2 Cohomology theories

The second part of these notes deals with the notions of homology and, mainly, with the dual one of cohomology. The definition of groups of singular homology and cohomology with coefficients in any commutative group can be given, in whole generality, for any topological space, with a particular computational ease for the CW complexes (§2.1). Then we define the de Rham cohomology for a $\mathcal{C}^\infty$ manifold (§2.2), a construction which will be shown to be isomorphic to the singular cohomology with real coefficients. The Mayer-Vietoris principle, analogously to Van Kampen's theorem for the fundamental group, decomposes the problem of computing cohomology on the open subsets of a cover (§2.3). After having studied orientation and integration on manifold, until the theorem of Stokes (§2.4) and Poincaré lemma for $\mathbb{R}^n$ (§2.5), we show that in a manifold of finite type the cohomology satisfies finiteness and Poincaré duality (§2.6). We then examine the degree of a smooth and proper map between $\mathcal{C}^\infty$ manifolds (§2.7), the Künneth formula for the cohomology of a product of manifolds (§2.8) and finally the cohomological theory of Čech, which provides the same results than the one of de Rham (§2.9).

The main references for the notes of this second part are Bott-Tu [1] and Hatcher [8].

2.1 Singular homology and cohomology of a topological space

We start by presenting some sketch of the construction of the groups of *singular homology and cohomology* for a topological space X with coefficients in an abelian group R.[54] Then we treat the special case of CW complexes.

2.1.1 Singular homology and cohomology

Let us deal first with the basic case of integer coefficients.

Definition 2.1.1. Let $k \in \mathbb{N}$. The *standard k-simplex* $\Delta_k \subset \mathbb{R}^k$ is the convex hull of the origin p_0 and the vectors $p_1, \dots, p_k$ of the canonical basis, i.e.

$$\Delta_k = \{x \in \mathbb{R}^k : x_i \geq 0 \text{ for any } 1 \leq i \leq k,\ \sum_{i=1}^{k} x_i \leq 1\}.$$

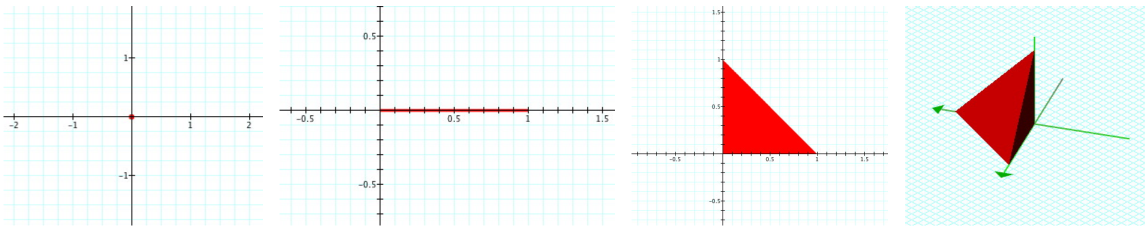

Figure 13: The standard simplexes Δ_k for $k = 0, 1, 2, 3$.

A *singular k-simplex* in X is any continuous map $\sigma : \Delta_k \to X$ (in particular, if $X = \mathbb{R}^n$ one talks about *affine singular k-simplex* in $\mathbb{R}^n$); its image $|\sigma| = \sigma(\Delta_k) \subset X$ is called *support* of σ.[55] We shall denote by $S_k(X, \mathbb{Z})$ the abelian group with coefficients in $\mathbb{Z}$ generated by the singular k-simplexes, i.e. the set of finite formal sums of type $\sum_{\lambda \in \Lambda} a_\lambda \sigma_\lambda$ with Λ finite, $a_\lambda \in \mathbb{Z}$ and σ_λ a singular k-simplex for any $\lambda \in \Lambda$: such a sum is called *singular k-chain* in X.

It is clear that $\Delta_0 = \{0\}$ and $\Delta_1 = I$: hence, the singular 0- and 1-simplexes in X are nothing but points and paths in X, and $S_0(X, \mathbb{Z})$ (resp. $S_1(X, \mathbb{Z})$) is the set of finite formal sums $\sum_{i=1}^{n} a_i x_i$ (resp. $\sum_{i=1}^{n} a_i \gamma_i$) where the x_i are points (resp. the γ_i are paths) of X. A singular simplex in X could also be degenerated, because the only thing required is continuity (for example, the image in X of a 2-simplex could collapse to a path or a point).

[54] If, as it often happens, R has also the structure of commutative ring, the constructions which follow will give rise to R-modules and R-linear morphisms.

[55] As it happens for paths, by abuse of language it is usual to forget about the difference between a singular simplex and its support.

Now let us define a *boundary* morphism for these groups:

$$\partial_k : S_k(X,\mathbb{Z}) \to S_{k-1}(X,\mathbb{Z}).$$

The idea is to send a singular k-simplex into its "oriented boundary", and then to extend to k-chains "by $\mathbb{Z}$-linearity". To start, note that Δ_k has $k+1$ faces, which are singular $(k-1)$-simplexes: for $i = 0,\dots,k$ define $\iota_k^i : \Delta_{k-1} \to \Delta_k$ by sending the k points $\{p_0,\dots,p_{k-1}\}$ of $\mathbb{R}^{k-1}$ into the k points $\{p_0,\dots,\widehat{p_i},\dots,p_k\}$ of $\mathbb{R}^k$ and then by extending the definition to their convex combinations by $\mathbb{R}$-linearity. Then, given a singular k-simplex σ in X, define the *ith face* of σ as the singular $(k-1)$-simplex $\sigma^{(i)} = \sigma \circ \iota_k^i$ in X, and set $\partial_k\sigma = \sum_{i=0}^k (-1)^i \sigma^{(i)}$; finally, given a singular k-chain $\sum_{\lambda\in\Lambda} a_\lambda \sigma_\lambda$ in X, set $\partial_k\left(\sum_{\lambda\in\Lambda} a_\lambda\sigma_\lambda\right) = \sum_{\lambda\in\Lambda} a_\lambda(\partial_k\sigma_\lambda)$.

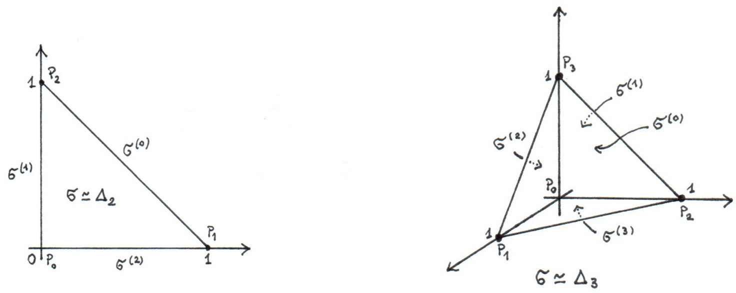

Figure 14: The faces of the standard simplexes Δ_2 and Δ_3.

Proposition 2.1.2. *One has* $\partial_k \circ \partial_{k+1} = 0$ (for $k = 1, 2, \dots$).

Proof. Exercise (it is enough to check the statement on the singular simplexes; for example, in the case $k = 1$ —see Figure 14(a)— one has $\partial_2(\sigma) = \sigma^{(0)} - \sigma^{(1)} + \sigma^{(2)}$ and hence $\partial_1(\partial_2(\sigma)) = \partial_1(\sigma^{(0)}) - \partial_1(\sigma^{(1)}) + \partial_1(\sigma^{(2)}) = (p_2 - p_1) - (p_2 - p_0) + (p_1 - p_0) = 0$). □

By Proposition 2.1.2 one has a complex (of chains) of abelian groups

$$\cdots \to S_{k+1}(X,\mathbb{Z}) \xrightarrow{\partial_{k+1}} S_k(X,\mathbb{Z}) \xrightarrow{\partial_k} S_{k-1}(X,\mathbb{Z}) \to \cdots \to S_1(X,\mathbb{Z}) \xrightarrow{\partial_1} S_0(X,\mathbb{Z}) \to 0. \tag{2.1}$$

Now, if R is any abelian group, by applying the functor $R \otimes_{\mathbb{Z}} \cdot$ to (2.1) one obtains the complex

$$\cdots \to S_{k+1}(X,R) \xrightarrow{\partial_{k+1}} S_k(X,R) \xrightarrow{\partial_k} S_{k-1}(X,R) \to \cdots \to S_1(X,R) \xrightarrow{\partial_1} S_0(X,R) \to 0, \tag{2.2}$$

where $S_k(X,R) := R \otimes_{\mathbb{Z}} S_k(X,\mathbb{Z})$ and the morphisms ∂_k are defined in practice as in the case of integral coefficients, replacing $\mathbb{Z}$ with R.

Definition 2.1.3. The *singular k-cycles* (resp. *k-boundaries*) in X are the elements of the subgroup $Z_k(X,R) = \ker\partial_k$ (resp. $B_k(X,R) = \operatorname{im}\partial_{k+1}$) of $S_k(X,R)$. The *singular homology* of X with coefficients in R is the homology (see Appendix A.2) of the complex (2.2):

$$H_k(X,R) = \frac{Z_k(X,R)}{B_k(X,R)}.$$

Remark 2.1.4. Note that all 0-chains are 0-cycles (i.e., $Z_0(X,R) = S_0(X,R)$). On the other hand the 1-cycles are precisely the loops, so both the first homology group (with integer coefficients) and the fundamental group are formed by equivalence classes of loops; however the differences between the two constructions are quite deep. Let us explain this point. If γ is a path in X from x_0 to x_1 and ψ is a path from x_1 to x_0, then in the fundamental group (which always needs a basepoint) we must distinguish between $\gamma\cdot\psi$ (a loop based at x_0) and $\psi\cdot\gamma$ (a loop based at x_1), although (if one just forgets about basepoints) both $\gamma\cdot\psi$ and $\psi\cdot\gamma$ represent in fact the same 1-cycle: in other words, in homology theory the paths γ and ψ commute to each other[56], while in homotopy theory they do not. Moreover, the equivalence relation in homology —i.e. to differ by a 1-boundary— is different from the one in homotopy —i.e. to be homotopic with fixed basepoint: in fact, the former is weaker than the latter (see the arguments written in the proof of Proposition 2.1.8).

