\cdots\hat{dx_j}\cdots\,dx_n$, for $f_j : \partial \mathbb{H}^n \to \mathbb{H}^n$ is the embedding map one has $\iota^*\omega = (-1)^n \left(\iota^*\omega\right)$ by definition of induced orientation on $\partial \mathbb{H}^n$, and this proves the statement. (3) Finally, in the general case write $\omega = \sum_{\lambda \in \Lambda} \rho^\lambda \omega$: by linearity it is enough to prove the problem for any $\rho^\lambda \omega$. This form has compact support (being $\text{supp}(\rho^\lambda \omega) \subset \text{supp}(\rho^\lambda) \cap \text{supp}(\omega)$) contained in $U$. Since $U_\lambda$ is diffeomorphic to $\mathbb{R}^n$ or $\mathbb{H}^n$, the theorem is still valid in it for what has been said before. Therefore one has $\int_U d(\rho^\lambda \omega) = \int_{U_\lambda} d(\rho^\lambda \omega) = \int_{\partial U_\lambda} \rho^\lambda \omega = \int_{\partial X} \rho^\lambda \omega$. □

Examples. (0) The Fundamental Theorem of Calculus in one variable, used to prove the Theorem of Stokes, is then embedded as a particular case in it. If $X = [a, b] \subset \mathbb{R}$, the induced orientation of $\partial X = \{a, b\}$ is given by assigning $-1$ to $a$ and $+1$ to $b$: hence, given $f \in C^\infty([a, b])$, one has $\int_a^b f(x)\,dx = \int_{[a, b]} df = (84)$In particular, this obviously happens when $\omega \in \Omega^k_c(X)$.

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\[
\int_{(a,b)} f = f(b) - f(a). \quad (1) \text{ Let } g : \mathbb{R}^n \to \mathbb{R} \text{ be a defining function, } X = g^{-1}(\mathbb{R}_{<0}) \text{ and } \partial X = g^{-1}(0) \text{ with the induced orientation (see the example at p. 71), and let } \omega = \sum_{j=1}^n (-1)^{n-j} f_j \, dx_1 \cdots \bar{dx}_j \cdots dx_n \in \Omega^{n-1}_c(X). \text{ One has } d\omega = (-1)^{n-1} \text{div}(f) dx_1 \cdots dx_n \text{ (where } \text{div}(f) = \sum_{j=1}^n \frac{\partial f_j}{\partial x_j}), \text{ and hence one obtains that } \int_X d\omega = (-1)^{n-1} \int_X \text{div}(f) dx_1 \cdots dx_n. \text{ On the other hand, locally } \partial X = g^{-1}(0) \text{ is the graph of a function } F : \mathbb{R}^{n-1} \to \mathbb{R}, \text{ i.e. } h : \mathbb{R}^{n-1} \to \mathbb{R}^n, h(x) = (x, F(x)) \text{ parametrizes } X; \text{ for what has been said in the example at p. 73(2), and recalling the definition of induced orientation on the boundary, one has } \int_{\partial X} \omega = (-1)^{n-1} \int_{\partial X} (f \cdot n) \theta_{\partial X}, \text{ where } n = \frac{\nabla g}{|\nabla g|} \text{ is the outward normal versor and } \theta_{\partial X} \text{ is the volume form obtained by restricting to } \partial X \text{ the } (n-1)\text{-form } \theta = \frac{1}{|\nabla g|} \sum_{j=1}^n (-1)^{j+1} \frac{\partial g}{\partial x_j} \, dx_1 \cdots \bar{dx}_j \cdots dx_n \text{ on } \mathbb{R}^n, \text{ for which one has } \frac{1}{|\nabla g|} \, dg \wedge \theta = dx_1 \cdots dx_n. \text{ From Stokes' theorem one hence gets the Divergence theorem:}
\[
\int_X \text{div}(f) \, dx_1 \cdots dx_n = \int_{\partial X} (f \cdot n) \theta_{\partial X}.
\]
In particular, for } f(x) = \frac{x}{n} \text{ one gets}
\[
\text{Vol}_n(X) = \frac{1}{n} \int_{\partial X} (n(x) \cdot x) \theta_{\partial X}.
\]
2.5 Poincaré lemmas

Let us compute the cohomology groups of \( \mathbb{R}^n \), by obtaining as a byproduct the invariance of the cohomology of de Rham under homotopy.

