2.6 Finiteness, Poincaré duality, relations with singular cohomology

Let us show some other important consequences of the results of Mayer-Vietoris.

Definition 2.6.1. Let X be a $\mathcal{C}^\infty$ manifold without boundary, of dimension n. An open cover $\{U_\lambda : \lambda \in \Lambda\}$ of X is said to be *good* if any finite intersection $U_{\lambda_1} \cap \cdots \cap U_{\lambda_r}$ is diffeomorphic to $\mathbb{R}^n$. A manifold endowed with a finite good cover is said to be *of finite type.*

Lemma 2.6.2. *Any $\mathcal{C}^\infty$ manifold admits a good cover. In particular, any compact $\mathcal{C}^\infty$ manifold is of finite type.*

Proof. We give only the idea of the proof, referring to texts of differential geometry (e.g. Spivak [14]) for more details. Assign to X a riemannian metric[87]. $\langle \cdot\,,\cdot \rangle$, and choose a cover made by geodesically convex open subsets[88] with respect to $\langle \cdot\,,\cdot \rangle$: such open subsets being diffeomorphic to $\mathbb{R}^n$, as well as anyone of their non empty finite intersections, such a cover is good. □

Theorem 2.6.3. *The cohomology and the cohomology with compact support of a $\mathcal{C}^\infty$ manifold of finite type is finite dimensional.*

Proof. Let X be a $\mathcal{C}^\infty$ manifold of finite type, of dimension n, and let us prove the result for the cohomology (the proof for the cohomology with compact support is similar) by recurrence on the cardinality of a finite good cover of X. If X is diffeomorphic a $\mathbb{R}^n$ it is enough to recall Poincaré lemma; then, if $\{U_1, \dots, U_{p+1}\}$ is a good cover of X, set $U = U_1 \cup \cdots \cup U_p$ and $V = U_{p+1}$ (note that a good cover for $U \cap V$ is $\{U_1 \cap U_{p+1}, \dots, U_p \cap U_{p+1}\}$). From the exact sequence of Mayer-Vietoris

$$\cdots \longrightarrow H^{m-1}(U\cap V) \xrightarrow{\alpha_{m-1}} H^m(X) \xrightarrow{\beta_m} H^m(U) \oplus H^m(V) \longrightarrow \cdots$$

we get that $H^m(X) \simeq \ker(\beta_m) \oplus \mathrm{im}(\beta_m) \simeq \mathrm{im}(\alpha_{m-1}) \oplus \mathrm{im}(\beta_m)$: in particular, since $H^{m-1}(U\cap V)$, $H^m(U)$ and $H^m(V)$ have finite dimension by the inductive hypothesis, also $H^m(X)$ has finite dimension. □

From now on, X is a oriented manifold without boundary of finite type, of dimension n. By Theorem 2.6.3, we know that $H^\bullet(X)$ and $H^\bullet_c(X)$ have finite dimension.

Lemma 2.6.4. *There is a natural bilinear form*

$$\int_X : H^\bullet(X) \otimes_{\mathbb{R}} H_c^{n-\bullet}(X) \to \mathbb{R}, \qquad ([\omega],[\theta]) \mapsto \int_X (\omega \wedge \theta)\,. \tag{2.24}$$

Proof. This integration form is clearly bilinear; we must only check that it is well-posed. Namely, let $\omega \in Z^m(X)$ and $\theta \in Z_c^{n-m}(X)$: then, setting $\omega' = \omega + d\tau$ and $\theta' = \theta + d\psi$ one has $\omega' \wedge \theta' = \omega \wedge \theta + d\chi$ where $\chi = (-1)^m \omega \wedge \psi + \tau \wedge \theta + \tau \wedge d\psi \in \Omega_c^{n-1}(X)$, and then Stokes' theorem (since $\partial X = \varnothing$) implies that $\int_X (\omega \wedge \theta) = \int_X (\omega' \wedge \theta')$. □

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[87] Recall that a riemannian manifold is a $\mathcal{C}^\infty$ manifold endowed with a Riemannian metric, which is a global $\mathcal{C}^\infty$ section $\langle \cdot\,,\cdot \rangle$ of $\odot^2 T^*X$ (in other words, for any $x \in X$ it is assigned a metric $\langle \cdot\,,\cdot \rangle_x$ on T_xX such that, if A and B are two vector fields $\mathcal{C}^\infty$ on X —i.e., global $\mathcal{C}^\infty$ sections of TX— the function $x \mapsto \langle A(x), B(x) \rangle_x$ is in $\mathcal{C}^\infty(X)$). Any $\mathcal{C}^\infty$ manifold can be endowed with a riemannian structure: given an atlas $\{(U_\lambda, \varphi_\lambda) : \lambda \in \Lambda\}$, a riemannian metric $\langle \cdot\,,\cdot \rangle_\lambda$ on any U_λ and a partition of unity $\{\rho_\lambda : \lambda \in \Lambda\}$ subordinate to $\{U_\lambda : \lambda \in \Lambda\}$, consider $\sum_{\lambda\in\Lambda} \rho_\lambda \langle \cdot\,,\cdot \rangle_\lambda$.

[88] Any point of a riemannian manifold admits a geodesically convex neighborhood (recall that a subset of a riemannian manifold is called *geodesically convex* if any two points of it can be joint by a geodesic contained in it).

Recall that a bilinear form $b : V \otimes_{\mathbb{R}} W \to \mathbb{R}$ naturally induces a morphism $V \to W^*$, which sends $v \in V$ into $b(v, \cdot\,) \in W^*$; if both spaces have the same finite dimension, such a morphism is an isomorphism if and only if b is nondegenerate, i.e. if and only if $b(v, W) = 0$ implies $v = 0$, and $b(V, w) = 0$ implies $w = 0$.

Proposition 2.6.5. *The integration form* (2.24) *defines, for any pair of open subsets $U, V \subset X$, a morphism between the sequence of Mayer-Vietoris* (2.15) *and the dual sequence of Mayer-Vietoris with compact support* (2.18)*:*

$$\begin{tikzcd}
\cdots \to H^{m-1}(U\cap V) \arrow[r, "\delta"] \arrow[d, "\varepsilon_{m-1}\int_{U\cap V}"'] & H^m(U\cup V) \arrow[r, "r"] \arrow[d, "\varepsilon_m \int_{U\cup V}"'] & H^m(U)\oplus H^m(V) \arrow[r, "\mu"] \arrow[d, "\varepsilon_m(\int_U \oplus \int_V)"'] & H^m(U\cap V) \to \cdots \arrow[d, "\varepsilon_m \int_{U\cap V}"'] \\
\cdots \to H_c^{n-m+1}(U\cap V)^* \arrow[r, "\delta_c^*"] & H_c^{n-m}(U\cup V)^* \arrow[r, "\sigma^*"] & H_c^{n-m}(U)^* \oplus H_c^{n-m}(V)^* \arrow[r, "\mu_c^*"] & H_c^{n-m}(U\cap V)^* \to \cdots
\end{tikzcd}$$

where $\varepsilon_m = (-1)^{\left[\frac{m-1}{2}\right]}$, δ and δ_c (resp. r, μ, σ, μ_c) *are the maps induced by the morphisms of coboundary* (resp. *by restriction, restriction and difference, extension by zero and sum, extension by zero and embedding map with sign*) *and $(\,\cdot\,)^*$ indicates the trasposed map.*

Proof. The verification that the second and third square of the diagram commute is immediate from the definitions.[(89)] As for the first square, let $\omega \in Z^{m-1}(U\cap V)$ and $\tau \in Z_c^{n-m}(U\cup V)$. Recalling (2.16) and (2.19) and that $\delta\omega$ has the support in $U\cap V$, and observing that $d(\rho_V\omega) = (d\rho_V)\wedge\omega$ and $d(\rho_V\tau) = (d\rho_V)\wedge\tau$, one has $\int_{U\cup V}\delta(\omega)\wedge\tau = \int_{U\cap V}\delta(\omega)\wedge\tau = -\int_{U\cap V}(d\rho_V)\wedge\omega\wedge\tau$ and $\delta_c^*\left(\int_{U\cap V}\omega\wedge\,\cdot\,\right)(\tau) = \int_{U\cap V}\omega\wedge(\delta_c\tau) = \int_{U\cap V}\omega\wedge(d\rho_V)\wedge\tau = (-1)^{m-1}\int_{U\cup V}(d\rho_V)\wedge\omega\wedge\tau$, and hence there is a difference of sign given by $(-1)^m$; the presence of the factor ε_m as in the diagram ensures the equality, since one has always $m + \left[\frac{m-2}{2}\right] \equiv \left[\frac{m-1}{2}\right]$ modulo 2. □

Lemma 2.6.6. (Five Lemma) *Consider a commutative diagram of abelian groups and morphisms*

$$\begin{tikzcd}
A_1 \arrow[r, "f_1"] \arrow[d, "\alpha"] & A_2 \arrow[r, "f_2"] \arrow[d, "\beta"] & A_3 \arrow[r, "f_3"] \arrow[d, "\gamma"] & A_4 \arrow[r, "f_4"] \arrow[d, "\delta"] & A_5 \arrow[d, "\varepsilon"] \\
B_1 \arrow[r, "g_1"] & B_2 \arrow[r, "g_2"] & B_3 \arrow[r, "g_3"] & B_4 \arrow[r, "g_4"] & B_5
\end{tikzcd}$$

where the rows are exact sequences and $\alpha, \beta, \delta, \varepsilon$ are isomorphisms. Then also γ is an isomorphism.