Example. Let us compute the homology of the space with one point $\{\mathrm{pt}\}$. Note that $S_k(\{\mathrm{pt}\},R)\simeq R$ for any $k\geq 0$ (namely there is only the constant k-simplex $\sigma_k(\Delta_k)\equiv \mathrm{pt}$). Now, σ_k has $(k+1)$ faces all equal to σ_{k-1}, and hence $\partial_k(\sigma_k)=\sum_{i=1}^{k+1}(-1)^i\sigma_{k-1}$ (for $k\geq 1$): therefore $\partial_k : R\to R$ is zero for k odd and an isomorphism for k even. This implies that $H_0(\{\mathrm{pt}\},R)\simeq R$ and $H_k(\{\mathrm{pt}\},R)=0$ for $k>0$.

Remark 2.1.5. More generally, given a subset $A\subset X$ one can consider the groups of relative singular k-chains $S_k(X/A,R) := S_k(X,R)/S_k(A,R)$: the morphisms $\partial_k : S_{k+1}(X,R)\to S_k(X,R)$ then induce morphisms $\partial_k : S_{k+1}(X/A,R)\to S_k(X/A,R)$, and so one obtains the complex of relative singular k-chains, whose homology $H_k(X/A,R)$ is called *relative singular homology* with coefficients in R. From the short exact sequence of complexes of chains $0\to S_\bullet(A,R)\to S_\bullet(X,R)\to S_\bullet(X/A,R)\to 0$ we get a long exact sequence of homology (see Appendix A.2)

$$\cdots\to H_{k+1}(X/A,R)\to H_k(A,R)\to H_k(X,R)\to H_k(X/A,R)\to H_{k-1}(A,R)\to\cdots \quad (2.3)$$

Now let us try to simplify our study. To start, observe that it is enough to bound to arcwise connected spaces:

Lemma 2.1.6. *If $X=\bigsqcup_{\alpha\in A}X_\alpha$ is the decomposition of X in arcwise connected components, then for any k one has $H_k(X,R)\simeq\bigoplus_{\alpha\in A}H_k(X_\alpha,R)$.*

Proof. The singular k-simplexes are always arcwise connected, hence $S_k(X,R)$ decomposes in the direct sum of its subgroups $S_k(X_\alpha,R)$. The boundary morphisms send any $S_k(X_\alpha,R)$ into $S_{k-1}(X_\alpha,R)$, hence also their kernels and images decompose in the same way: it follows that also the groups of homology decompose. □

As a second step, since any abelian group R is a $\mathbb{Z}$-module, we may ask if and how the groups $H_k(X,\mathbb{Z})$ (homology with integer coefficients) determine the groups $H_k(X,R)$. One indeed proves that:[57]

[56] Also the additive notation used in homology, i.e. $\gamma+\psi=\psi+\gamma$, suggest this abelian situation.

[57] See Appendix A.2 for the notion of Tor, and especially A.2.3 for $\mathrm{Tor}^{\mathbb{Z}}$.

Proposition 2.1.7. (Universal coefficients formula for the singular homology). *For any k one has an exact sequence, which splits* non functorially*:*

$$0 \to H_k(X,\mathbb{Z}) \otimes_{\mathbb{Z}} R \to H_k(X,R) \to \mathrm{Tor}_1^{\mathbb{Z}}(R, H_{k-1}(X,\mathbb{Z})) \to 0. \tag{2.4}$$

In particular one has $H_k(X,R) \simeq H_k(X,\mathbb{Z}) \otimes_{\mathbb{Z}} R$ if at least one out of R and $H_{k-1}(X,\mathbb{Z})$ has no torsion.

Hence we may substantially bound to study the singular homology with integer coefficients of arcwise connected spaces. Let us investigate more in detail the cases $k = 0, 1$.

Proposition 2.1.8. *Let X be an arcwise connected space. Then:*

$$H_0(X,\mathbb{Z}) \simeq \mathbb{Z}, \qquad\qquad H_1(X,\mathbb{Z}) \simeq \frac{\pi_1(X)}{[\pi_1(X),\pi_1(X)]}.$$

Proof. (1) As we have seen, $S_0(X,\mathbb{Z})$ is the set of finite formal sums $\sum_{i=1}^n a_i x_i$. So there is a natural morphism ("degree") $\psi : S_0(X,\mathbb{Z}) \to \mathbb{Z}$ given by $\psi(\sum_{i=1}^n a_i x_i) = \sum_{i=1}^n a_i$. Consider a path $\sigma \in S_1(X,\mathbb{Z})$ between x_0 and x_1: since $\partial_1\sigma = x_1 - x_0$, one has $\psi(\partial_1\sigma) = 0$. Hence it is induced a morphism, clearly surjective, $H_0(X,\mathbb{Z}) = S_0(X,\mathbb{Z})/\operatorname{im}(\partial_1) \to \mathbb{Z}$. If X is arcwise connected, ψ is also injective: namely, if $\sum_{i=1}^n a_i = 0$ then choose paths σ_i from a base point x_0 to the various x_i, and note that $\partial_1(\sum_{i=1}^n a_i\sigma_i) = \sum_{i=1}^n a_i x_i - (\sum_{i=1}^n a_i)x_0 = \sum_{i=1}^n a_i x_i$. (2) We saw that the singular 1-simplexes are the paths. Hence there are two possible equivalence relations between them: the one of homotopy rel ∂I (which we shall denote by $\simeq$) and the one of homology, i.e. to differ by a 1-boundary (which we shall denote by $\sim$). Note that (i) if σ is a constant path then $\sigma \sim 0$, (ii) $\sigma_1 \simeq \sigma_2$ implies $\sigma_1 \sim \sigma_2$, (iii) $\sigma_1 \cdot \sigma_2 \sim \sigma_1 + \sigma_2$, (iv) $\sigma^{-1} \sim -\sigma$ (where we recall that $\sigma^{-1}(t) := \sigma(1-t)$). [58] Thanks to these remarks, for a given base point $x_0 \in X$ one obtains a morphism of groups $\alpha : G := \pi_1(X,x_0) \to H_1(X,\mathbb{Z})$. Let us give a cycle $\sum_i a_i\sigma_i \in Z_1(X,\mathbb{Z})$; up to repeating the paths σ_i we may suppose that $a_i = \pm 1$. Since $\partial_1(\sum_i \sigma_i) = \sum_i \partial_1\sigma_i = \sum_i(\sigma_i(1) - \sigma_i(0)) = 0$, given a σ_i there is surely some other σ_j such that $\sigma_i \cdot \sigma_j$ is defined, and this will be used to replace the two; continuing this procedure we may assume that all σ_i are loops, let us say of base point x_i. If γ_i is a path from x_0 to x_i, one has $\gamma \cdot \sigma_i \cdot \gamma^{-1} \sim \sigma_i$ for (iii) and (iv), and hence we may assume that all σ_i are loops based at x_0. Using again (iii) we may combine all σ_i's in a unique loop σ: this shows that α is surjective. Being $[G,G] \subset \ker(\alpha)$ (this is obvious, since the homology $H_1(X,\mathbb{Z})$ is an abelian group), it is induced a surjective morphism $\tilde{\alpha} : G/[G,G] \to H_1(X,\mathbb{Z})$: now let us construct a morphism $\tilde{\psi} : H_1(X,\mathbb{Z}) \to G/[G,G]$, which will be shown to be the inverse of $\tilde{\alpha}$. For any $x \in X$ fix a path γ_x from x_0 to x, with $\gamma_{x_0} = c_{xo}$; given then a path σ in X, let us define $\psi(\sigma) = [\gamma_{\sigma(0)} \cdot \sigma \cdot \gamma_{\sigma(1)}^{-1}] \in G$, extending then by $\mathbb{Z}$-linearity to a morphism $\psi : S_1(X,\mathbb{Z}) \to G/[G,G]$.[59] Noting that $\psi(B_1(X,\mathbb{Z})) = \{1\}$ mod $[G,G]$,[60] one obtains an induced morphism $\tilde{\psi} : H_1(X,\mathbb{Z}) \to G/[G,G]$. Therefore, if σ is a loop one has $(\tilde{\psi} \circ \tilde{\alpha})([\sigma]) = \tilde{\psi}(\sigma + B_1(X,\mathbb{Z})) = [\gamma_{x_0} \cdot \sigma \cdot \gamma_{x_0}^{-1}] = [\sigma]$, i.e. $\tilde{\psi} \circ \tilde{\alpha} = \mathrm{id}_{G/[G,G]}$. We are left with showing that $\tilde{\alpha} \circ \tilde{\psi} = \mathrm{id}_{H_1(X,\mathbb{Z})}$. Note that the assignation $x \mapsto \gamma_x$ extends, again by $\mathbb{Z}$-linearity, to a morphism $\gamma : S_0(X,\mathbb{Z}) \to S_1(X,\mathbb{Z})$; and then one proves that, given a 1-chain $\sigma \in S_1(X,\mathbb{Z})$, the class $\tilde{\alpha}(\psi(\sigma))$ in

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[58] (i) is clear, since $\sigma = c_{x_0} = \partial_2(\varphi)$ with φ the 2-singular simplex constantly equal to x_0. For (ii), given a homotopy $h : I^2 \to X$ we cut I^2 along the diagonal obtaning two 2-simplexes φ_i; then, by calculating $\partial_2(\varphi_2 - \varphi_1)$ one obtains $\sigma_2 - \sigma_1$ plus two constant paths in edges, which are boundaries by (i). For (iii), let $\sigma : \Delta_2 \to X$ be obtained by the composition of the orthogonal projection of Δ_2 on the edge $[p_0,p_2]$ followed by the map $\sigma_1 \cdot \sigma_2 : [p_0,p_2] \simeq I \to X$: then $\partial_2\sigma = \sigma_2 - (\sigma_1 \cdot \sigma_2) + \sigma_1$. Finally, (iv) follows from the others, since $\sigma + \sigma^{-1} \sim \sigma \cdot \sigma^{-1} \sim 0$.