We identify \( \mathbb{R}^{n+1} \cong \mathbb{R}^n \times \mathbb{R} \), denoting the variables by \((x, t)\). Let \( \pi : \mathbb{R}^{n+1} \to \mathbb{R}^n \) be the first projection \((\pi(x, t) = x)\) and \( s_0 : \mathbb{R}^n \to \mathbb{R}^{n+1} \) be its zero section \((s_0(x) = (x, 0))\). Correspondingly, there are morphisms of complexes

\[
\pi^* : \Omega^\bullet(\mathbb{R}^n) \to \Omega^\bullet(\mathbb{R}^{n+1}), \quad s_0^* : \Omega^\bullet(\mathbb{R}^{n+1}) \to \Omega^\bullet(\mathbb{R}^n),
\]

as well as the respective morphisms in cohomology \( H^\bullet \pi^* : H^\bullet(\mathbb{R}^n) \to H^\bullet(\mathbb{R}^{n+1}) \) and \( H^\bullet s_0^* : H^\bullet(\mathbb{R}^{n+1}) \to H^\bullet(\mathbb{R}^n) \).

**Proposition 2.5.1.** \( \pi^* \) and \( s_0^* \) are inverse one to each other in the homotopy category \( K(\mathbb{R}) \) (see Appendix A.2).

**Proof.** Since \( \pi \circ s_0 = \text{id}_{\mathbb{R}^{n+1}} \), one obtains immediately that \( s_0^* \circ \pi^* = \text{id}_{\Omega^\bullet(\mathbb{R}^n)} \). We are left with showing that \( \pi^* \circ s_0^* \) is homotopic to \( \text{id}_{\Omega^\bullet(\mathbb{R}^{n+1})} \), i.e. that there exists a morphism of complexes \( k^* : \Omega^\bullet(\mathbb{R}^{n+1}) \to \Omega^\bullet(\mathbb{R}^{n+1}) \) such that \( \xi^m = \text{id}_{\Omega^m(\mathbb{R}^{n+1})} - \pi^* \circ s_0^* = d^{m-1} \circ k^m + k^{m+1} \circ d^m \):

\[
\cdots \longrightarrow \Omega^{m-1}(\mathbb{R}^{n+1}) \xrightarrow{\xi_{m-1}} \Omega^m(\mathbb{R}^{n+1}) \xrightarrow{k^m} \Omega^m(\mathbb{R}^{n+1}) \xrightarrow{k^{m+1}} \Omega^{m+1}(\mathbb{R}^{n+1}) \xrightarrow{\xi_m} \cdots.
\]

Any \( \Omega^m(\mathbb{R}^{n+1}) \) can be written as linear combination of forms of type \( f(x, t) dx_1 \) (with \(|J| = m\)) and \( f(x, t) dx_1 dt \) (with \(|J| = m-1\)): let us define \( k^m : \Omega^m(\mathbb{R}^{n+1}) \to \Omega^{m-1}(\mathbb{R}^{n+1}) \) by \( k^m(f(x, t) dx_1 dt) = (-1)^{m-1} \left( \int_0^1 f(x, \tau) d\tau \right) dx_1 dt \) and then extending by \( \mathbb{R} \)-linearity. It will be enough to check the homotopy relation on forms of this type. If \( \omega = f(x, t) dx_1 dt \) with \(|J| = m\), one has \( (\text{id}_{\Omega^m(\mathbb{R}^{n+1})} - \pi^* \circ s_0^*)(\omega) = [f(x, t) - f(x, 0)] dx_1 = k^{m+1}(d^m \omega) = (d^{m-1} \circ k^m + k^{m+1} \circ d^m)(\omega) \); if \( \omega = f(x, t) dx_1 dt \) with \(|J| = m-1\), since \( s_0^* \omega = 0 \) one has \( (\text{id}_{\Omega^{m-1}(\mathbb{R}^{n+1})} - \pi^* \circ s_0^*)(\omega) = \omega = (d^{m-1} \circ k^m + k^{m+1} \circ d^m)(\omega) = d^{m-1} \left( \int_0^1 f(x, \tau) d\tau \right) dx_1 dt \).