Proof. Exercise. □

Theorem 2.6.7. (Poincaré duality) *Let X be a oriented manifold* without boundary*, of dimension n. Then the bilinear integration form* (2.24) *is nondegenerate, and hence one has*

$$H^\bullet(X) \simeq H_c^{n-\bullet}(X)^*.$$

(Note that the hypothesis of finite type has been removed.)

Proof. We start by assuming thate X is of finite type. Let $U, V \subset X$ be two open subset. By Proposition 2.6.5 and Lemma 2.6.6, if the statement holds for U, V and $U\cap V$ it will hold also for $U\cup V$. Hence it shall be once more enough to argue by recurrence on the cardinality of a finite good cover of X, exactly as in the proof of Theorem 2.6.3. In the general case, we refer to a text of topology differential, for example Greub-Halperin-Vanstone [7]. □

[(89)] Recall also that, given two vector spaces V_1 and V_2 on a field k, then $V_1^* \oplus V_2^*$ is naturally identified with $(V_1 \oplus V_2)^*$ by sending $(\alpha, \beta) \in V_1^* \oplus V_2^*$ into $\gamma \in (V_1 \oplus V_2)^*$ defined as $\gamma(v, w) = \alpha(v) + \beta(w)$.

Example. Let $X = \bigsqcup_{i\in\mathbb{N}} X_i$, where the X_i are oriented $\mathcal{C}^\infty$ manifolds of finite type. One then has $H^\bullet(X) = \prod_{i\in\mathbb{N}} H^\bullet(X_i)$, while $H_c^\bullet(X) = \bigoplus_{i\in\mathbb{N}} H_c^\bullet(X_i)$, which implies $H_c^\bullet(X)^* = \prod_{i\in\mathbb{N}} H_c^\bullet(X_i)^*$ (the dual of a direct sum is isomorphic to the direct product of the duals, see Appendix A.2): since $H^\bullet(X_i) \simeq H_c^{n-\bullet}(X_i)^*$, this agrees with Poincaré duality. Note that, if X is of finite type, it holds also $H_c^\bullet(X) \simeq H^{n-\bullet}(X)^*$ (the bidual of a vector space of finite dimension is canonically isomorphic to the space itself), but this is false in general: as in the case $X = \bigsqcup_{i\in\mathbb{N}} X_i$ above, the dual of a direct product is not necessarily isomorphic to the direct sum of the duals.

Hence let us show that the generator of the cohomology with compact support of top degree of a orientable manifold is a "bump form" as we saw for $\mathbb{S}^1$ (see the example at p. 67).

Corollary 2.6.8. *If X is a connected oriented manifold without boundary of dimension n, one has*

$$H_c^n(X) = \mathbb{R};$$

a generator of $H_c^n(X)$ will be a n-form on X with compact support, as small as one likes, with total integral 1.

Proof. Recall that $H^0(X) = \mathbb{R}$, then apply Poincaré duality. Finally, a n-form of the type described above is surely non zero in $H_c^n(X)$ (otherwise, since $\partial X = \varnothing$, its total integral should be zero). □

Remark 2.6.9. Given a manifold without boundary X, the fact that Poincaré duality fails to hold is one of the simplest methods for proving that such manifold is not orientable. For example, if one has a compact manifold without boundary, we said above that it has finite dimensional cohomology (in this case obviously coinciding with the cohomology with compact support), and so if such manifold is orientable then Poincaré duality implies that such dimensions must be symmetric (i.e. $\dim H^j(X) = \dim H^{n-j}(X)$ for $j = 0, \dots, n = \dim(X)$). But, for example, this does not happen for the Möbius band, the Klein bottle and the real projective of even dimension (see Exemples at pp. 77 and 88), which therefore can not be orientable.

Let us show that, as repeatedly announced, the cohomology of de Rham is isomorphic to the singular cohomology with coefficients in $\mathbb{R}$.

Theorem 2.6.10. (de Rham) *Let X be a $\mathcal{C}^\infty$ manifold. For any open subset $U \subset X$ one has a isomorphism in $\mathfrak{Mod}_{\deg}(\mathbb{R})$*

$$H^\bullet(U) \xrightarrow{\sim} H^\bullet(U, \mathbb{R}). \tag{2.25}$$

Proof. We show the sketch of the proof, referring for example to Bredon [2] for more details. Consider the singular simplexes $\mathcal{C}^\infty$ in X (i.e., those simplexes $\sigma : \Delta_k \to X$ which extend to $\mathcal{C}^\infty$ functions in some open neighborhood of Δ_k) and the $\mathcal{C}^\infty$ chains generated by them: since their boundary is a $\mathcal{C}^\infty$ chain, the $\mathcal{C}^\infty$ chains form a subcomplex of (2.2). Let us dualize it, so obtaining the complex of sheaves $S^\bullet_{X,\mathcal{C}^\infty}(\mathbb{R})$ of $\mathcal{C}^\infty$ singular cohomology. Since a $\mathcal{C}^\infty$ k-form can be integrated on a compact $\mathcal{C}^\infty$ submanifold of dimension k with boundary, for any k it is defined a morphism of sheaves $\Omega^k_X \to S^k_{X,\mathcal{C}^\infty}(\mathbb{R})$; thanks to Stokes' theorem, such morphism commutes with the differential, giving rise to a morphism of complexes of sheaves $\Omega^\bullet_X \to S^\bullet_{X,\mathcal{C}^\infty}(\mathbb{R})$ and hence, given any open subset $U \subset X$, to a morphism of graded $\mathbb{R}$-modules $\psi : H^\bullet(U) \to H^\bullet_{\mathcal{C}^\infty}(U, \mathbb{R})$, where $H^\bullet_{\mathcal{C}^\infty}(U, \mathbb{R})$ is the singular cohomology with coefficients in $\mathbb{R}$ obtained considering only the simplexes $\mathcal{C}^\infty$. The already used inductive argument, based on Mayer-Vietoris principle, Poincaré lemma and the Five Lemma, allows one to show that ψ is a isomorphism in the case where X is of finite type. The compatibility of both cohomologies with disjoint unions allows to remove the hypothesis of finite type; finally, one proves analogously that the natural morphism $\chi : H^\bullet(U, \mathbb{R}) \to H^\bullet_{\mathcal{C}^\infty}(U, \mathbb{R})$ is a isomorphism. The desired isomorphism is $\chi^{-1} \circ \psi$. □

Remark 2.6.11. Theorem 2.6.10 provides a direct proof that two different cohomology theories (in this case, de Rham cohomology and singular cohomology) give the same results; other similar comparisons are Theorem 2.1.19 (between cellular homology and singular homology) and Corollary 2.9.6 (between Čech cohomology and de Rham cohomology). All these results provide an *ad hoc* approach for each case; an alternative —and, by many aspects, preferable— *axiomatic* approach is to state a few axioms that a reasonable homology/cohomology theory should satisfy (for example the invariance under homotopy equivalence), and then to prove that all homology/cohomology theories satisfying those axioms must give the same result at least on a sufficiently large class of spaces, e.g. on CW complexes. For this approach we refer to the classical book of Eilenberg and Steenrod [4].

2.7 Degree

Let X and Y be oriented connected manifolds without boundary of dimension n, $f : X \to Y$ a proper $\mathcal{C}^\infty$ map. There is a pull-back morphism

$$H^n f^* : H^n_c(Y) \to H^n_c(X), \qquad H^n f^*([\omega]) = [f^*\omega].$$

We have seen (Corollary 2.6.8) that $H^n_c(X) = \mathbb{R}$ and $H^n_c(Y) = \mathbb{R}$.

Definition 2.7.1. The *degree* of f is $\deg(f) = \int_X f^*\omega$, where $[\omega]$ is a generator of $H^n_c(Y)$ with $\int_Y \omega = 1$.