[59] If G is a group and T a set, any function $f : T \to G$ gives rise to a morphism of groups $\tilde{f} : \mathbb{Z}^{(T)} \to G/[G,G]$ by setting $\tilde{f}(\sum_{i=1}^n a_i t_i) = f(t_1)^{a_1} \cdot \dots \cdot f(t_n)^{a_n} \mod [G,G]$ (namely, since $\mathbb{Z}^{(T)}$ is abelian, only the definition in the abelianization of G makes sense).

[60] By linearity,it is enough to show that if $\varphi : \Delta_2 \to X$ is a 2-simplex the $\psi(\partial\varphi) = 1 \mod [G,G]$. Set $\sigma(p_i) = y_i$, $f = \sigma^{(2)}$, $g = \sigma^{(0)}$, $h = (\sigma^{(1)})^{-1}$: then $\psi(\partial\varphi) = \psi(\sigma^{(0)} - \sigma^{(1)} + \sigma^{(2)}) = \psi(g - h^{-1} + f) = \psi(f)\psi(g)\psi(h^{-1})^{-1} = \dots = [\gamma_{y_0} \cdot f \cdot \gamma_{y_1}^{-1} \cdot \gamma_{y_1} \cdot g \cdot \gamma_{y_2}^{-1} \cdot (\gamma_{y_0} \cdot h^{-1} \cdot \gamma_{y_2}^{-1})^{-1}] = [f \cdot g \cdot h] = 1$ because $f \cdot g \cdot h \sim c_{y_0}$.

$H_1(X,\mathbb{Z})$ is represented by $\sigma - \gamma_{\partial\sigma}$.[61] In particular, if σ is a 1-cycle (i.e., if $\sigma \in Z_1(X,\mathbb{Z})$) this yields that the class $\tilde{\alpha}(\psi(\sigma))$ is represented by the same σ, and this completes the proof. □

Remark 2.1.9. The isomorphism $H_1(X,\mathbb{Z}) \simeq \frac{\pi_1(X)}{[\pi_1(X),\pi_1(X)]}$ is known as *Hurewicz theorem*. This theorem, of fundamental importance in homotopy theory, in its most general form states also that if there exists $n \geq 2$ such that $\pi_i(X) = 0$ for any $i = 1,\dots,n-1$ then it holds also $H_i(X,\mathbb{Z}) = 0$ for any $i = 1,\dots,n-1$, and it is defined a isomorphism of abelian groups $H_n(X,\mathbb{Z}) \simeq \pi_n(X)$.

If $f : X \to Y$ is a continuous map, then it is naturally induced a morphism of complexes $S_\bullet f : S_\bullet(X,R) \to S_\bullet(Y,R)$, and hence a morphism of graded $\mathbb{Z}$-modules $H_\bullet f : H_\bullet(X,R) \to H_\bullet(Y,R)$, and it holds $H_\bullet(g \circ f) = H_\bullet g \circ H_\bullet f$ and $H_\bullet(\mathrm{id}_X) = \mathrm{id}_{H_\bullet(X,R)}$: in other words, it is induced a covariant functor

$$H_\bullet : \mathfrak{Top} \to \mathfrak{Mod}_{\mathrm{deg}}(\mathbb{Z}).$$

The functoriality shows immediately that

Corollary 2.1.10. *Homeomorphic spaces have the same homology.*

Actually, similarly to the fundamental group, there is an invariance under homotopy:

Proposition 2.1.11. *Homotopic maps induce the same morphism in homology.*

Proof. If $h : X \times I \to Y$ is a homotopy between $f = h_0$ and $g = h_1$, then the morphism of complexes of chains $S_\bullet g - S_\bullet f$ is homotopic to zero (see Appendix A.2) and hence $H_\bullet g = H_\bullet f$. Namely, the idea (for the details see [8, §2.1])) is to see $\Delta_k \times I$ as a union of $k+1$ $(k+1)$-simplexes where each one has a k-simplex in common with the next one,[62] and to construct in this way a morphism $(K_h)_k : S_k(X,\mathbb{Z}) \to S_{k+1}(Y,\mathbb{Z})$ for which then one verifies that $(S_\bullet g - S_\bullet f)_k = (\partial_Y)_{k+1} \circ (K_h)_k + (K_h)_{k-1} \circ (\partial_X)_k$. □

Corollary 2.1.12. *Homotopically equivalent spaces have the same homology* (in particular, if X is contractible then it holds $H_0(X,R) \simeq R$ and $H_k(X,R) = 0$ for $k > 0$).

Applying the functor $\mathrm{Hom}_{\mathbb{Z}}(\,\cdot\,,R)$ to the complex of chains (2.1) one obtains the dual complex:

$$(2.5)\qquad 0 \to S^0(X,R) \xrightarrow{d^0} S^1(X,R) \to \cdots \to S^{k-1}(X,R) \xrightarrow{d^{k-1}} S^k(X,R) \xrightarrow{d^k} S^{k+1}(X,R) \to \cdots$$

where $S^k(X,R) = \mathrm{Hom}_{\mathbb{Z}}(S_k(X,\mathbb{Z}),R)$ and d^k is the morphism transposed to ∂_{k+1}.[63] This is then a complex of cochains in $\mathfrak{Mod}(\mathbb{Z})$.

[61] By linearity, it is enough to check the statement for a 1-simplex; then it holds $\tilde{\alpha}(\psi(\sigma)) = \tilde{\alpha}([\gamma_{\sigma(0)} \cdot \sigma \cdot \gamma_{\sigma(1)}^{-1}] = \gamma_{\sigma(0)} \cdot \sigma \cdot \gamma_{\sigma(1)}^{-1} + B_1(X,\mathbb{Z}) = \gamma_{\sigma(0)} + \sigma + \gamma_{\sigma(1)}^{-1} + B_1(X,\mathbb{Z}) = \gamma_{\sigma(0)} + \sigma - \gamma_{\sigma(1)} + B_1(X,\mathbb{Z})$, where the second-last equality follows from (iii) and the last one from (iv).

[62] For example, a square is divided by a diagonal in two triangles (i.e., 2-simplexes) with an edge in common; and a prism with triangular base, if we consider the three planes containing each a vertex of the inferior base and the opposed edge of the superior base, gets divided in three tetragons (i.e., 3-simplexes) pairwise with a face in common.

[63] If R is a commutative unitary ring, the adjunction between the functor of extension of coefficients and the "forgetful" functor (see Appendix A.2) gives a natural isomorphism of R-modules $S^k(X,R) = \mathrm{Hom}_{\mathbb{Z}}(S_k(X,\mathbb{Z}),R) \simeq \mathrm{Hom}_R(S_k(X,R),R)$.

Definition 2.1.13. The *singular k-cocycles* (resp. *k-coboundaries*) in X are the elements of the subgroups $Z^k(X,R) = \ker d^k$ (resp. $B^k(X,R) = \operatorname{im} d^{k-1}$) of $S^k(X,R)$. The *singular cohomology* of X with coefficients in R is the cohomology of the complex (of cochains) (2.5):

$$H^k(X,R) = \frac{Z^k(X,R)}{B^k(X,R)}.$$

The singular cohomology enjoys properties similar to those of singular homology (among them, the additivity with respect to arcwise connected components and the fact that $H^0(X,R) \simeq R$ if R is arcwise connected[(64)]). This last fact follows also from the following formula, which is analogous to the one for singular homology[(65)]:

Proposition 2.1.14. (Universal coefficients formula for the singular cohomology). *For any k one has an exact sequence, which splits* non functorially*:*

$$(2.6) \qquad 0 \to \operatorname{Ext}^1_{\mathbb{Z}}(H_{k-1}(X,\mathbb{Z}),R) \to H^k(X,R) \to \operatorname{Hom}_{\mathbb{Z}}(H_k(X,\mathbb{Z}),R) \to 0.$$

In particular it holds $H^k(X,R) \simeq \operatorname{Hom}_{\mathbb{Z}}(H_k(X,\mathbb{Z}),R)$ if $H_{k-1}(X,\mathbb{Z})$ is a free group or R is divisible.