**Corollary 2.5.2.** If \( X \) is a \( C^\infty \) manifold, then \( H^\bullet(X \times \mathbb{R}) \cong H^\bullet(X) \).

**Proof.** The proof of Proposition 2.5.1 is still valid also replacing \( \mathbb{R}^n \) by \( X \).

**Corollary 2.5.3.** (Poincaré lemma) For any \( n \in \mathbb{N} \) it holds \( H^0(\mathbb{R}^n) = \mathbb{R} \) and \( H^j(\mathbb{R}^n) = 0 \) for any \( j \neq 0 \).

**Proof.** As we have seen, the result is true for \( n = 1 \); by Proposition 2.5.1 we get that \( H^0 s_0^* \) is an isomorphism between \( H^0(\mathbb{R}^{n+1}) \) and \( H^0(\mathbb{R}^n) \) (with inverse \( H^0 \pi^* \)), and the induction proceeds. \( \square \)

(85) Let \( X \) be a \( C^\infty \) manifold of dimension \( n \). The Poincaré lemma states that the complex of de Rham of sheaves on \( X \) is exact except in degree zero: namely (see Appendix) the exactness of a complex of sheaves is equivalent to the exactness of the complex in any fiber, and any point of a manifold has a basis of open neighborhoods diffeomorphic to \( \mathbb{R}^n \). Since the locally constant functions are only the \( C^\infty \) functions with zero differential, one has an exact sequence of sheaves \( 0 \to \mathcal{C}^0_X \to \mathcal{C}^1_X \to \cdots \to \mathcal{C}^n_X \). By Poincaré lemma, we may claim that the complex of de Rham is a resolution of the sheaf \( \mathbb{R}_X \) in \( \mathcal{C}(\mathbb{R}) \), i.e. the morphism
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Corollary 2.5.4. If \( f, g : X \to Y \) are two homotopic maps of \( C^\infty \) manifolds, then \( H^\bullet f^* = H^\bullet g^* \). In particular, homotopically equivalent \( C^\infty \) manifolds have the same cohomology of de Rham.

Proof. Let \( h : X \times \mathbb{R} \to Y \) be a \( C^\infty \) map such that \( h(x, t) = f(x) \) for \( t \leq 0 \) and \( h(x, t) = g(x) \) for \( t \geq 1 \). If \( \pi : X \times \mathbb{R} \to X \) is the first projection and \( s_j : X \to X \times \mathbb{R} \) is its section \( s_j(x) = (x, j) \) (where \( j = 0, 1 \)), one hence has \( f = h \circ s_0 \) and \( g = h \circ s_1 \); therefore \( f^* = s_0^* \circ h^* \) and \( g^* = s_1^* \circ h^* \), hence \( H^\bullet f^* = H^\bullet s_0^* \circ H^\bullet h^* \) and \( H^\bullet g^* = H^\bullet s_1^* \circ H^\bullet h^* \). On the other hand, from the proof of Proposition 2.5.1 we know that \( H^\bullet s_0^* \) and \( H^\bullet s_1^* \) are isomorphisms which invert \( H^\bullet h^* \), and hence are equal. This gives \( H^\bullet f^* = H^\bullet g^* \).

Hence, for example, contractible manifolds have the cohomology of one point (i.e. \( H^0 = \mathbb{R} \) and \( H^j = 0 \) for \( j \neq 0 \)), the vector bundles have the cohomology of the base and \( H^\bullet (\mathbb{R}^{n+1}) \approx H^\bullet (S^n) \). Now we compute the cohomology of the sphere \( S^n \).

Proposition 2.5.5. It holds \( H^0(S^n) = H^n(S^n) = \mathbb{R} \) and \( H^j(S^n) = 0 \) for \( j \neq 0, n \).

Proof. For \( n = 1 \) see the example at p. 67. Now let \( n \geq 2 \), and set \( U_{\pm} = \{ x \in S^n : x_{n+1} \geq \mp \frac{1}{2} \} \subset S^n \) (open neighborhoods of the upper and lower hemisphere respectively). Since \( U_{\pm} \) are diffeomorphic to \( \mathbb{R}^n \), we get \( H^j(U_{\pm}) = \mathbb{R} \) and \( H^j(U_{\pm}) = 0 \) for \( j \neq 0 \). The equation \( \{ x \in S^n : x_{n+1} = 0 \} \subset S^{n-1} \) is a strong deformation retract of \( U_{+} \cap U_{-} \), hence by the Corollary 2.5.4 one has \( H^\bullet(U_{+} \cap U_{-}) = H^\bullet(S^{n-1}) \). The statement can be proven by recurrence using the sequence of Mayer-Vietoris (2.15).