Proposition 2.7.2. *The real number* $\deg(f)$ *is well-posed and invariant under* proper *homotopies.*(90)

Proof. The well-posedness follows from Stokes' theorem: if $\omega' = \omega + d\sigma$, one has $\int_X f^*\omega' = \int_X f^*\omega$ because $\int_X f^* d\sigma = \int_X d(f^*\sigma) = \int_{\partial X} f^*\sigma = 0$. For the proof of the invariance under proper homotopies we follow the trace of Guillemin-Pollack [8, pag. 189] in the case of compact manifolds. (1) In general, let S and T be oriented manifolds without boundary of dimension k with S boundary of a manifold W, and let $u : S \to T$ be a proper map which has a smooth and proper extension to all of W (i.e. there exists a smooth and proper map $U : W \to T$ such that u is the restriction of U to $S = \partial W$): then $\int_S u^*\omega = 0$ for any k-form ω with compact support on T. Namely, since $U = u$ on S, by Stokes one has $\int_S u^*\omega = \int_{\partial W} U^*\omega = \int_W d(U^*\omega) = \int_W U^*(d\omega) = 0$ (because ω is a k-form on the manifold T of dimension k, hence $d\omega = 0$). (2) Now let $f, g : X \to Y$ be two proper maps which are homotopic by a proper homotopy $h : X \times I \to Y$, with $f = h_0$ and $g = h_1$. The map $\partial h : \partial(X \times I) = (X \times \{1\}) - (X \times \{0\}) \to Y$ (where the "minus" means "union with opposite orientation"), identifying $X \times \{1\}$ and $X \times \{0\}$ with X, operates on $X \times \{0\}$ as f and on $X \times \{1\}$ as g. Applying (1) to $S = (X \times \{1\}) - (X \times \{0\})$, $T = Y$, $W = X \times I$, $U = h$ and $u = \partial h$ one then has $0 = \int_{(X\times\{1\})-(X\times\{0\})} (\partial h)^*(\omega) = \int_{X\times\{1\}} (\partial h)^*(\omega) - \int_{X\times\{0\}} (\partial h)^*(\omega) \simeq \int_X g^*\omega - \int_X f^*\omega$. $\square$

Remark 2.7.3. Note that, by the invariance of the degree, it is important that *the homotopy be a proper map* (of course this does not cause any problem in the case where X and Y are compact manifolds). For example, let $X = Y = \mathbb{R}$, $f = \mathrm{id}_\mathbb{R}$ and $g = -\mathrm{id}_\mathbb{R}$; both f and g are proper maps, but there does not exist any proper homotopy between them.(91) In fact it turns out that these maps have different degree (as we shall see soon, one has $\deg(\mathrm{id}_\mathbb{R}) = 1$ and $\deg(-\mathrm{id}_\mathbb{R}) = -1$).

Now we try to better understand $\deg(f)$. Given a morphism of $\mathcal{C}^\infty$ manifolds $f : X \to Y$, a *critical point* for f is a point of X in which the differential is not surjective, and a *critical value* for f is the image of some critical point. A point of Y which is not a critical value for f is said *regular value* for f: in particular, any $y \in Y \setminus f(X)$ is a regular value for f. Let us recall an important theorem which will be useful in the sequel, referring for example to Guillemin-Pollack [8] for the proof.

Theorem 2.7.4. (Sard) *The set of critical values for f has measure zero*(92) *in Y.*

(90) i.e., if $g : X \to Y$ is another proper $\mathcal{C}^\infty$ map and $h : X \times I \to Y$ is a *proper* homotopy between f and g, then $\deg(f) = \deg(g)$.

(91) A continuous and proper map φ from $\mathbb{R}$ to itself must necessarily satisfy either $\lim_{x\to+\infty} \varphi(x) = +\infty$ or $\lim_{x\to+\infty} \varphi(x) = -\infty$; would there exist a proper homotopy $h : \mathbb{R} \times I \to \mathbb{R}$ with $h_0 = \mathrm{id}_\mathbb{R}$ and $h_1 = -\mathrm{id}_\mathbb{R}$, since $h_0 = \mathrm{id}_\mathbb{R}$ by continuity it should be $\lim_{x\to+\infty} h(x,t) = +\infty$ for any $t \in I$, but this is in contrast to $h_1 = -\mathrm{id}_\mathbb{R}$. On the other hand, a (non proper) homotopy between $\mathrm{id}_\mathbb{R}$ and $-\mathrm{id}_\mathbb{R}$ is the affine one $h(x,t) = (1-2t)x$: note that $h^{-1}(0) = (\mathbb{R} \times \{\frac{1}{2}\}) \cup (\{0\} \times I)$ is non compact.

(92) A subset $B \subset Y$ has *measure zero* if there exists a countable family $\{(V_m, \psi_m) : m \in \mathbb{N}\}$ of charts in Y such that $B \subset \bigcup_{m\in\mathbb{N}} V_m$ and $\psi_m(V_m \cap B)$ has Lebesgue measure zero in $\mathbb{R}^n$ for any $m \in \mathbb{N}$.

If f is a $\mathcal{C}^\infty$ diffeomorphism from a connected open neighborhood U of $x_0 \in X$ to a (connected) open neighborhood V of $f(x_0) \in Y$, one defines $\mathrm{sign}(f)(x_0) = \pm 1$ according to the fact that $df(x_0) : T_{x_0}X \xrightarrow{\sim} T_{f(x_0)}Y$ preserves or reverses the orientations. The function $\mathrm{sign}(f)$ is constant on U; if $(U_\lambda, \varphi_\lambda)$ (resp. (V_μ, ψ_μ)) is a oriented local chart in X (resp. Y) containing x_0 (resp. $f(x_0)$), one has $\mathrm{sign}(f)(x_0) = \mathrm{sign}(\psi_\mu \circ f \circ \varphi_\lambda^{-1})(u_0)$, where $\varphi_\lambda(x_0) = u_0 \in \mathbb{R}^n$. From the affine case we easily get that if $\omega \in \Omega_c^n(V)$ then one has $f^*\omega \in \Omega_c^n(U)$ and for any $x_0 \in U$ it holds

$$\int_U f^*\omega = \mathrm{sign}(f)(x_0) \int_V \omega. \tag{2.26}$$

Proposition 2.7.5. *If f is not surjective then* $\deg(f) = 0$ *, and in general it holds*

$$\deg(f) \;=\; \sum_{x \in f^{-1}(y)} \mathrm{sign}(f)(x) \quad \in \mathbb{Z},$$

where y is any regular value for f.

Proof. We start by proving that $\deg(f) = 0$ if f is not surjective. Let $y \in Y \setminus f(X)$ and let V be an open neighborhood of y with $V \cap f(X) = \varnothing$:[93] choosing a generator $[\omega]$ of $H_c^n(Y)$ with the support of ω compact in V and $\int_V \omega = 1$ (see Corollary 2.6.8), one has $f^*\omega = 0$ and hence $\deg(f) = 0$. So let f be surjective, and let y be a regular value for f (which exists by Sard theorem): by the Implicit Function theorem f is a local homeomorphism in the points of $f^{-1}(y)$ (in these points $df(x)$ is surjective, hence an isomorphism), therefore the fiber $f^{-1}(y)$ is discrete, and even finite (f is proper). Let then $f^{-1}(y) = \{x_1, \dots, x_k\}$, U_{x_j} an open neighborhood of x_j with $f|_{U_{x_j}} : U_{x_j} \xrightarrow{\sim} V_{x_j} := f(U_{x_j})$, $V = \bigcap_{j=1}^k V_{x_j}$ (open neighborhood of y), $U_j = f^{-1}(V) \cap U_{x_j}$ (open neighborhood of x_j): one has $f|_{U_j} : U_j \xrightarrow{\sim} f(U_j) = V$ for any $j = 1, \dots, k$. If $[\omega]$ is a generator of $H_c^n(Y)$ with support of ω compact in V and $\int_V \omega = 1$, then $f^*\omega$ is a n-form with compact support in $\bigsqcup_{j=1}^k U_j$ (see Figure 21): recalling (2.26), one has $\int_X f^*\omega = \sum_{j=1}^k \int_{U_j} f^*\omega = \sum_{j=1}^k \mathrm{sign}(f|_{U_j})(x_j)$. □

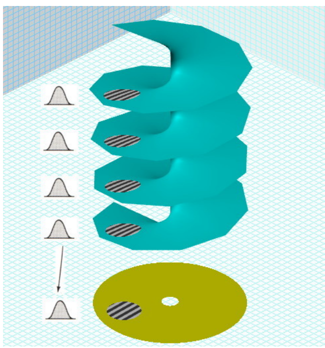


Figure 21: Pulling back a bump form by a local diffeomorphism.