If $f : X \to Y$ is a continuous map, it is naturally induced a morphism (pull-back) of complexes $S^\bullet f : S^\bullet(Y,R) \to S^\bullet(X,R)$, and hence a morphism of graded $\mathbb{Z}$-modules $H^\bullet f : H^\bullet(Y,R) \to H^\bullet(X,R)$: so one obtains a contravariant functor

$$H^\bullet : \mathfrak{Top} \to \mathfrak{Mod}_{\deg}(R)$$

which is once again invariant under homotopy:

Proposition 2.1.15. *Homotopic maps induce the same morphism in cohomology. In particular, homotopically equivalent spaces have the same cohomology.*

In particular let $j : U \to X$ be the inclusion of an open subset (endowed with the topology induced from X): then $S^\bullet j : S^\bullet(X,R) \to S^\bullet(U,R)$ is the restriction morphism. For any $k \in \mathbb{N}$ one the obtains a sheaf $S^k_X(R)$ (see Appendix A.4) given by $\Gamma(U; S^k_X(R)) = S^k(U,R)$ and, from (2.5), a complex of sheaves

$$(2.7) \qquad S^\bullet_X(R): \quad 0 \to S^0_X(R) \xrightarrow{d^0} S^1_X(R) \to \cdots \to S^{k-1}_X(R) \xrightarrow{d^{k-1}} S^k_X(R) \xrightarrow{d^k} S^{k+1}_X(R) \to \cdots.$$

Remark 2.1.16. (1) As abelian group, a field k of characteristic zero is obviously without torsion, and it is also divisible[(66)]: from (2.4) and (2.6) it follows that

$$H_j(X,k) \simeq k \otimes_{\mathbb{Z}} H_j(X,\mathbb{Z}), \qquad H^j(X,k) \simeq \operatorname{Hom}_{\mathbb{Z}}(H_j(X,\mathbb{Z}),k).$$

These k-vector spaces are dual to each other (see Note 115, p. 101); in particular, they have the same dimension. **(2)** If X is a topological space whose homology groups $H_j(X,\mathbb{Z})$

[(64)] Namely $H^0(X,R) \simeq \operatorname{Hom}_{\mathbb{Z}}(S_0(X,R)/\operatorname{im}(\partial_1),R) = \operatorname{Hom}_{\mathbb{Z}}(H_0(X,\mathbb{Z}),R) \simeq \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z},R) \simeq R$.

[(65)] See Appendix A.2 for the notion of Ext, and especially A.2.3 for $\operatorname{Ext}_{\mathbb{Z}}$.

[(66)] because it contains $\mathbb{Z}$ and hence $\mathbb{Q}$, and hence it is a $\mathbb{Q}$-vector space.

are finitely generated and vanish for j large enough,[(67)] one can define the *Euler-Poincaré characteristic* of X as the alternating sum (in which rk denotes the rank[(68)])

$$\chi(X) := \sum_{j\in\mathbb{Z}}(-1)^j \mathrm{rk}\, H_j(X,\mathbb{Z}). \tag{2.8}$$

Since homology groups are invariant under homotopic equivalence (in particular, under homeomorphisms), also $\chi(X)$ is homotopically invariant. For what has been said just above in (1), if k is a field of characteristic zero then $\chi(X) = \sum_{j\in\mathbb{Z}}(-1)^j \dim_k H_j(X,k) = \sum_{j\in\mathbb{Z}}(-1)^j \dim_k H^j(X,k)$.

2.1.2 CW complexes

As a particularly significant example of computation of the singular homology, we now briefly present the *cellular complexes* or *CW complexes*, which are a generalization of the classical notion of "triangularizable spaces" (see Examples at p. 57) due to J.H.C. Whitehead; we refer for example to Hatcher [8, §2.2] for the missing proofs.

We call $\dot{\mathbb{B}}^n$ (open ball) a *n-cell*; to "attach a n-cell" to a topological space T means to provide a continuous map $\phi : \mathbb{S}^{n-1} \to T$ and then to consider the quotient space $(T \sqcup \mathbb{B}^n)/\sim$, where "$\sim$" identifies $x \in \partial\mathbb{B}^n \simeq \mathbb{S}^{n-1}$ with $\phi(x) \in T$. A *CW complex* is a topological space X obtained by successively attaching to a discrete set (whose elements are seen as 0-cells) a family of n-cells, for increasing n, without being forced to do that necessarily for any n and with the possibility of attaching any number (possibly infinitely many) for any n that we have decided to consider; one of these (potentially infinite) constructions which lead to X is called a *CW structure* on X. Fixed one of them, the "nth intermediate stage" $X^n \subset X$, where we have already attached all m-cells with $m \leq n$ but not yet the next ones, will be called *n-skeleton* of X (relative to the fixed CW structure). Let us resume the procedure:

(1) start from a discrete set X^0, the 0-cells of X;

(2) inductively, form the n-skeleton X^n from X^{n-1} by attaching n-cells $D^n_\alpha \simeq \dot{\mathbb{B}}^n$ to it by means of continuous maps $\phi_\alpha : \mathbb{S}^{n-1} \to X^{n-1}$: i.e., X^n is the quotient space of $X^{n-1} \sqcup \bigsqcup_\alpha D^n_\alpha$ via the identification $x \sim \phi_\alpha(x)$, where $x \in \partial D^n_\alpha \simeq \mathbb{S}^{n-1}$;

(3) to $X = \bigcup_n X^n$ is assigned the *weak topology*, for which $A \subset X$ is open (closed) if and only if $A \cap X^n$ is open (closed) in X^n for any $n \in \mathbb{N}$.

(67) Such a space is usually called *of finite type*. Note that —excepted the case of a space with a finite number of points— the groups of chains $S_j(X,\mathbb{Z})$ are far from being finitely generated, and therefore one can not apply Proposition A.2.5, since $\sum_{j\in\mathbb{Z}}(-1)^j \mathrm{rk}\, S_j(X,\mathbb{Z})$ does not make sense. On the other hand, as we shall see soon in the framework of CW complexes, the groups of cellular chains (see (2.9)) are free of finite rank, and Proposition A.2.5 will be applied.

(68) The rank $\mathrm{rk}\,(G)$ of a finitely generated abelian group G is the (finite) number of components isomorphic to $\mathbb{Z}$ in any decomposition of G as a direct sum of cyclic subgroups. So, for example, $\mathrm{rk}\,(\mathbb{Z}^n) = n$ and $\mathrm{rk}\,(\mathbb{Z}/n\mathbb{Z}) = 0$.

If there exists $n \in \mathbb{N}$ such that $X^k = X$ for $k \geq n$, X the complex will be called *of finite dimension*, and in such case its *dimension* $\dim X$ will be defined as the minimum of such n;[(69)] this happens in particular if X is *finite*, i.e. if it has a finite number of cells[(70)]. A *subcomplex* of X is a subset obtained by attaching cells of X and containing the closure of anyone of these cells; it is easy to prove that a subcomplex is closed in X, and that it is itself a CW complex.

Proposition 2.1.17. *A finite CW complex is compact*[(71)]. *Conversely, any compact subspace of a CW complex is contained in a finite subcomplex* (in particular, the closure of any cell meets only a finite number of other cells[(72)]).

Examples. **(1)** $X = [0, 1]$ is a finite CW complex of dimension 1 obtained by attaching a 1-cell to $X^0 = \{0, 1\}$ (with $\phi : \mathbb{S}^0 = \{\pm 1\} \to X^0$, $\phi(-1) = 0$, $\phi(1) = 1$). **(2)** If $X = [0, 1[$ is a CW complex, then surely it can not be finite[(73)]. Actually a CW structure for X is obtained by setting $X^0 = \{P_n = \frac{n-1}{n} : n \in \mathbb{N} = \mathbb{Z}_{\geq 1}\}$ and attaching, for $n \in \mathbb{N}$, the 1-cell D^1_n between the points P_n and P_{n+1}, i.e. by means of $\phi_n : \{\pm 1\} \to X^0$, $\phi_n(-1) = P_n$, $\phi_n(1) = P_{n+1}$. Hence X has dimension 1. **(3)** A bouquet of circles is a CW complex of dimension 1 obtained by attaching 1-cells to $X^0 = \{\mathrm{pt}\}$ (with ϕ constant). **(4)** (See Figure 15(c) for the case $n = 2$) $X = \mathbb{S}^n$ is a finite CW complex of dimension n obtained by attaching a n-cell to $X^0 = \{\mathrm{pt}\}$ (with ϕ constant). Its k-skeletons are $\{\mathrm{pt}\}$ (for $0 \leq k < n$) and X (for $k \geq n$). **(5)** (See Figure 15(a-b) for the case $n = 2$) The cube $X = [0, 1]^n$ is a finite CW complex of dimension n with $\binom{n}{k} 2^{n-k}$ k-cells ($0 \leq k \leq n$). Its k-skeletons are formed by the graph of its vertexes ($k = 0$), of its edges ($k = 1$), and so on. Also its boundary $\partial X = X^{n-1}$ is a finite CW complex, of dimension $n-1$; thinking to "inflate" ∂X until it becomes $\mathbb{S}^{n-1}$, the CW structure of ∂X gives rise to a CW structure of $\mathbb{S}^{n-1}$ alternative to the simpler one of (3). **(6)** (See Figure 15(d)) The projective space $X = \mathbb{P}^n$ is obtained by attaching a n-cell to $\mathbb{P}^{n-1}$ (with $\phi : \mathbb{S}^{n-1} \to \mathbb{P}^{n-1}$ the Hopf map). Hence X has a k-cell for any $0 \leq k \leq n$. **(7)** The complex projective space $X = \mathbb{P}^n(\mathbb{C})$ is obtained by attaching a $2n$-cell to $\mathbb{P}^{n-1}(\mathbb{C})$: hence it has a k-cell for any $0 \leq k \leq 2n$ even. **(8)** As said above, the *triangularizable spaces* are the classical notion at the origin of CW complexes: let us briefly recall what they are. A "simplicial complex" in $\mathbb{R}^n$ is a family K of affine singular simplexes in $\mathbb{R}^n$ such that (1) if σ is in K, also its faces $\sigma^{(i)}$ are in K, and (2) if σ and τ are in K, then $|\sigma| \cap |\tau|$ is either the support of a face both of σ and of τ, or it is empty (in other words, any simplex in K is univoquely determined by the set of its vertexes: there are not two different simplexes with the same set of vertexes); the "support" of K is $|K| = \bigcup\{|\sigma| : \sigma \in K\} \subset \mathbb{R}^n$. A topological space X is said "triangularizable", or also "polyhedron", if there exists a homeomorphism $|K| \xrightarrow{\sim} X$ for some simplicial complex K; such homeomorphism (which, as it is easy to verify, defines a CW structure on X) is said to be a "triangularization" of X.[(74)]

[(69)] Note that, if X has finite dimension, the condition (3) is superfluous.

