Topologia 2 (Master degree in Mathematics - ALGANT)

Program of the course – Instructions for the exam

University of Padova - 2010/11

Lecturer: Corrado Marastoni - maraston@math.unipd.it

(Final version of 09/12/2010)

Program of the course

1. Fundamental group

• **Homotopy.** Paths and arcwise connected spaces. Homotopy, homotopy relatively to a subset. Contractible spaces. Quotient functions and quotient topology. Functions defined, or with values, in a sphere. Retract, (weak, strong) deformation retract. Non contractibility of the sphere and Brouwer's Fixed Point theorem.

• **Fundamental group.** Groupoid. Fundamental groupoid and fundamental group of a topological space. Simply connected spaces. Functoriality and invariance by homotopy. Fundamental group of a retract and of a topological product. Fundamental group of the circle $\mathbb{S}^1$ and of other discrete quotients of topological groups. Theorem of Van Kampen: general formulation and particular cases, applications.

• **Covering spaces.** Generalities about fiber bundles. Covering spaces and local homeomophisms. Liftings; covering spaces lift paths and homotopies with uniqueness. Monodromy lemma and consequences. Characteristic subgroup of a covering space. Monodromy action. Lifting criterion. Existence and uniqueness of a covering space with prescribed characteristic subgroup. Covering automorphisms, normal covering spaces, universal cover. Applications: a special case of Borsuk-Ulam theorem, the fundamental group of a topological group, of a manifold, of classical real groups.

2. Cohomology theories

• **Singular homology and cohomology, CW complexes.** Singular homology and cohomology, also relative to a subset. Universal coefficients formulas. The zeroth homology group; Hurewicz theorem for the first homology group. Invariance by homotopy equivalence. Euler characteristic of a space of finite type. CW complexes and cellular homology.

• **Cohomology of de Rham.** Exterior algebra and construction of the de Rham complex on a smooth manifold. Cohomology of de Rham. Cohomology of de Rham with compact support. Zeroth de Rham cohomology group. The Mayer-Vietoris principle with or without compact support, and associated long exact sequences in cohomology. Oriented manifolds, induced orientation on the boundary. Integration on manifolds, Stokes' theorem. Poincaré lemmas with or without compact support, invariance by homotopy equivalence. Finiteness and Poincaré duality. Theorem of de Rham about the isomorphism between singular and de Rham cohomology. Degree of a smooth proper map. Künneth formula and applications.

• **Cohomology of Čech.** Cohomology of Čech and isomorphism with the de Rham cohomology; applications.

A. Appendices

Some elements of various theories (categories, modules on a ring, homological algebra, multilinear algebra, sheaves, manifolds), necessary to describe the above notions, have been provided and treated whenever needed.

Instructions for the exam

• The exam will be an oral trial, organized into two parts.

1. In the first part, of theoretical type, the student will be required to appropriately define notions and objects and to state and prove results.
2. In the second part, of practical type, the student will be required to compute the fundamental group and/or the homology/cohomology groups of some concrete examples.

• As for the first part, the student should study carefully all notions (definitions, results with statement and proofs, examples and so on) contained in the notes published in the web page of the course

`http://www.math.unipd.it/~maraston/Topologia2` ,

with the following exceptions.

1. **Fundamental group.** $*$ *pp. 5-6:* Remark 1.1.5 and its footnotes are not required. $*$ *p. 15:* Proposition 1.3.8: no proof. $*$ *p. 18:* Corollary 1.4.3: no proof. $*$ *p. 18:* Lemmas 1.4.5 and 1.4.6: no proof. $*$ *pp. 22-23:* Theorem 1.5.3: no proof of the well-posedness of ψ. $*$ *p. 34:* Lemma 1.6.13: no proof of the continuity of $\tilde{h}$. $*$ *p. 45:* Proposition 1.7.4: no proof.
2. **Cohomology theories.** $*$ *pp. 53-54:* Proposition 2.1.8: only the proof for the zeroth homology. $*$ *p. 54:* Proposition 2.1.11: no proof. $*$ *p. 73:* Theorem 2.4.10: no proof. $*$ *p. 81:* Theorem 2.6.10: no proof. $*$ *pp. 90-91-92:* the section about Čech cohomology is not required.

A. Appendices. $*$ *Modules on a ring:* the part of this section about double complexes is not required. $*$ *Sheaves:* the notes from the definition of F^+ until the end of the section are not necessary. $*$ *Manifolds:* the proofs in this section are not required.

• To take the exam it is necessary to arrange an appointment in a reasonable advance with the lecturer (email `maraston@math.unipd.it`); in this sense, the official dates published in the web server should be intended only as formal dates useful to make the registration procedure easier. On the day of the exam the student will be asked to show his/her *libretto universitario* (Padova University booklet) or, if temporarily unavailable, an official ID (e.g. passport). Anyway, the registration procedure will require both the booklet and the presence in the UNIWEB online list, and therefore it could be possible that, when lacking one or both of these requisites, this procedure can not be completed in the same day of the exam but some time later.