Example. **(1)** Given $n \in \mathbb{N}$, consider $f_n : \mathbb{C} \to \mathbb{C}$, $f_n(z) = z^n$. We saw that f_n is a n-sheet covering of $\mathbb{C}^\times$, hence a proper map; the regular values are those of $\mathbb{C}^\times$. Let us choose for example 1, and consider $f_n^{-1}(1) = \{w_j : 1 \le j \le n\}$, the nth roots of unity. Since (see Remark 1.4.1) $J_{f_n}(w_j) = |f_n'(w_j)|^2 = n^2 > 0$, the differentials $df_n(w_j)$ preserve the orientations: hence $\deg(f_n) = 1 + \cdots + 1 = n$. The same holds for

[93] Any proper function between metrizable spaces is closed (exercise).

the restriction $f_n|_{\mathbb{S}^1} : \mathbb{S}^1 \to \mathbb{S}^1$. On the other hand, $\phi_n := f_n|_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ has degree 0 for n even (it enough to observe that is not surjective; or that $\phi_n^{-1}(1) = \{\mp 1\}$ and $\phi_n'(\mp 1) = \mp n \lessgtr 0$ so that $\deg(\phi_n) = (-1) + (+1) = 0$), and degree 1 for n odd (namely $\phi_n^{-1}(1) = \{1\}$, and $\phi_n'(1) = n > 0$). **(2)** Consider $f, g : \mathbb{S}^n \to \mathbb{S}^n$ ($n \geq 2$), with $f = \mathrm{id}_{\mathbb{S}^n}$ and $g = -\mathrm{id}_{\mathbb{S}^n}$ (the antipodal map): they are auto-diffeomorphisms, hence all $y \in \mathbb{S}^n$ are regular values. Let us choose for example $y = N = e_{n+1}$ (the North pole). Since $\mathrm{sign}(f)(N) = 1$ (namely $df(N) = \mathrm{id}_{T_N\mathbb{S}^n}$), one has $\deg(f) = 1$; on the other hand, setting $S = -N$ (the South pole), $dg(S)$ sends the positive basis $(-1)^{n+1}(e_1, \dots, e_n)$ of $T_S\mathbb{S}^n$ (see the example at p. 71) into the basis $(-1)^{n+1}(-e_1, \dots, -e_n) = -(e_1, \dots, e_n)$ of $T_N\mathbb{S}^n$. Recalling that a positive basis of $T_N\mathbb{S}^n$ is $(-1)^n(e_1, \dots, e_n)$, this implies that $\deg(g) = (-1)(-1)^n = (-1)^{n+1}$. It follows that, if n is even, by Proposition 2.7.2 $f = \mathrm{id}_{\mathbb{S}^n}$ can not be smoothly homotopic to $g = -\mathrm{id}_{\mathbb{S}^n}$, and hence these maps can not be homotopic at all, again by Whitney approximation theorem on smooth manifolds (we remind the cited reference Lee [12, Theorems 10.21 and 10.22]). In particular, this provides an answer to the pending question about the "combing" of spheres (Corollary 1.1.20).

2.8 Künneth formula

Let us study the cohomology of the product of manifolds.

Let X and Y be two $\mathcal{C}^\infty$ manifolds, $\pi : X \times Y \to X$ and $\rho : X \times Y \to Y$ the projections. Viewing $\Omega^\bullet(X)$ and $\Omega^\bullet(Y)$ as graded vector spaces, $\Omega^\bullet(X) \otimes_{\mathbb{R}} \Omega^\bullet(Y)$ has a structure of graded vector space given by $(\Omega^\bullet(X) \otimes_{\mathbb{R}} \Omega^\bullet(Y))^m = \bigoplus_{j\in\mathbb{Z}} \big(\Omega^j(X) \otimes_{\mathbb{R}} \Omega^{m-j}(Y)\big)$ (see (A.2)), and one obtains a morphism of graded vector spaces by setting

$$\Omega^\bullet(X) \otimes_{\mathbb{R}} \Omega^\bullet(Y) \to \Omega^\bullet(X \times Y), \qquad \omega \otimes \theta \mapsto \pi^*\omega \wedge \rho^*\theta.$$

Now, if $d\omega = 0$ and $d\theta = 0$ then $d(\pi^*\omega \wedge \rho^*\theta) = d(\pi^*\omega) \wedge \rho^*\theta + (-1)^{\deg(\omega)}\pi^*\omega \wedge d(\rho^*\theta) = \pi^*(d\omega) \wedge \rho^*\theta + (-1)^{\deg\omega}\pi^*\omega \wedge \rho^*(d\theta) = 0$; moreover, if $d\omega = 0$ and $\theta = d\tau$ then $\pi^*\omega \wedge \rho^*\theta = (-1)^{\deg(\omega)} d(\pi^*\omega \wedge \rho^*\tau)$ and similarly if $\omega = d\sigma$ and $d\theta = 0$ then $\pi^*\omega \wedge \rho^*\theta = d(\pi^*\sigma \wedge \rho^*\theta)$. Hence it is induced a map of cohomology[(94)]

$$\psi : H^\bullet(X) \otimes_{\mathbb{R}} H^\bullet(Y) \to H^\bullet(X \times Y),$$

which we now show to be an isomorphism if at least one of the two manifolds is of finite type.

Theorem 2.8.1. (Künneth formula) *Let X and Y be two $\mathcal{C}^\infty$ manifolds, and assume that X is of finite type. Then*

$$H^\bullet(X \times Y) \simeq H^\bullet(X) \otimes_{\mathbb{R}} H^\bullet(Y),$$

i.e.

$$H^m(X \times Y) \simeq \bigoplus_{j\in\mathbb{Z}} \big(H^j(X) \otimes_{\mathbb{R}} H^{m-j}(Y)\big) \qquad \textit{for any } m \in \mathbb{Z}.$$

Proof. We shall use once more the argument of induction on the cardinality of a finite good cover of X, the base case (i.e. $X \simeq \mathbb{R}^n$) being true by Poincaré lemma. Let $U, V \subset X$ be open, and fix any integer $m \in \mathbb{Z}$. Applying to the sequence of Mayer-Vietoris (2.15) the exact functor $\cdot \otimes_{\mathbb{R}} H^{m-j}(Y)$ and considering the direct sum for $j \in \mathbb{Z}$, one obtains the exact sequence

$$\cdots \to \bigoplus_{j\in\mathbb{Z}} \big(H^{j-1}(U\cap V) \otimes_{\mathbb{R}} H^{m-j}(Y)\big) \to \bigoplus_{j\in\mathbb{Z}} \big(H^{j}(U\cup V) \otimes_{\mathbb{R}} H^{m-j}(Y)\big) \to \bigoplus_{j\in\mathbb{Z}} \Big[\big(H^{j}(U) \otimes_{\mathbb{R}} H^{m-j}(Y)\big) \oplus$$
$$\oplus \big(H^{j}(V) \otimes_{\mathbb{R}} H^{m-j}(Y)\big)\Big] \to \bigoplus_{j\in\mathbb{Z}} \big(H^{j}(U\cap V) \otimes_{\mathbb{R}} H^{m-j}(Y)\big) \to \bigoplus_{j\in\mathbb{Z}} \big(H^{j+1}(U\cup V) \otimes_{\mathbb{R}} H^{m-j}(Y)\big) \to \cdots$$

The commutativity of the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \bigoplus_{j\in\mathbb{Z}} \big(H^j(U\cup V) \otimes_{\mathbb{R}} H^{m-j}(Y)\big) \arrow[r] \arrow[d, "\psi^m_{U\cup V}"] & \bigoplus_{j\in\mathbb{Z}} \Big[\big(H^j(U) \otimes_{\mathbb{R}} H^{m-j}(Y)\big) \oplus \big(H^j(V) \otimes_{\mathbb{R}} H^{m-j}(Y)\big)\Big] \arrow[r] \arrow[d, "\psi^m_U \oplus \psi^m_V"] & \bigoplus_{j\in\mathbb{Z}} \big(H^j(U\cap V) \otimes_{\mathbb{R}} H^{m-j}(Y)\big) \arrow[r, "\oplus_j \delta_j"] \arrow[d, "\psi^m_{U\cap V}"] & \bigoplus_{j\in\mathbb{Z}} \big(H^{j+1}(U\cup V) \otimes_{\mathbb{R}} H^{m-j}(Y)\big) \arrow[r] \arrow[d, "\psi^{m+1}_{U\cup V}"] & \cdots \\
\cdots \arrow[r] & H^m((U\cup V)\times Y) \arrow[r] & H^m(U\times Y) \oplus H^m(V\times Y) \arrow[r] & H^m((U\cap V)\times Y) \arrow[r, "\delta"] & H^{m+1}((U\cup V)\times Y) \arrow[r] & \cdots
\end{tikzcd}$$