[(70)] Of course, these notions and properties do not depend on the particular CW structure considered.

[(71)] Namely, to attach a cell preserves compactness.

[(72)] Actually the name *CW complex* comes from the reference both to the just mentioned property of *Closure-finiteness* and to the term *Weak* (which denotes weak topology).

[(73)] otherwise, by Proposition 2.1.17 it should be compact, but it is not.

[(74)] In presence of a triangularization, the calcolation of the homology gets semplified by using a "ad hoc" theory, called "simplicial homology", which can be proven to give the same results of the singular homology (on the other hand, as we shall see soon, also the presence of the more general CW structure gives the same advantages, thanks to the "cellular homology"). Historically the triangularization procedure is the more classical one, but almost always it results to be much more complicated than the decomposition in cells: actually, to triangularize a space could be not simple at all. For example, while for a square $[0, 1]^2$ the disadvantage is still relative (it is enough to provide four vertexes, five edges and two triangles versus one 0-cell, one 1-cell and one 2-cell), one shows that for a torus $(\mathbb{S}^1)^2$ it is necessary to consider at least

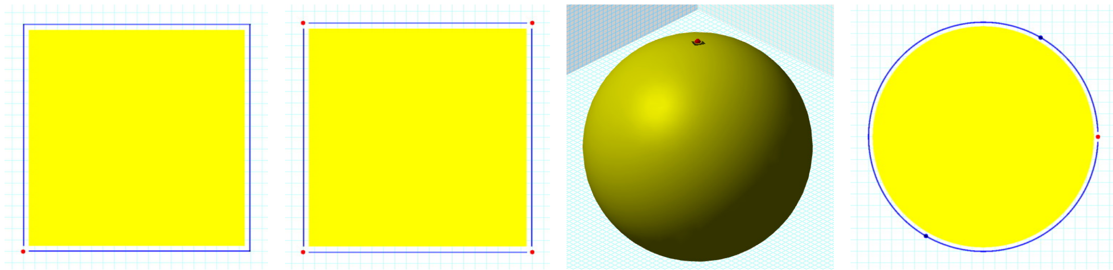

Figure 15: In these pictures the 0-cells are represented in red, the 1-cells in blue and the 2-cells in yellow. **(a-b)** Two CW structures on the square. **(c)** The standard CW structure of $\mathbb{S}^2$. **(d)** The standard CW structure (with the Hopf map) of the projective plane $\mathbb{P}^2$.

The decomposition of a CW complex in n-skeletons turns out to be particularly useful for the calcolation of the homology. Let us start with the following lemma.

Lemma 2.1.18. *Let X be a CW complex.*

(i) *The relative homology* (see Remark 2.1.5) $H_k(X^n/X^{n-1}, \mathbb{Z})$ *vanishes for $k \neq n$, and is a free abelian group of rank equal to the number of n-cells of X when $k = n$.*

(ii) $H_k(X^n, \mathbb{Z}) = 0$ *for any* $k > n$ (in particular, if X is of finite dimension then $H_k(X, \mathbb{Z}) = 0$ for any $k > \dim X$).

(iii) *The inclusion $X^n \hookrightarrow X$ induces a isomorphisms $H_k(X^n, \mathbb{Z}) \xrightarrow{\sim} H_k(X, \mathbb{Z})$ for $k < n$.*

Given a CW complex, one can naturally construct a complex (of *cellular chains*) of abelian groups

$$\cdots \to H_{n+1}(X^{n+1}/X^n, \mathbb{Z}) \xrightarrow{d_{n+1}} H_n(X^n/X^{n-1}, \mathbb{Z}) \xrightarrow{d_n} H_{n-1}(X^{n-1}/X^{n-2}, \mathbb{Z}) \to \cdots \tag{2.9}$$

where the abelian group $H_n(X^n/X^{n-1}, \mathbb{Z})$ is in degree n and, by the Lemma 2.1.18(i), is free of rank equal to the number of n-cells of X, and d_n is the composition of natural maps $H_n(X^n/X^{n-1}, \mathbb{Z}) \to H_{n-1}(X^{n-1}, \mathbb{Z}) \to H_{n-1}(X^{n-1}/X^{n-2}, \mathbb{Z})$, see (2.3). The homology groups of this complex, called *cellular homology groups* of X, are denoted by $H_k^{CW}(X, \mathbb{Z})$.

Theorem 2.1.19. *One has $H_k^{CW}(X, \mathbb{Z}) \simeq H_k(X, \mathbb{Z})$ for any $k \in \mathbb{Z}$.*

Corollary 2.1.20. *Let X be a CW complex.*

(i) *If X has a finite number N of n-cells, then $H_n(X, \mathbb{Z})$ has at most N generators* (in particular, if for a certain n the complex X has no n-cells then $H_n(X, \mathbb{Z}) = 0$).

(ii) *If X has no cells in adjacent dimensions, then the groups $H_n(X, \mathbb{Z})$ are free of rank equal to the number of n-cells of X.*

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seven vertexes, twenty-one edges and fourteen triangles versus one 0-cell, two 1-cells and one 2-cell, while for the projective space $\mathbb{P}^2$ there are at least six vertexes, fifteen edges and ten triangles versus one 0-cell, one 1-cell and one 2-cell (see e.g. Hatcher [8, §2.1], Bredon [2, p. 248]).

(iii) *If X is finite, then its Euler-Poincaré characteristic* (see (2.8)) *can be computed as*

$$\chi(X) = \sum_{j=0}^{\dim X} (-1)^j \sharp\{j\text{-cells of } X\} \tag{2.10}$$

and it is independent from the particular CW structure considered on X.

Proof. (i) By Lemma 2.1.18(i), $H_n(X^n/X^{n-1},\mathbb{Z})$ is a free abelian group with N generators, hence it follows that its subgroup $\ker(d_n)$ and the quotient of the latter $H_n^{CW}(X,\mathbb{Z}) \simeq H_n(X,\mathbb{Z})$ have at most N generators. (ii) Since there are no two consecutive non zero $H_n(X^n/X^{n-1},\mathbb{Z})$, the morphisms d_n of the complex of cellular homology are forced to be zero, hence $H_n(X,\mathbb{Z}) \simeq H_n^{CW}(X,\mathbb{Z}) \simeq H_n(X^n/X^{n-1},\mathbb{Z})$, and by Lemma 2.1.18(i) the latter is free of rank equal to the number of n-cells of X. (iii) If X is finite, the groups $H_j^{CW}(X,\mathbb{Z})$ are of finite rank and vanish for $j > n = \dim X$, hence $\chi(X) = \sum_j (-1)^j \mathrm{rk}\, H_j(X,\mathbb{Z}) = \sum_{j=0}^n (-1)^j \mathrm{rk}\, H_j^{CW}(X,\mathbb{Z})$ is defined. But from Proposition A.2.5 applied to the complex of cellular homology we get also $\chi(X) = \sum_{j=0}^n (-1)^j \mathrm{rk}\, H_j(X^j/X^{j-1},\mathbb{Z})$, and (2.10) follows from Lemma 2.1.18(i); the independence of $\chi(X)$ from the CW structure considered on X is clear, since any two of them give homeomorphic topological spaces, and hence the same homology groups. □