Now we are able to answer to a question still pending from the first part of the course (Theorem 1.2.5).

Corollary 2.5.6. \( S^n \) is not contractible.

Proof. By what has been shown in the proof of Theorem 1.2.5, thanks to Weierstrass (density of \( C^\infty \) in \( C^0 \) for the topology of uniform convergence on compact subsets) it is enough to show that there does not exist retractions \( B^{n+1} \to S^n \) of class \( C^\infty \). If a such a map would exist, by Proposition 2.2.9 it would induce an injective map \( H^\bullet(S^n) \to H^\bullet(B^{n+1}) \); but this is absurd by Proposition 2.5.5.

Let us study the cohomology with compact support in the general case of manifolds. Let \( X \) be a \( C^\infty \) manifold of dimension \( n, \pi : X \times \mathbb{R} \to X \) the first projection. Since \( \pi \) is not proper, the pull-back \( \pi^* \) does not respect compactness. So let us consider the map of integration along the fiber

\[
\pi_* : \Omega^\bullet_c(X \times \mathbb{R}) \to \Omega^\bullet_c(X),
\]

of complexes induced by the monomorphism \( \alpha \) is a quasi-isomorphism between the complex associated to \( \mathbb{R}X \) and the complex of de Rham (see Appendix):

\[
\begin{array}{cccccccc}
r_X : & \cdots & 0 & \longrightarrow & \mathbb{R}X & \longrightarrow & 0 & \longrightarrow & \cdots \\
\alpha^* \downarrow & & & & & & & \\
\Omega^\bullet_X : & \cdots & 0 & \longrightarrow & \Omega^\infty_X & \longrightarrow & \Omega^1_X & \longrightarrow & \cdots \\
\end{array}
\]

(\( \bullet \))If two \( C^\infty \) maps are homotopic (as continuous functions), there exists also a \( C^\infty \) homotopy between them (just use the density theorem of Weierstrass on every chart of an atlas...).

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defined as follows: observing that any form \( \omega \in \Omega^m_c(X \times \mathbb{R}) \) can be written as linear combination of forms of type \( f(x, t)\pi^*\theta \) or \( f(x, t)\pi^*\phi \wedge dt \) (with \( f \in C_\infty(X \times \mathbb{R}), \theta \in \Omega^m(X) \) and \( \phi \in \Omega^{m-1}(X) \)), set \( \pi_+(f(x, t)\pi^*\theta) = 0, \pi_+(f(x, t)\pi^*\phi \wedge dt) = \left( \int_{\mathbb{R}} f(x, t)dt \right) \phi \) extending then by \( \mathbb{R} \)-linearity. One verifies directly (exercise) that \( \pi_* \) commutes with \( d \), and hence that it is a morphism of complexes: it is then induced a map in cohomology \( H^\bullet \pi_* : H^\bullet_c(X \times \mathbb{R}) \to H^\bullet_c(X) \).

On the other hand, choose any \( e = e(t) \, dt \in \Omega^1_c(\mathbb{R}) \) with \( \int_{\mathbb{R}} e = 1 \) and consider the map

\[
\epsilon_* : \Omega^\bullet_c(X) \to \Omega^\bullet_c(X \times \mathbb{R})
\]

given by \( \epsilon_*(\theta) = \pi_\theta \wedge e \): also \( \epsilon_* \) commutes with \( d \) (exercise) and hence induces a map in cohomology \( H^\bullet \epsilon_* : H^\bullet_c(X \times \mathbb{R}) \to H^\bullet_c(X) \).

**Proposition 2.5.7.** \( \pi_* \) and \( \epsilon_* \) are inverse to each other in the homotopy category \( K(\mathbb{R}) \).