(94) Alternatively, $\Omega^\bullet(X) \otimes_{\mathbb{R}} \Omega^\bullet(Y)$ is a complex by considering the differential induced from those of $\Omega^\bullet(X)$ and $\Omega^\bullet(Y)$, i.e. $d(\omega \otimes \theta) = d\omega \otimes \theta + (-1)^{\deg(\omega)}\omega \otimes d\theta$; for what we have just seen, such differential commutes with the morphism $\omega \otimes \theta \mapsto \pi^*\omega \wedge \rho^*\theta$ which therefore, as morphism of complexes, induces a morphism between the cohomologies; then, since $\otimes_{\mathbb{R}}$ is an exact functor, we get $H^\bullet(\Omega^\bullet(X) \otimes_{\mathbb{R}} \Omega^\bullet(Y)) \simeq H^\bullet(X) \otimes_{\mathbb{R}} H^\bullet(Y)$.

follows immediately from the definitions for what concerns the first two squares. As for the third (which contains the morphism of coboundary δ), let $[\omega]\otimes[\theta]\in H^j(U\cap V)\otimes_{\mathbb{R}} H^{m-j}(Y)$ and note that, given a partition of unity $\{\rho_U,\rho_V\}$ in $U\cup V$ subordinate to $\{U,V\}$, the pull-backs $\{\pi^*\rho_U,\pi^*\rho_V\}=\{\rho_U\circ\pi,\rho_V\circ\pi\}$ are a partition of unity in $(U\cup V)\times Y$ subordinate to $\{U\times Y,V\times Y\}$: recalling the definitions, one hence has $\delta(\psi^m_{U\cap V}(\omega\otimes\theta))=\delta(\pi^*\omega\wedge\rho^*\theta)=d((\pi^*\rho_U)(\pi^*\omega\wedge\rho^*\theta))=d((\pi^*\rho_U)(\pi^*\omega)\wedge\rho^*\theta)=d(\pi^*(\rho_U\omega))\wedge\rho^*\theta=\pi^*d(\rho_U\omega)\wedge\rho^*\theta=\pi^*\delta_j(\omega)\wedge\rho^*\theta=\psi^{m+1}_{U\cup V}(\delta_j(\omega\otimes\theta))$. Now, by the Five Lemma 2.6.6 the statement is true for $U\cup V$ if it is true separately for U, V and $U\cap V$, and the induction proceeds. □

Remark 2.8.2. It is also possible to show that the Künneth formula still holds with the slightly weaker hypothesis that $H^\bullet(X)$ be finite dimensional.

The morphism $\psi: H^\bullet(X)\otimes_{\mathbb{R}} H^\bullet(Y)\to H^\bullet(X\times Y)$ induces a morphism

$$\psi_c: H^\bullet_c(X)\otimes_{\mathbb{R}} H^\bullet_c(Y)\to H^\bullet_c(X\times Y),$$

which is actually a isomorphism, without further hypotheses:

Theorem 2.8.3. (Künneth formula with compact support) *Let X and Y be two $\mathcal{C}^\infty$ manifolds. Then*

$$H^\bullet_c(X\times Y)\simeq H^\bullet_c(X)\otimes_{\mathbb{R}} H^\bullet_c(Y),$$

i.e.

$$H^m_c(X\times Y)\simeq\bigoplus_{j\in\mathbb{Z}}\left(H^j_c(X)\otimes_{\mathbb{R}} H^{m-j}_c(Y)\right)\qquad \textit{for any } m\in\mathbb{Z}.$$

Proof. If one of the two manifols is of finite type we can argue by induction as in the proof just done, the base case being true by the Poincaré lemma with compact support.(95) For the proof of the general case, which uses arguments similar to those previously seen in the proof of Künneth formula and also some more refined considerations of topology, we refer for example to Greub-Halperin-Vanstone [7]. □

Definition 2.8.4. If X is a $\mathcal{C}^\infty$ manifold of dimension n with $H^\bullet(X)$ (resp. $H^\bullet_c(X)$) of finite dimension, the number

$$b^j(X)=\dim_{\mathbb{R}} H^j(X)\qquad (\text{resp. } b^j_c(X)=\dim_{\mathbb{R}} H^j_c(X))$$

is called the *jth Betti number* (resp. *jth Betti number with compact support*) of X, and

$$P_X(t)=\sum_{j=0}^n b^j(X)\,t^j\qquad (\text{resp. } P_{X,c}(t)=\sum_{j=0}^n b^j_c(X)\,t^j\ )$$

the *Poincaré polynomial* (resp. *Poincaré polynomial with compact support*) of X.

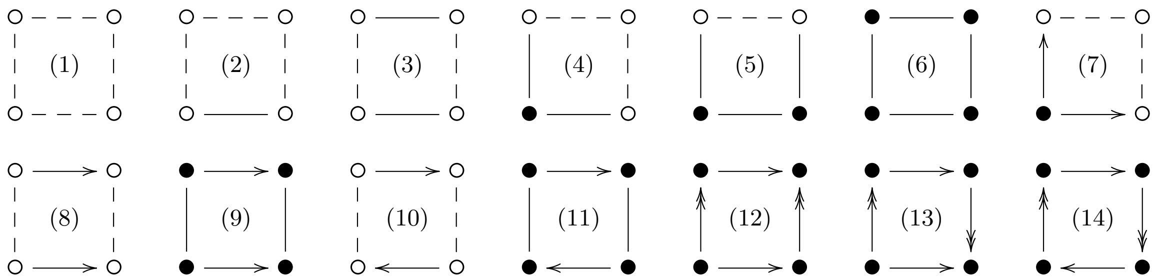
From Künneth formulas we get the following

Corollary 2.8.5. *If X and Y have finite dimensional cohomology, then*

$$P_{X\times Y}(t)=P_X(t)P_Y(t),\qquad P_{X\times Y,c}(t)=P_{X,c}(t)P_{Y,c}(t).$$

(95) Note that if X and Y are of finite type and orientable, the result can also be obtained by Poincaré duality from the Künneth formula.

Examples. **(1)** Consider $\mathbb{T}^n = (\mathbb{S}^1)^n$ (the n-dimensional torus): one has $P_{\mathbb{T}^n}(t) = P_{\mathbb{S}^1}(t)^n = (1+t)^n$, hence $b^j(\mathbb{T}^n) = \binom{n}{j}$. More generally, let $n_1, \dots, n_k \in \mathbb{N}$, and $M = \prod_{j=1}^k \mathbb{S}^{n_j}$: then $P_M(t) = \prod_{j=1}^k (1 + t^{n_j})$. **(2)** Let X_k be the closed unitary disc of $\mathbb{R}^2$ with k internal points $x_1, \dots, x_k$ removed. Since X is a strong deformation retract of the plane with k holes Π_k, it has the same cohomology of the latter (i.e., as we have seen, $H^0(X_k) = \mathbb{R}$, $H^1(X_k) = \mathbb{R}^k$, and $H^2(X_k) = 0$). As for the cohomology with compact support (in this case, due to the presence of boundary points, it is not possible to apply Poincaré duality), let us argue by induction on k in order to prove that $H^0_c(X_k) = 0$, $H^1_c(X_k) = \mathbb{R}^{k-1}$ and $H^2_c(X_k) = 0$. If $k = 1$ one has $X_1 \simeq \mathbb{R}_{\geq 0} \times \mathbb{S}^1$, and hence by Künneth one has $H^\bullet_c(X_1) = (0,0,0)$; then if $k \geq 2$ we can apply the sequence of Mayer-Vietoris with compact support for $X = X_{k-1}$, $U = X_k$ and V a small open disc which covers x_k but not $x_1, \dots, x_{k-1}$ (note that V is diffeomorphic to $\mathbb{R}^2$ and $U \cap V$ to $\mathbb{R} \times \mathbb{S}^1$, hence $H^\bullet_c(V) = (0,0,\mathbb{R})$ and $H^\bullet_c(U \cap V) = (0, \mathbb{R}, \mathbb{R})$). **(3)** Let us consider the following polygons, which give rise to surfaces with or without boundary. Anytime we say "bordered", we mean that the edges represented with a continuous line and without arrows belong to the surface, while the dashed edges do not; a white (resp. black) vertex does not belong (resp. belongs) to the surface; finally, the edges marked with arrows of the same type should be identified by respecting the direction of the arrows (this is equivalent to passing to a quotient space where the pair of sticked points —vertexes included— are identified to a single point). Of course, in the case of presence of non smooth boundary points, to make such surfaces into $\mathcal{C}^\infty$ ones it will be enough to make the wedged points into smooth ones through a suitable homeomorphism (like the one which makes a square homeomorphic to a ball).