Examples. **(1)** The complex of cellular chains of $X = [0,1]$ is $0 \to \mathbb{Z} \xrightarrow{d_1} \mathbb{Z}^2 \to 0$, and since X is connected it must be $H_0(X,\mathbb{Z}) \simeq \mathbb{Z}$, which implies that d_1 is injective: hence $H_1(X,\mathbb{Z}) = 0$. So it holds $\chi(X) = \mathrm{rk}\, H_0(X,\mathbb{Z}) - \mathrm{rk}\, H_1(X,\mathbb{Z}) = \sharp(\text{vertexes}) - \sharp(\text{edges}) = 1$. **(2)** For $X = [0,1[$ the complex becomes $0 \to \mathbb{Z}^{(\mathbb{N})} \xrightarrow{d_1} \mathbb{Z}^{(\mathbb{N})} \to 0$, with $d_1(D_n^1) = P_{n+1} - P_n$: hence[75] $\ker(d_1) = 0$ and $\mathrm{coker}(d_1) \simeq \mathbb{Z}$, and we get that $H_0(X,\mathbb{Z}) = \mathbb{Z}$ and $H_1(X,\mathbb{Z}) = 0$, as it is well known.[76] **(3)** If X is the bouquet of n circles, the complex of cellular chains is $0 \to \mathbb{Z}^n \xrightarrow{d_1} \mathbb{Z} \to 0$; since X is connected, it must be $H_0(X,\mathbb{Z}) \simeq \mathbb{Z}$, which implies $d_1 = 0$: hence $H_1(X,\mathbb{Z}) = \mathbb{Z}^n$, and $\chi(X) = 1 - n$. **(4)** If $n \geq 2$, from Corollary 2.1.20(ii) one immediately gets $H_j(\mathbb{S}^n,\mathbb{Z}) \simeq \mathbb{Z}$ (if $j = 0, n$) and zero otherwise; then $\chi(\mathbb{S}^n) = 1 + (-1)^n$. **(5)** Let $X = [0,1]^n$. Since X is contractible, it must be $\chi(X) = \chi(\{\mathrm{pt}\}) = 1$. On the other hand, for (2.10) it must be also $\chi(X) = \sum_{j=0}^n (-1)^j \binom{n}{j} 2^{n-j}$, and this is true since $\sum_{j=0}^n (-1)^j \binom{n}{j} 2^{n-j} = (-1)^n \sum_{j=0}^n (-1)^j \binom{n}{j} 2^j = (-1)^n \sum_{j=0}^n \binom{n}{j} (-2)^j = (-1)^n [1 + (-2)]^n = 1$. Similarly, ∂X is homeomorphic to $\mathbb{S}^{n-1}$ and hence $\chi(\partial X) = \chi(\mathbb{S}^{n-1}) = 1 + (-1)^{n-1}$: namely, also for (2.10) one has $\chi(\partial X) = \sum_{j=0}^{n-1} (-1)^j \binom{n}{j} 2^{n-j} = \left(\sum_{j=0}^n (-1)^j \binom{n}{j} 2^{n-j}\right) - (-1)^n = 1 - (-1)^n = 1 + (-1)^{n-1}$. **(6)** The complex of cellular chains of the projective space $X = \mathbb{P}^n$ is $0 \to \mathbb{Z} \xrightarrow{d_n} \mathbb{Z} \to \cdots \to \mathbb{Z} \xrightarrow{d_1} \mathbb{Z} \to 0$. One can calculate that d_k is zero for k odd, and the multiplication by 2 for k even: one then obtains that $H_k(X,\mathbb{Z})$ is isomorphic to $\mathbb{Z}$ if $k = 0$ or if ($k = n$, n odd); to $\mathbb{Z}/2\mathbb{Z}$ if ($0 < k < n$, k odd); and zero otherwise. Hence it holds $\chi(X) = 1$ (if n is even) and $\chi(X) = 1 - 1 = 0$ (if n is odd): namely, since X is formed by one k-cell for any $0 \leq k \leq n$, this is also what says (2.10) (i.e. $\chi(X) = 1 - 1 + 1 - 1 + \cdots$, with $n+1$ summands). Using another method, to think $\mathbb{P}^2$ as a square with the identification of antipodal boundary points (see Examples 2.8) gives to it another CW structure with two 0-cells, two 1-cells and one 2-cell: therefore, once more by (2.10) one has that $\chi(\mathbb{P}^2) = 2 - 2 + 1 = 1$. **(7)** For the complex projective space $X = \mathbb{P}^n(\mathbb{C})$, from Corollary 2.1.20(ii) one gets that $H_k(X,\mathbb{Z}) \simeq \mathbb{Z}$ for ($0 \leq k \leq 2n$, k even) and zero otherwise; it holds $\chi(X) = n + 1$ (note that $\mathbb{P}^1(\mathbb{C}) \simeq \mathbb{S}^2$, and namely $\chi(\mathbb{P}^1(\mathbb{C})) = 2 = \chi(\mathbb{S}^2)$) **(8)** Consider any polyhedral surface X in $\mathbb{R}^3$ homeomorphic to $\mathbb{S}^2$.[77] Then $\chi(X) = \chi(\mathbb{S}^2) = 2$; on the other hand, by (2.10) it holds

[75] It holds $d_1(\sum_{n\in\mathbb{N}} a_n D_n^1) = \sum_{n\in\mathbb{N}} a_n(P_{n+1} - P_n) = -a_1 P_1 - \sum_{n\geq 2}(a_n - a_{n-1})P_n$: if this quantity is zero then $-a_1 = 0$, $a_2 - a_1 = 0$, ... which implies $\ker(d_1) = 0$; if one aims to solve the equality $-a_1 P_1 - \sum_{n\geq 2}(a_n - a_{n-1})P_n = \sum_{n\in\mathbb{N}} b_n P_n$ (where, say, $b_n = 0$ for any $n > N$) one finds $a_0 = -b_0$, $a_1 = -b_0 - b_1$, ..., $a_j = -\sum_{n=1}^N b_n$ for any $j \geq N$, which forces $\sum_{n=1}^N b_n = 0$ (otherwise the a_n's would not be "almost all zero", as it is prescribed by the direct sum): this says that $\mathrm{coker}(d_1) \simeq \mathbb{Z}$.

[76] In this case, of course, one would have better used the homotopic invariance of the singular homology.

[77] For example the external surfaces of a cube, of a tetrahedron, of a prism or pyramid with polygonal base, of a parallelepiped with a pyramidal hole in the upper face,...

also $\chi(X) = V - E + F$, where V (resp. E, F) indicates the number of vertices (resp. edges, faces) of X. We deduce the well-known Euler relation $V - E + F = 2$ (for example, $8 - 12 + 6$ for the external surface of a cube, $4 - 6 + 4$ for the one of a tetrahedron, ...).

2.2 Cohomology of de Rham

The singular cohomology is defined for any topological space; now we deal with a different cohomological construction (with coefficients in $R = \mathbb{R}$)[78] for $\mathcal{C}^\infty$ manifolds, due to G. de Rham. This construction will give rise to $\mathbb{R}$-vector spaces isomorphic to the spaces of singular cohomology with coefficients in $\mathbb{R}$ defined above (see Theorem 2.6.9).

Let X be a $\mathcal{C}^\infty$ manifold of dimension n, $\{(U_\lambda, \varphi_\lambda) : \lambda \in \Lambda\}$ a $\mathcal{C}^\infty$ atlas, $\pi : T^*X \to X$ the cotangent bundle. Consider the mth external power of T^*X for $m = 0, \dots, n$:

$$\wedge^m(T^*X) = \{(x, \alpha) : x \in X,\ \alpha \in \wedge^m(T^*_x X)\}.$$

The pair $(\wedge^m(T^*X), \wedge^m\pi)$, where $\wedge^m\pi : \wedge^m(T^*X) \to X$ is the natural projection, is a vector bundle of rank $\binom{n}{m}$ on X: in particular, $\wedge^0(T^*X) = X \times \mathbb{R}$ and $\wedge^1(T^*X) = T^*X$.

Definition 2.2.1. The sheaf Ω^m_X of $\mathcal{C}^\infty$ *differential m-forms* on X is the sheaf of $\mathbb{R}$-vector spaces on X of $\mathcal{C}^\infty$ sections of the real vector bundle $(\wedge^m(T^*X), \wedge^m\pi)$.

Therefore, given an open subset $U \subset X$, a differential m-form $\omega \in \Omega^m_X(U)$ assigns in a smooth way to any $x \in U$ an alternating m-form $\omega(x)$ on T_xX and, if $U \subset V$, one has a restriction map $\Omega^m_X(V) \to \Omega^m_X(U)$. Note that in particular it holds $\Omega^0_X = \mathcal{C}^\infty_X$.

Remark 2.2.2. In the case of an open subset $W \subset \mathbb{R}^n$, let u_i be the coordinate functions (which form a basis of $T^*_tW \simeq (\mathbb{R}^n)^*$ for any $t \in W$). For any $t \in W$ it holds $du_i(t) = u_i$, hence (see Example A.3) a differential m-form $\mathcal{C}^\infty$ on W can be uniquely written as $\omega = \sum_{|I|=m} f_I du_I$, where $f_I \in \mathcal{C}^\infty(W)$ and $du_I = du_{i_1} \wedge \cdots \wedge du_{i_m}$ with $1 \le i_1 < \cdots < i_m \le n$. Hence, on $U \cap U_\lambda$ a $\omega \in \Omega^m_X(U)$ can be uniquely written as $\omega = \sum_{|I|=m} f_I dx_{\lambda,I}$, where $f_I \in \mathcal{C}^\infty(U)$ and $dx_{\lambda,I} = dx_{\lambda,i_1} \wedge \cdots \wedge dx_{\lambda,i_m}$.

For $\omega \in \Omega^m_X(U)$ and $\theta \in \Omega^p_X(U)$, the *external product* (or *wedge product*) $\omega \wedge \theta \in \Omega^{m+p}_X(U)$ will be defined pointwise, i.e. $(\omega \wedge \theta)(x) = \omega(x) \wedge \theta(x)$ for any $x \in U$, and will enjoy the same formal properties shown in Appendix A.3: in particular it holds $\omega \wedge \theta = (-1)^{mp}\theta \wedge \omega$. This gives to the sheaf of graded $\mathbb{R}$-vector spaces

$$\Omega^\bullet_X = \bigoplus_{m=0}^{n} \Omega^m_X$$

the structure of sheaf of graded $\mathbb{R}$-algebras, with the multiplication

$$\cdot \wedge \cdot : \Omega^\bullet_X \otimes_{\mathbb{R}} \Omega^\bullet_X \to \Omega^\bullet_X.$$

Example. If $\omega = f_1\, dt_1 + f_2\, dt_2 + f_3\, dt_3 \in \Omega^1_{\mathbb{R}^3}(U)$ and $\theta = g_{12}\, dt_1dt_2 + g_{13}\, dt_1dt_3 + g_{23}\, dt_2dt_3 \in \Omega^2_{\mathbb{R}^3}(U)$, one has $\omega \wedge \theta = (f_1g_{23} - f_2g_{13} + f_3g_{12})dt_1dt_2dt_3 \in \Omega^3_{\mathbb{R}^3}(U)$.