**Proof.** One sees immediately that \( \pi_* \circ \epsilon_* = \text{id}_{\Omega^\bullet_c(X)} \); to show that \( \epsilon_* \circ \pi_* \) is homotopic to \( \text{id}_{\Omega^\bullet(X \times \mathbb{R})} \), we define \( k^m : \Omega^m(X \times \mathbb{R}) \to \Omega^{m-1}(X \times \mathbb{R}) \) by extending by linearity \( k^m(f(x, t)\pi^*\phi) = 0 \) (with \( f \in C_\infty(X \times \mathbb{R}) \) and \( \theta \in \Omega^m(X) \)) and \( k^m(f(x, t)\pi^*\phi \wedge dt) = (-1)^{m-1} \left( \int_{-\infty}^\infty f(x, \tau) \, d\tau - \int_{-\infty}^\infty \int_{\mathbb{R}} f(x, t) \, dt \right) \pi^*\phi \) (with \( f \in C_\infty(X \times \mathbb{R}) \) and \( \phi \in \Omega^{m-1}(X) \)). It is then a computational exercise to verify that \( \text{id}_{\Omega^\bullet(X \times \mathbb{R})} - \epsilon_* \circ \pi_* = d^{m-1} \circ k^m + k^{m+1} \circ d^m \).

**Corollary 2.5.8.** If \( X \) is a \( C_\infty \) manifold, then \( H^\bullet_c(X \times \mathbb{R}^m) \cong H^\bullet_c(X) \).

**Proof.** From Proposition 2.5.7 we get that \( H^\bullet(X \times \mathbb{R}) \) is isomorphic to \( H^\bullet_c(X \times \mathbb{R}) \) and \( H^{m-1}_c(X) \) (with inverse \( H^\bullet \epsilon_* \)); by applying repeatedly this fact, one gets the statement.

**Corollary 2.5.9.** (Poincaré lemma with compact support) For any \( n \in \mathbb{N} \) one has \( H^n_c(\mathbb{R}^n) = \mathbb{R} \) and \( H^n_c(\mathbb{R}^n) = 0 \) for any \( j \neq n \).

**Proof.** Choose \( X = \{pt\} \) and \( m = n \) in Corollary 2.5.8.

**Remark 2.5.10.** Let us find a generator for \( H^n_c(\mathbb{R}^n) \). By applying \( n \) times the map \( \epsilon_* \) to the function \( 1 \) on \( \mathbb{R}^n \), one obtains the \( n \)-form with compact support \( \alpha \, dx_1 \cdots dx_n \) on \( \mathbb{R}^n \), where \( \alpha(x) = \prod_{i=1}^n e(x_i) \). Note that \( \int_{\mathbb{R}^n} \alpha \, dx_1 \cdots dx_n = 1 \), and that — up to a translation by a vector — \( \text{supp}(\alpha) \) can be a compact as small as one likes at the neighborhood of any point of \( \mathbb{R}^n \) (it is a bump form, in the sense already seen for \( S^1 \) in the example at p. 67).

**Examples.**

1. \textit{(Difference of affine spaces)} Given \( n, m \in \mathbb{N} \) with \( n > m \), consider \( \mathbb{R}^n \simeq \mathbb{R}^m \times \mathbb{R}^{n-m} \) with coordinates \( x = (x', x'') \), and identify \( \mathbb{R}^m \) with the subspace \( \{x \in \mathbb{R}^n : x'' = 0\} \). The space \( X = \mathbb{R}^n \setminus \mathbb{R}^m \) is diffeomorphic to \( \mathbb{R}^m \times \mathbb{R}_+^n \times \mathbb{S}^{n-m-1} \) by \( x \mapsto (x', |x''|, x''/|x''|) \): using \( \log : \mathbb{R}_+^n \to \mathbb{R} \) one has \( X \simeq \mathbb{R}^{m+1} \times \mathbb{S}^{n-m-1} \). Therefore one gets \( H^0(X) = H^{n-1}(X) = \mathbb{R} \) and \( H^j(X) = 0 \) for \( j \neq 0, n - 1 \) (if \( m \leq n - 2 \)), or \( H^0(X) = \mathbb{R}^2 \) and \( H^j(X) = 0 \) for \( j \neq 0 \) (if \( m = n - 1 \)); moreover \( H^{m+1}(X) = H^m_c(X) = \mathbb{R}^2 \) and \( H^j(X) = 0 \) for \( j \neq m - 1 \), \( n \) (if \( m \leq n - 2 \)), or \( H^m_c(X) = \mathbb{R}^2 \) and \( H^j(X) = 0 \) for \( j \neq n \) (if \( m = n - 1 \)).