(1) *Square without boundary*: homeomorphic to $\mathbb{R}^2$; orientable manifold without boundary, contractible; the Betti numbers are $b^\bullet = (1,0,0)$, and those with compact support are $b^\bullet_c = (0,0,1)$.

(2) *Square bordered at one edge*: homeomorphic to $\mathbb{R}_{\geq 0} \times \mathbb{R}$; orientable manifold with boundary, contractible; $b^\bullet = (1,0,0)$, $b^\bullet_c = (0,0,0)$ (Künneth, or Poincaré lemma).

(3) *Square bordered on two opposed edges*: homeomorphic to $I \times \mathbb{R}$; orientable manifold with boundary, contractible; $b^\bullet = (1,0,0)$, $b^\bullet_c = (0,1,0)$ (Künneth, or Poincaré lemma).

(4) *Square bordered on two contiguous edges*: homeomorphic to $(\mathbb{R}_{\geq 0})^2$; orientable manifold with boundary, contractible; $b^\bullet = (1,0,0)$, $b^\bullet_c = (0,0,0)$ (Künneth).

(5) *Square bordered on three edges*: homeomorphic to $I \times \mathbb{R}_{\geq 0}$; orientable manifold with boundary, contractible; $b^\bullet = (1,0,0)$, $b^\bullet_c = (0,0,0)$ (Künneth). (Note that (4) and (5) are homeomorphic.)

(6) *Bordered square*: homeomorphic to I^2, or to $\mathbb{B}^2$; compact orientable manifold with boundary, contractible; $b^\bullet = b^\bullet_c = (1,0,0)$.

(7) *"Roast chestnut wrapping"*: homeomorphic to $\mathbb{R}^2$; orientable manifold without boundary, contractible; $b^\bullet = (1,0,0)$, $b^\bullet_c = (0,0,1)$.

(8) *Annulus without boundary*: homeomorphic to $\mathbb{S}^1 \times \mathbb{R}$; orientable manifold without boundary, not contractible; it deformation-retracts on the central circle $\simeq \mathbb{S}^1$, hence $b^\bullet = (1,1,0)$ and $b^\bullet_c = (0,1,1)$ by Poincaré lemmas.

(9) *Bordered annulus*: homeomorphic to $\mathbb{S}^1 \times I$; compact orientable manifold with boundary, not contractible; it deformation-retracts on the central circle $\simeq \mathbb{S}^1$, hence $b^\bullet = b^\bullet_c = (1,1,0)$.

(10) *Möbius band without boundary*: non orientable manifold without boundary; it deformation-retracts on the central circle $\simeq \mathbb{S}^1$, hence $b^\bullet = (1,1,0)$; on the other hand one has $b_c^\bullet = (0,0,0)$.[(96)]

(11) *Bordered Möbius band*: compact non orientable manifold with boundary, not contractible; it deformation-retracts on the central circle $\simeq \mathbb{S}^1$, hence $b^\bullet = b_c^\bullet = (1,1,0)$.

(12) *Torus* $\mathbb{T}^2$: homeomorphic to $(\mathbb{S}^1)^2$; compact orientable manifold without boundary, not contractible; $b^\bullet = b_c^\bullet = (1,2,1)$ (Künneth).

(13) *Klein bottle*: compact non orientable manifold without boundary, not contractible; in its well-known 3-dimensional representation, the "Klein bottle" is a bottle with a hole in the bottom and with the neck which penetrates at one side of the bottle until glueing its end with the hole in the bottom. In the representation as square, such "glued hole" corresponds to the loop denoted by with "$\twoheadrightarrow$". One can compute that $b^\bullet = b_c^\bullet = (1,1,0)$.[(97)]

(14) *Real projective plane* $\mathbb{P}^2$: compact non orientable manifold without boundary, non contractible. In this representation, the identified points are the points at infinity (a copy of the real projective line $\mathbb{P}^1$). One computes that $b^\bullet = b_c^\bullet = (1,0,0)$.[(98)]

Note that, in the framework of manifolds without boundary, the Poincaré duality is respected in the case of orientability (1-7-8-12) and not respected in the case of non orientability (10-13-14).

[(96)] Apply the sequence of Mayer-Vietoris with compact support with U an open piece of Möbius band which should be a neighborhood of the edge "$\rightarrow$" (in the figure, U appears to be formed by two horizontal bands, one above and one below, which include the identified edge: although it could appear to be disconnected, in fact it is homeomorphic to $\mathbb{R}^2$) and V an open central piece (homeomorphic to $\mathbb{R}^2$) of the band, which should slightly overlap U on the two extremities: hence $U \cap V$ is homeomorphic to two disjoint copies of $\mathbb{R}^2$. In this case the linear map $\varphi : \mathbb{R}^2 \simeq H_c^2(U \cap V) \to H_c^2(U) \oplus H_c^2(V) \simeq \mathbb{R}^2$ is an isomorphism: namely, if $(1,0)$ and $(0,1)$ represent respectively a positive bump form generating $H_c^2(U \cap V)_+$ and $H_c^2(U \cap V)_-$ then $\varphi(1,0) = (-1,1)$ and $\varphi(0,1) = (-(-1),1) = (1,1)$ (in fact, when sticking the two edges to create U, the bump form on $(U \cap V)_-$ changes sign with respect to the one of $(U \cap V)_+$ because of the twisting structure of the Möbius band). It is interesting to note that the same argument can be applied to the annulus without boundary (8), the only difference being that in the latter case the linear map φ has rank 1 (because, in the previous notation, both $(1,0)$ and $(0,1)$ are mapped into $(-1,1)$).

[(97)] Choose U and V similarly to (10), i.e. U is an open piece of bottle which is a neighborhood of the edge "$\rightarrow$" identified by keeping the orientation (in the figure, U appear to be formed by two horizontal bands of the same tickness, one above and one below, which includes the edge "$\rightarrow$": U is only apparently disconnected, and is in fact homeomorphic to a Möbius band without boundary) and V a horizontal central piece of band, symmetric and open (also homeomorphic to a Möbius band without boundary) , which slightly overlaps U on the two extremities: the intersection $U \cap V$ is then homeomorphic to a ring without boundary of type (8). Both the sequences of Mayer-Vietoris (usual and with compact support, in fact the manifold is compact) give the above result. Of course, one could come to the same result also with a different choice of U and V: for example, taking as U an open piece of the bottle which is a neighborhood of the edge "$\twoheadrightarrow$" identified by reversing the orientation (such U is homeomorphic to an annulus without boundary), and as V a vertical central open piece (itself homeomorphic to an annulus without boundary) of the band, which slightly overlaps U on both extremities: the intersection $U \cap V$ is then homeomorphic to two disjoint copies of the annulus without boundary.

[(98)] Let $X = \mathbb{P}^2$, $Y \simeq \mathbb{P}^1$ the manifold of points at infinity and p any point of the affine plane $X \setminus Y \simeq \mathbb{R}^2$; consider the open subset $U = X \setminus \{p\}$ and $V = X \setminus Y$. Now, U deformation-retracts to $Y \simeq \mathbb{P}^1$, while, as it has been said, $V \simeq \mathbb{R}^2$; using the sequence of Mayer-Vietoris (either the usual or with compact support, since the manifold is compact), one obtains the result. As we saw in the examples at p. 77, the same argument with $\mathbb{B}^n$ in the place of the square and $\mathbb{S}^{n-1}$ with identified antipodal points allows to prove that $b^\bullet(\mathbb{P}^n) = b_c^\bullet(\mathbb{P}^n) = (1,0,\dots,0,1)$ (if n is odd, with 1 in degree 0 and n: it is a compact orientable manifold) and $= (1,0,\dots,0,0)$ (if n is even: it is a compact non orientable manifold).

2.9 Cohomology of Čech

We aim to compute the de Rham cohomology of a $\mathcal{C}^\infty$ manifold by using arguments of combinatorial nature on any good cover of the manifold.