If $f : X \to Y$ is a $\mathcal{C}^\infty$ map, the *pull-back* of differential forms associated to f is the morphism of sheaves on Y

$$f^* : \Omega^\bullet_Y \to f_*\Omega^\bullet_X$$

[78] Viewing $\mathcal{C}^\infty$ as functions with values in $\mathbb{C}$ (as it is usual to do) instead than only in $\mathbb{R}$ as we do here, one naturally obtains the same construction with values in $\mathbb{C}$ instead than in $\mathbb{R}$.

defined pointwise: if $V \subset Y$ is open, $\omega \in \Omega^m_Y(V)$, $x \in f^{-1}(V)$ and $u_1, \dots, u_m \in T_xX$, one defines $f^*\omega \in \Omega^m_X(f^{-1}(V))$ by $(f^*\omega)(x) = df^*_x(\omega(x))$, i.e.

$$(f^*\omega)(x)(u_1, \dots, u_m) = \omega(f(x))(df_x(u_1), \dots, df_x(u_m)).$$

(If $\iota : U \to X$ is the embedding of an open subset, ι^* is simply the restriction map.) In local coordinates, let $X \supset U_\lambda \xrightarrow{\sim} \mathbb{R}^n$, $Y \supset V_\mu \xrightarrow{\sim} \mathbb{R}^p$ and $f : U_\lambda \to V_\mu$: then, written $f = (f_1, \dots, f_p)$ and $\omega = \sum_{|I|=m} \omega_I dy_{\mu,I}$, one has

$$f^*\omega = \sum_{|I|=m} (\omega_I \circ f) df_I, \tag{2.11}$$

where $df_I = df_{i_1} \wedge \cdots \wedge df_{i_m}$ with $df_{i_j} = \frac{\partial f_{i_j}}{\partial x_{\lambda,1}} dx_{\lambda,1} + \cdots + \frac{\partial f_{i_j}}{\partial x_{\lambda,n}} dx_{\lambda,n}$. In particular, if $f : U \xrightarrow{\sim} V$ is a diffeomorphism of open subsets of $\mathbb{R}^n$ and $\omega = dy_1 \wedge \cdots \wedge dy_n$, then, using (A.7) and (2.11), one obtains

$$f^*(dy_1 \wedge \cdots \wedge dy_n) = \det(J_f)\, dx_1 \wedge \cdots \wedge dx_n. \tag{2.12}$$

The pull-back commutes with the external product, and hence one obtains:

Proposition 2.2.3. $\Omega^\bullet$ *is a contravariant functor from* $\mathcal{C}^\infty$ *to the category of graded* $\mathbb{R}$-*algebras, which assigns to any* $X \in \mathcal{C}^\infty$ *the graded* $\mathbb{R}$-*algebra* $\Omega^\bullet_X(X)$ *and to any morphism* $f : X \to Y$ *of* $\mathcal{C}^\infty$ *manifolds the pull-back morphism* $f^* : \Omega^\bullet_Y(Y) \to \Omega^\bullet_X(X)$.

Let us construct a *differential* morphism for the differential forms, which generalizes the classical differential of functions and allows to interpret $\Omega^\bullet_X$ also as complex of sheaves on X. We start by observing that

Proposition 2.2.4. *If* $f : X \to Y$ *is a map of* $\mathcal{C}^\infty$ *manifolds, then* f^* *commutes with the differential of functions.*

Proof. If $\alpha : Y \to \mathbb{R}$ is $\mathcal{C}^\infty$, $x \in X$, $v \in T_xX$, then $(f^*d\alpha)(x)(v) = d\alpha(f(x))(df_x(v)) = d(\alpha \circ f)(x)(v)$, i.e. $f^*d\alpha = d(\alpha \circ f) = d(f^*\alpha)$. □

Now we define the differential in the affine case.

Proposition 2.2.5. *For any open* $U \subset \mathbb{R}^n$ *there exists one and only one application*

$$D : \Omega^\bullet_{\mathbb{R}^n}(U) \to \Omega^{\bullet+1}_{\mathbb{R}^n}(U)$$

such that:

(1) D *is* $\mathbb{R}$-*linear;*

(2) $D(\omega \wedge \theta) = D\omega \wedge \theta + (-1)^m \omega \wedge D\theta$ *for any* $\omega \in \Omega^m_{\mathbb{R}^n}(U)$ *and* $\theta \in \Omega^p_{\mathbb{R}^n}(U)$*;*

(3) $D \circ D = 0$*;*

(4) $D : \Omega^0_{\mathbb{R}^n}(U) = \mathcal{C}^\infty(U) \to \Omega^1_{\mathbb{R}^n}(U)$ *is the usual differential of functions* d.

If $\omega = \sum_{|I|=m} \omega_I dx_I \in \Omega^m_{\mathbb{R}^n}(U)$*, such application is*

$$D\omega = \sum_{|I|=m} d\omega_I \wedge dx_I \qquad \in \Omega^{m+1}_{\mathbb{R}^n}(U). \tag{2.13}$$

Proof. The fact that (2.13) satisfies the conditions (1) and (4) is obvious, while (2) and (3) can be verified by a direct calculation (exercise: we may assume in (2) that $\omega = \omega_I dx_I$ and $\theta = \theta_J dx_J$, in (3) that ω be either a function or $\omega = dx_I$). Finally let δ be another operator satisfying the conditions (1)–(4). We first remark that for any I with $|I| = m$ it holds $\delta(dx_I) = 0$ (arguing by induction on m: if $m = 1$ then by (4) and (3) $\delta(dx_i) = \delta(\delta x_i) = 0$; then, assuming the statement be true until $m-1$, the induction proceeds to m by (2)): therefore, setting $\omega = \sum_{|I|=m} \omega_I dx_I$, we have $\delta\omega = \sum_{|I|=m}[\delta\omega_I \wedge dx_I + (-1)^1 \omega_I \delta(dx_I)] = \sum_{|I|=m} \delta\omega_I \wedge dx_I = D\omega$. $\square$

From now on, D shall be denoted by d.

Corollary 2.2.6. *If $f : V \xrightarrow{\sim} U$ is a diffeomorphism between open subsets of $\mathbb{R}^n$, then d commutes with f^*.*

Proof. Let $\delta = (f^{-1})^* \circ d \circ f^*$: it satisfies (1), (2), (3) (exercise) and (4) (see Proposition 2.2.4), hence $\delta = d$ by Proposition 2.2.5. $\square$

Let X be a manifold, $U \subset X$ an open subset, $\omega \in \Omega^m_X(U)$. Given a chart $(U_\lambda, \varphi_\lambda)$ with $U \cap U_\lambda \neq \varnothing$, define $d\omega$ on $U \cap U_\lambda$ by $\varphi_\lambda^* d((\varphi_\lambda^{-1})^*\omega)$. By Corollary 2.2.6, this is a good definition: if $U \cap U_\lambda \cap U_\mu \neq \varnothing$, set $f = \varphi_\mu \circ \varphi_\lambda^{-1}$ one has $\varphi_\lambda^* d((\varphi_\lambda^{-1})^*\omega) = \varphi_\lambda^* d(f^*(\varphi_\mu^{-1})^*\omega) = \varphi_\lambda^* f^* d((\varphi_\mu^{-1})^*\omega) = \varphi_\mu^* d((\varphi_\mu^{-1})^*\omega)$, where the second equality comes from Corollary 2.2.6. This differential inherits the properties (1)–(4) from the affine case; we now show that d commutes with the pull-back.

Proposition 2.2.7. *If $f : X \to Y$ is a morphism of $\mathcal{C}^\infty$ manifolds, then $d \circ f^* = f^* \circ d$.*

Proof. Let $V \subset Y$ be an open subset and $\omega \in \Omega^m_Y(V)$: let us show that $d(f^*\omega) = f^*(d\omega)$ in $\Omega^{m+1}_X(f^{-1}(V))$. From Proposition 2.2.4 we know that this is true when $m = 0$, and can be verified by recurrence if $\omega = d\phi$ (namely $d\omega = 0$ and $d(f^*\omega) = d(f^*d\phi) = d(d(f^*\phi)) = 0$); moreover, if $\omega = \theta \wedge \varepsilon$ and the commutation holds for θ and ε, it holds also for ω (use the property (2)). Now, since the definition of d is local, we may bound to the case where $\omega = \sum_{|I|=m} \omega_I dy_I$, and then this is true for what has been said previously. $\square$

Let X be a $\mathcal{C}^\infty$ manifold of dimension n. By Proposition 2.2.7, d commutes with the restriction. Hence d is a morphism of sheaves, and since $d \circ d = 0$ the following definition makes sense.

Definition 2.2.8. The *complex of de Rham* of sheaves on X is

$$\Omega^\bullet_X : \qquad 0 \to \mathcal{C}^\infty_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n-1}_X \xrightarrow{d} \Omega^n_X \to 0.$$

Given an open subset $U \subset X$, the elements of the subspace

$$Z^m_X(U) = \{\omega \in \Omega^m_X(U) : d\omega = 0\} \qquad (\text{resp. } B^m_X(U) = d\big(\Omega^{m-1}_X(U)\big) \subset Z^m_X(U))$$

(resp. the m-cocycles and m-coboundaries in U) are called *closed* (resp. *exact*) differential m-forms on U. The *de Rham cohomology* on U is the cohomology of the complex (of cochains) $\Omega^\bullet_X$ calcolated in U, i.e. the $\mathbb{R}$-vector spaces

$$H^m_X(U) = \frac{Z^m_X(U)}{B^m_X(U)}.$$

It is clear that the cohomology of de Rham is concentrated between the degrees 0 and n.