2. \textit{(Plane with \( k \) holes)} Applying the results just seen and arguing by recurrence with Mayer-Vietoris using the same choice of open subsets as in the example at p. 25, it is immediate to prove that, for the plane with \( k \) holes \( \Pi_k = \mathbb{R}^2 \setminus \{x_1, \ldots, x_k\} \) (or, analogously, the sphere with \( k + 1 \) holes \( \mathbb{S}^2 \setminus \{y_1, \ldots, y_{k+1}\} \), which is diffeomorphic to it) the cohomology groups are \( H^0(\Pi_k) = \mathbb{R}, H^1(\Pi_k) = \mathbb{R}^k \), and \( H^2(\Pi_k) = 0 \), while the cohomology groups with compact support are \( H^0_c(\Pi_k) = \mathbb{R}, H^1_c(\Pi_k) = \mathbb{R}^k \), and \( H^2_c(\Pi_k) = \mathbb{R} \).

\(^{(87)}\) Note that one can assume \( \text{supp}(\epsilon) \subset |x - \varepsilon, x + \varepsilon| \) for any choice of \( \varepsilon > 0 \).
Notes on Algebraic Topology

(3) (Real projective space) The real projective space \( X = \mathbb{P}^n \) is the compact manifold defined as the quotient of \( \mathbb{B}^n \) by the relation identifying antipodal points on the boundary \( S^{n-1} \) (obtaining in this way the “hyperplane at infinity” \( X' \), diffeomorphic to \( \mathbb{P}^{n-1} \)). • Let us prove (as we already mentioned in the examples of p. 59) that the singular homology \( H_k(X, \mathbb{Z}) \) is isomorphic to \( \mathbb{Z} \) if \( k = 0 \) or if \( (k = n, n \text{ odd}) \); to \( \mathbb{Z}/2\mathbb{Z} \) if \( (0 < k < n, k \text{ odd}) \); and zero otherwise. For \( n = 1 \) this is true since \( \mathbb{P}^1 \) is diffeomorphic to \( S^1 \). Then the general case follows by induction on \( n \) by using the Mayer-Vietoris sequence (2.22) with \( U = X \setminus \{0\} \) (which deformation-retracts to \( X' \simeq \mathbb{P}^{n-1} \)) and \( V = X \setminus X' \simeq \mathbb{B}^n \) (contractible): note that \( U \cap V \) is homotopically equivalent to \( S^{n-1} \). To understand what could be a generator of the intermediate homology groups \( H_k(X, \mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z} \) for \( k \) odd, let us check \( G = H_1(\mathbb{P}^2, \mathbb{Z}) \). Any loop in \( \mathbb{P}^2 \) not joining antipodal (identified) points of the hyperplane at infinity \( X' \) is a 1-boundary (i.e. is a boundary of a surface), hence is trivial in \( G \): a generator of \( G \) is then shown in Figure 20. • The singular cohomology \( H^k(X, \mathbb{Z}) \) is isomorphic to \( \mathbb{Z} \) if \( k = 0 \) or if \( (k = n, n \text{ odd}) \); to \( \mathbb{Z}/2\mathbb{Z} \) if \( (0 < k \leq n, k \text{ even}) \); and zero otherwise. This can be proven either by recurrence with the Mayer-Vietoris sequence (2.22), or using the Universal coefficients formula (2.6) (recall that \( \text{Ext}^1(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \)). • The cohomology of de Rham \( H^k(X) \) is isomorphic to \( \mathbb{R} \) if \( k = 0 \) or if \( (k = n, n \text{ odd}) \), and zero otherwise: once more this can be proven by recurrence with the Mayer-Vietoris sequence (2.15). Note that the same result holds for the singular cohomology \( H^k(X, \mathbb{R}) \) (by the Universal coefficients formula (2.6)), and this coincidence is a general fact (see Theorem 2.6.9).

Figure 20: The real projective plane \( \mathbb{P}^2 \): the polygon, three classical embeddings in \( \mathbb{R}^3 \) (the cross-cap, the Roman surface and the Boy surface) and a generator of the first homology group \( H_1(\mathbb{P}^2, \mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z} \).