Let X be a $\mathcal{C}^\infty$ manifold, $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ a countable open cover of X. Given $q+1$ natural numbers $k_0 < \cdots < k_q$, we set for short $U_{k_0,\dots,k_q} = \bigcap_{i=0}^{q} U_{k_j}$. The embedding map $\iota_j : U_{k_0,\dots,k_q} \to U_{k_0,\dots,\widehat{k_j},\dots,k_q}$ (where, as usual, $\widehat{\cdot}$ denotes that the term is removed) induces the restriction map $\iota_j^* : \Omega^\bullet(U_{k_0,\dots,\widehat{k_j},\dots,k_q}) \to \Omega^\bullet(U_{k_0,\dots,k_q})$.

Definition 2.9.1. The space

$$C^q(\mathcal{U}, \Omega^p) = \prod_{k_0 < \cdots < k_q} \Omega^p(U_{k_0,\dots,k_q})$$

is called *space of q-cochains* of the cover $\mathcal{U}$ with values in Ω^p.

We have morphisms of complexes

$$\iota^* : \Omega^\bullet(X) \to \mathcal{C}^0(\mathcal{U}, \Omega^\bullet), \qquad \delta : C^q(\mathcal{U}, \Omega^\bullet) \to C^{q+1}(\mathcal{U}, \Omega^\bullet),$$

where ι^* is the product of the restriction maps $\Omega^\bullet(X) \to \Omega^\bullet(U_{k_0})$ (for $k_0 \in \mathbb{N}$) and, given a q-cochain $\omega = (\omega_{k_0,\dots,k_q})_{k_0 < \cdots < k_q}$,

$$\delta(\omega)_{k_0,\dots,k_{q+1}} = \sum_{j=0}^{q+1} (-1)^j \iota_j^* \omega_{k_0,\dots,\widehat{k_j},\dots,k_{q+1}} \qquad \in \Omega^\bullet(U_{k_0,\dots,k_{q+1}}).$$

It should be clear that in fact we are generalizing the sequences of Mayer-Vietoris.

Proposition 2.9.2. *The generalized sequence of Mayer-Vietoris*

$$C^\bullet(\mathcal{U}, \Omega^\bullet) : \qquad 0 \to \Omega^\bullet(X) \xrightarrow{\iota^*} \mathcal{C}^0(\mathcal{U}, \Omega^\bullet) \xrightarrow{\delta} C^1(\mathcal{U}, \Omega^\bullet) \xrightarrow{\delta} C^2(\mathcal{U}, \Omega^\bullet) \xrightarrow{\delta} \cdots$$

is exact in $\mathbf{C}(\mathbb{R})$.

Proof. One verifies directly that the sequence is a complex, and that $\mathrm{im}(\iota^*) = \ker(\delta)$ (exercise). In order to prove that it is in fact an exact sequence, it is enough to show that the identity is homotopic to zero, i.e. to find an operator $k : C^\bullet(\mathcal{U}, \Omega^\bullet) \to C^{\bullet-1}(\mathcal{U}, \Omega^\bullet)$ such that $\mathrm{id}_{C^\bullet(\mathcal{U},\Omega^\bullet)} = k \circ \delta + \delta \circ k$. Consider a partition of unity $\{\rho_n : n \in \mathbb{N}\}$ subordinate to $\mathcal{U}$ and, setting $\omega \in C^{q+1}(\mathcal{U}, \Omega^\bullet)$, let $k^{q+1}(\omega)_{k_0,\dots,k_q} = \sum_{n \in \mathbb{N}} \rho_n \omega_{n,k_0,\dots,k_q}$ (here we mean that the indexes of ω are alternating, and hence $\omega_{n,k_0,\dots,k_q} = 0$ if $n = k_j$ for some $0 \le j \le q$, and $\omega_{n,k_0,\dots,k_q} = (-1)^{j+1} \omega_{k_0,\dots,k_j,n,k_{j+1},\dots,k_q}$ if $k_j < n < k_{j+1}$): then $k^{q+1}\delta(\omega)_{k_0,\dots,k_q} = \sum_n \rho_n \delta(\omega)_{n,k_0,\dots,k_q} = (\sum_n \rho_n)\omega_{k_0,\dots,k_q} + \sum_{n,j} (-1)^{j+1} \rho_n \omega_{n,k_0,\dots,\widehat{k_j},\dots,k_q} = \omega_{k_0,\dots,k_q} - \delta k^q(\omega)_{k_0,\dots,k_q}$, as desired. $\square$

The cochains with values in $\Omega^\bullet$ form a double complex $(C^q(\mathcal{U}, \Omega^p), \delta^{p,q}, d^{p,q})$ where the row morphisms $\delta^{p,q} : C^q(\mathcal{U}, \Omega^p) \to C^{q+1}(\mathcal{U}, \Omega^p)$ have just been introduced, and the column morphisms $d^{p,q} : C^q(\mathcal{U}, \Omega^p) \to C^q(\mathcal{U}, \Omega^{p+1})$ are induced by the differential of forms; moreover, one clearly has $C^q(\mathcal{U}, \Omega^p) = 0$ for $p < 0$ or $q < 0$. Recall that, given such a double complex $X^{\bullet,\bullet}$, it is canonically defined a simple complex $s(X^{\bullet,\bullet})$ (see Appendix A.2).

Proposition 2.9.3. *For any open cover $\mathcal{U}$ of X, the generalized sequence of Mayer-Vietoris induces an isomorphism*

$$H^\bullet(X) \simeq H^\bullet(s(C^\bullet(\mathcal{U}, \Omega^\bullet)).$$

Proof. Follows from Proposition 2.9.2 and from Proposition A.2.4 in Appendix A.2 (the double complex $C^\bullet(\mathcal{U}, \Omega^\bullet)$ is augmented with the column given by the complex of de Rham). □

Now let us define the cohomology of Čech associated to the cover $\mathcal{U}$. For $q \geq 0$ let

$$C^q(\mathcal{U}, \mathbb{R}) = \{(f_{k_0,\cdots,k_q}) \in C^q(\mathcal{U}, \Omega^0) = \prod\nolimits_{k_0<\cdots<k_q} \mathcal{C}^\infty(U_{k_0,\cdots,k_q}) : df = 0\},$$

the space of functions locally constant on the intersections $U_{k_0,\cdots,k_q}$. The morphisms of the generalized sequence of Mayer-Vietoris induce morphisms $\delta : C^q(\mathcal{U}, \mathbb{R}) \to C^{q+1}(\mathcal{U}, \mathbb{R})$.

Definition 2.9.4. The *complex of Čech* is

$$C^\bullet(\mathcal{U}, \mathbb{R}) : \qquad 0 \to C^0(\mathcal{U}, \mathbb{R}) \xrightarrow{\delta} C^1(\mathcal{U}, \mathbb{R}) \xrightarrow{\delta} C^2(\mathcal{U}, \mathbb{R}) \to \cdots,$$

The *cohomology of Čech* of the cover $\mathcal{U}$ is the cohomology $H^\bullet(\mathcal{U}, \mathbb{R}) = H^\bullet(C^\bullet(\mathcal{U}, \mathbb{R}))$.

Proposition 2.9.5. *For any good cover $\mathcal{U}$ of X, it is induced a isomorphism*

$$H^\bullet(\mathcal{U}, \mathbb{R}) \simeq H^\bullet(s(C^\bullet(\mathcal{U}, \Omega^\bullet)).$$

Proof. Consider the double complex $C^\bullet(\mathcal{U}, \Omega^\bullet)$ augmented with the row $C^\bullet(\mathcal{U}, \mathbb{R})$ (note that, for any q, it holds $C^q(\mathcal{U}, \mathbb{R}) = \ker\left[d^{q,0} : C^q(\mathcal{U}, \Omega^0) \to C^q(\mathcal{U}, \Omega^1)\right]$. If $\mathcal{U}$ is a good cover of X, all columns are exact (the cohomology of the qth column is given by $\prod_{k_0<\cdots<k_q} H^\bullet(U_{k_0,\cdots,k_q})$), and the conclusion follows again from Proposition A.2.4 in Appendix A.2. □

Applying Propositions 2.9.3 and 2.9.5 one hence obtains:

Corollary 2.9.6. *For any good open cover $\mathcal{U}$ of X, one has a isomorphism*

$$H^\bullet(X) \simeq H^\bullet(\mathcal{U}; \mathbb{R}).$$

One immediately recovers the result on finiteness (Theorem 2.6.3): namely, if X has a finite good cover $\mathcal{U}$, then surely also $H^\bullet(\mathcal{U}, \mathbb{R})$ (and hence $H^\bullet(X)$) has finite dimension.

Remark 2.9.7. More generally, the notion of Čech cohomology could be defined for any topological space endowed with an open cover $\mathcal{U}$, with particularly satisfactory results when the cover is *good*, i.e. any open set in $\mathcal{U}$ as well as any finite intersection of open sets in $\mathcal{U}$ is contractible. We refer e.g. to Hatcher [9].

Examples. **(1)** Let $X = \mathbb{S}^1$, and consider a good cover $\mathcal{U}$ given by three open arcs U_j (with $j = 0, 1, 2$) which slightly overlap each other (see Figure 22(a)). The complex of Čech relative to $\mathcal{U}$ is

$$C^\bullet(\mathcal{U}, \mathbb{R}) : \qquad 0 \to C^0(\mathcal{U}, \mathbb{R}) \xrightarrow{\delta} C^1(\mathcal{U}, \mathbb{R}) \to 0;$$

in this case one has $C^0(\mathcal{U}, \mathbb{R}) = \{(\alpha_0, \alpha_1, \alpha_2) : \alpha_j \text{ constant on } U_j\} \simeq \mathbb{R}^3$ and $C^1(\mathcal{U}, \mathbb{R}) = \{(\beta_{01}, \beta_{02}, \beta_{12}) : \beta_{ij} \text{ constant on } U_{ij}\} \simeq \mathbb{R}^3$, while $\delta(\alpha)_{ij} = \alpha_j - \alpha_i$ and hence, with respect to the canonical bases one has

$\delta = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}$; since δ has rank 2, one has $H^0(\mathbb{S}^1) \simeq \ker(\delta) \simeq \mathbb{R}$ and $H^1(\mathbb{S}^1) \simeq \operatorname{coker}(\delta) \simeq \mathbb{R}$. Note that a 1-cocycle β is a coboundary if and only if $\beta_{02} = \beta_{01} + \beta_{12}$, hence a generator of $H^1(\mathbb{S}^1)$ is (the class of) the 1-cocycle $\beta = (1, 0, 0)$. **(2)** Let $X = \mathbb{S}^2$, and consider a good cover $\mathcal{U}$ given by the open northern emisphere U_0 and by three open spherical wedges U_j (with $j = 1, 2, 3$) which cover the closed southern emisphere and slightly overlap each other, in a way that each one of them contains the South pole $S = -e_3$ and a portion of the equator (see Figure 22(b)). The complex of Čech relative to $\mathcal{U}$ is

$$C^\bullet(\mathcal{U}, \mathbb{R}) : \qquad 0 \to C^0(\mathcal{U}, \mathbb{R}) \xrightarrow{\delta^0} C^1(\mathcal{U}, \mathbb{R}) \xrightarrow{\delta^1} C^2(\mathcal{U}, \mathbb{R}) \to 0;$$

it holds $C^0(\mathcal{U}, \mathbb{R}) = \{(\alpha_0, \alpha_1, \alpha_2, \alpha_3)\} \simeq \mathbb{R}^4$, $C^1(\mathcal{U}, \mathbb{R}) = \{(\beta_{01}, \beta_{02}, \beta_{03}, \beta_{12}, \beta_{13}, \beta_{23})\} \simeq \mathbb{R}^6$ and $C^2(\mathcal{U}, \mathbb{R}) = \{(\gamma_{012}, \gamma_{013}, \gamma_{023}, \gamma_{123}\} \simeq \mathbb{R}^4$, while $\delta^0(\alpha)_{ij} = \alpha_j - \alpha_i$ and $\delta^1(\beta)_{ijk} = \beta_{ij} - \beta_{ik} + \beta_{jk}$. With respect to the canonical bases one hence has

$$\delta^0 = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \qquad \delta^1 = \begin{pmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{pmatrix}$$

(note that $\delta^1 \circ \delta^0 = 0$). The maps δ^j have rank 3: hence both $H^0(\mathbb{S}^2) \simeq \ker(\delta^0)$ and $H^2(\mathbb{S}^2) \simeq \operatorname{coker}(\delta^1)$ have dimension $4 - 3 = 1$, and $H^1(\mathbb{S}^2) \simeq \ker(\delta^1)/\operatorname{im}(\delta^0)$ vanishes. **(3)** Let X be the plane with two holes. Thinking for example $X \simeq \mathbb{R}^2 \setminus \{(\pm 1, 0)\}$, a good cover $\mathcal{U}$ is given by the five open subsets $U_0 = \{|y| > 2(|x|-1)\}$, $U_1 = \{-(x-1) < y < 3(x-1)\}$, $U_2 = r_x(U_1)$, $U_3 = r_y(U_1)$ and $U_4 = r_x r_y(U_1)$, where $r_x(x, y) = (x, -y)$ and $r_y(x, y) = (-x, y)$ (see Figure 22(c)). The complex of Čech relative to $\mathcal{U}$ is hence given by $C^0(\mathcal{U}, \mathbb{R}) = \{(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4)\} \simeq \mathbb{R}^5$ and $C^1(\mathcal{U}, \mathbb{R}) = \{(\beta_{01}, \beta_{02}, \beta_{03}, \beta_{04}, \beta_{12}, \beta_{34})\} \simeq \mathbb{R}^6$, with $\delta^0(\alpha)_{ij} = \alpha_j - \alpha_i$ having rank 4: hence $H^0(X) \simeq \ker(\delta^0)$ has dimension $5 - 4 = 1$, $H^1(X) \simeq \operatorname{coker}(\delta^0)$ has dimension $6 - 4 = 2$ and $H^j(X) = 0$ for $j \geq 2$. Another good cover $\mathcal{U}'$ is given by $U'_0 = \{y > 0\}$, $U'_1 = \{y < 0\}$, $U'_2 = \{y < -x-1\}$, $U'_3 = \{y < x-1\}$ and $U'_4 = \{|x| < 1\}$ (see Figure 22(d)), whose complex of Čech is hence given by $C^0(\mathcal{U}', \mathbb{R}) = \{(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4)\} \simeq \mathbb{R}^5$, $C^1(\mathcal{U}', \mathbb{R}) = \{(\beta_{02}, \beta_{03}, \beta_{04}, \beta_{12}, \beta_{13}, \beta_{14}, \beta_{23}, \beta_{24}, \beta_{34})\} \simeq \mathbb{R}^9$, $C^2(\mathcal{U}', \mathbb{R}) = \{(\gamma_{123}, \gamma_{124}, \gamma_{134}, \gamma_{234})\} \simeq \mathbb{R}^4$, $C^3(\mathcal{U}', \mathbb{R}) = \{(\delta_{1234})\} \simeq \mathbb{R}$ with $\delta^0(\alpha)_{ij} = \alpha_j - \alpha_i$ of rank 4, $\delta^1(\beta)_{ijk} = \beta_{ij} - \beta_{ik} + \beta_{jk}$ of rank 3 and $\delta^2(\gamma)_{1234} = \gamma_{123} - \gamma_{134} + \gamma_{124} - \gamma_{234}$ of rank 1 (check that $\delta^{j+1} \circ \delta^j = 0$ for $j = 0, 1$): therefore, once more we get that $H^0(X) \simeq \ker(\delta^0)$ has dimension $5 - 4 = 1$, $H^1(X) \simeq \ker(\delta^1)/\operatorname{im}(\delta^0)$ has dimension $(9-3) - 4 = 2$, $H^2(X) \simeq \ker(\delta^2)/\operatorname{im}(\delta_1)$ has dimension $3 - 3 = 0$ and hence vanishes, as well as (obviously!) $H^3(X) \simeq \operatorname{coker}(\delta^2)$.

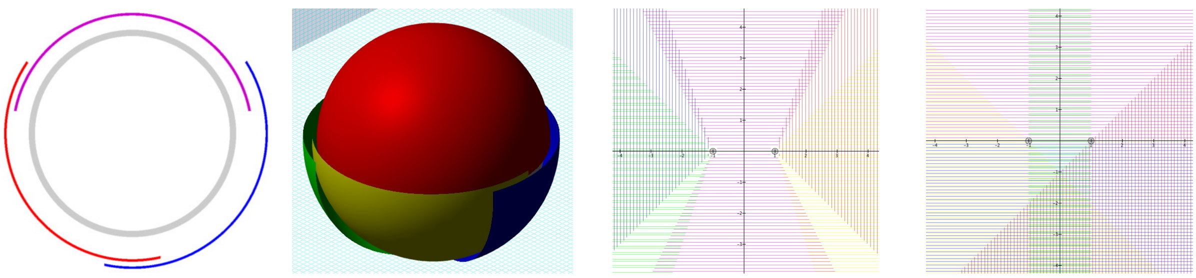


Figure 22: Čech covers for **(a)** the circle $\mathbb{S}^1$, **(b)** the sphere $\mathbb{S}^2$, **(c-d)** the plane with two holes.