Example. The classical example is the de Rham complex on $X = \mathbb{R}^3$. Given an open subset $W \subset \mathbb{R}^3$, the differential $d : \mathcal{C}^\infty_{\mathbb{R}^3}(W) \to \Omega^1_{\mathbb{R}^3}(W)$ is the usual differential $d\phi = \frac{\partial\phi}{\partial t_1}dt_1 + \frac{\partial\phi}{\partial t_2}dt_2 + \frac{\partial\phi}{\partial t_3}dt_3$; the differential

$d : \Omega^1_{\mathbb{R}^3}(W) \to \Omega^2_{\mathbb{R}^3}(W)$ is the *curl* operator $\mathrm{curl}(\alpha_1 dt_1 + \alpha_2 dt_2 + \alpha_3 dt_3) = (\frac{\partial\alpha_2}{\partial t_1} - \frac{\partial\alpha_1}{\partial t_2})dt_1\, dt_2 - (\frac{\partial\alpha_1}{\partial t_3} - \frac{\partial\alpha_3}{\partial t_1})dt_1\, dt_3 + (\frac{\partial\alpha_3}{\partial t_2} - \frac{\partial\alpha_2}{\partial t_3})dt_2\, dt_3$; the differential $d : \Omega^2_{\mathbb{R}^3}(W) \to \Omega^3_{\mathbb{R}^3}(W)$ is the *divergence* $\mathrm{div}(\rho_3 dt_1\, dt_2 - \rho_2 dt_1\, dt_3 + \rho_1 dt_2\, dt_3) = (\frac{\partial\rho_1}{\partial t_1} + \frac{\partial\rho_2}{\partial t_2} + \frac{\partial\rho_3}{\partial t_3})dt_1\, dt_2\, dt_3$.

Given a morphism $f : X \to Y$ of $\mathcal{C}^\infty$ manifolds, we have seen (Proposition 2.2.7) that the pull-back $f^* : \Omega^\bullet_Y \to f_*\Omega^\bullet_X$ is a morphism of complexes. Then for any open $V \subset Y$ it is naturally induced a functorial morphism of cohomology (see Appendix A.2)

$$H^\bullet f^* : H^\bullet_Y(V) \to H^\bullet_X(f^{-1}(V)).$$

Proposition 2.2.9. *Let $A \subset X$ be a $\mathcal{C}^\infty$ submanifold of X, $r : X \to A$ a $\mathcal{C}^\infty$ retraction. Then, denoted by $\iota : A \to X$ the canonical inclusion, the morphism $H^\bullet\iota^* : H^\bullet_X(X) \to H^\bullet_X(A)$ is surjective and the morphism $H^\bullet r^* : H^\bullet_X(A) \to H^\bullet_X(X)$ is injective.*

Proof. Since $r \circ \iota = \mathrm{id}_A$, one has $H^\bullet\iota^* \circ H^\bullet r^* = \mathrm{id}_{H^\bullet_X(A)}$, which implies the statement. □

We can construct another complex by considering the forms with compact support. Given an open subset $U \subset X$, we define

$$\mathcal{C}^\infty_{X,c}(U) = \{\phi \in \mathcal{C}^\infty_X(U) : \mathrm{supp}(\phi) \text{ is compact in } U\}.$$

Since $\mathrm{supp}(d\omega) \subset \mathrm{supp}(\omega)$, the complex of de Rham induces another complex with the forms with compact support: if $Z^m_{X,c}(U) := Z^m_X(U) \cap \Omega^m_{X,c}(U)$ and $B^m_{X,c}(U) := d\big(\Omega^{m-1}_{X,c}(U)\big) \subset Z^m_{X,c}(U)$, the *de Rham cohomology with compact support* on an open subset $U \subset X$ is

$$H^m_{X,c}(U) = \frac{Z^m_{X,c}(U)}{B^m_{X,c}(U)}.$$

Obviously, if X is a compact manifold then $H^m_{X,c}(X) = H^m_X(X)$.

Remark 2.2.10. Note that $\Omega^\bullet_c$ is not a functor on $\mathcal{C}^\infty$: namely the pull-back of a form with compact support does not have necessarily compact support (for example, the pull-back of functions by vector bundles). However, $\Omega^\bullet_c$ is a contravariant functor from $\mathcal{C}^\infty$ to the category of graded $\mathbb{R}$-algebras if one considers only the proper $\mathcal{C}^\infty$ morphisms[79]: hence, given a proper morphism $f : X \to Y$ of $\mathcal{C}^\infty$ manifolds, for any open $V \subset Y$ it is naturally induced a morphism of cohomology $H^\bullet_c f^* : H^\bullet_{Y,c}(V) \to H^\bullet_{X,c}(f^{-1}(V))$. Moreover, it is a covariant functor for the open inclusions: if $U \subset X$ is open and $\iota : U \to X$ is the inclusion, then $\iota_* : \Omega^\bullet_{U,c} \to \Omega^\bullet_{X,c}$ is the natural morphism of extension by zero, and induces a morphism of cohomology $H^\bullet_c\iota_* : H^\bullet_{U,c}(U) \to H^\bullet_{X,c}(X)$.

Remark 2.2.11. From the construction it is evident that the cohomology is invariant under diffeomorphisms: as a consequence, if $U \subset X$ we shall often write $H^\bullet(U)$ and $H^\bullet_c(U)$ instead of $H^\bullet_X(U)$ and $H^\bullet_{X,c}(U)$.

[79] I.e., the inverse image of any compact of Y is compact in X.

Remark 2.2.12. We can define the cohomology of de Rham also for the manifolds with boundary (see Appendix A.5): if V is an open subset of $\mathbb{H}^n$, the $\mathcal{C}^\infty$ functions on V are the restrictions to V of $\mathcal{C}^\infty$ functions defined on some open neighborhood of V in $\mathbb{R}^n$.[80] The same can be said for the cohomology of de Rham with compact support: the $\mathcal{C}^\infty$ functions on V with compact support are the restrictions to V of $\mathcal{C}^\infty$ functions defined on some open neighborhood of V in $\mathbb{R}^n$ such that the intersection of their support with V is compact.

The cohomology of manifolds will be the object of our study in the sequel. For the degree zero the computation is immediate.

Proposition 2.2.13. *$H^0(U)$ is the vector space of functions locally constant on U* (hence $\dim_{\mathbb{R}}(H^0(U))$ is the number of connected components of U)*.*[81] *Analogously, $H^0_c(U)$ is the vector space of functions locally constant on U with compact support* (hence $\dim_{\mathbb{R}}(H^0_c(U))$ is the number of compact connected components of U)*.*

Proof. It is a direct consequence of the definition. □

Examples. **(0)** It holds $H^0(\{\mathrm{pt}\}) = \mathbb{R}$ and $H^m(\{\mathrm{pt}\}) = 0$ for $m \neq 0$. **(1)** Let $X = \mathbb{R}$. Then $H^0(\mathbb{R}) = \mathbb{R}$; moreover $B^1_{\mathbb{R}}(\mathbb{R}) = \Omega^1_{\mathbb{R}}(\mathbb{R})$ since $\omega = f dx = dg$, with $g(x) = \int_0^x f(t)dt$, and so $H^1(\mathbb{R}) = 0$. Since there are no constant functions on $\mathbb{R}$ with compact support, one has $H^0_c(\mathbb{R}) = 0$. Then consider the integration map $\int_{\mathbb{R}} : \Omega^1_c(\mathbb{R}) \to \mathbb{R}$, which is a morphism of $\mathbb{R}$-vector spaces. We claim that $B^1_{\mathbb{R},c}(\mathbb{R}) = \ker(\int_{\mathbb{R}})$: if $\omega = df \in B^1_{\mathbb{R},c}(\mathbb{R})$ with $\mathrm{supp}(f)$ compact in $]a, b[$, then $\int_{\mathbb{R}} \omega = \int_a^b f'(x)dx = f(b) - f(a) = 0$; conversely, if $\omega = f\, dx \in \ker(\int_{\mathbb{R}})$, then $\omega = dg$ with $g(x) = \int_{-\infty}^x f(t)dt \in \mathcal{C}^\infty_c(\mathbb{R})$. Hence, $\int_{\mathbb{R}}$ being clearly surjective, one has $H^1_c(\mathbb{R}) = \Omega^1_c(\mathbb{R})/\ker(\int_{\mathbb{R}}) = \mathbb{R}$. Later se shall show (Poincaré lemmas) that for any $n \in \mathbb{N}$ it holds $H^0(\mathbb{R}^n) = H^n_c(\mathbb{R}^n) = \mathbb{R}$ and $H^m(\mathbb{R}^n) = H^{n-m}_c(\mathbb{R}^n) = 0$ for any $m \neq 0$. **(2)** Let $X = \mathbb{R}_{\geq 0}$, a manifold with boundary $\partial X = \{0\}$ (see Remark 2.2.12). One has $\Omega^0(X) = \mathcal{C}^\infty_{\mathbb{R}}(X) \simeq \{ f|_X : f \in \mathcal{C}^\infty_{\mathbb{R}}(\mathbb{R})\}$ and $\Omega^1(X) \simeq \{gdx : g \in \mathcal{C}^\infty_{\mathbb{R}}(X)\}$: hence $H^0(X) = \mathbb{R}$ and $H^1(X) = 0$, exactly as for $\mathbb{R}$. On the other hand one has $\Omega^0_c(X) = \mathcal{C}^\infty_{\mathbb{R},c}(X) \simeq \{ f|_X : f \in \mathcal{C}^\infty_{\mathbb{R}}(\mathbb{R}), \mathrm{supp}(f) \cap X \text{ compact}\}$ and $\Omega^1_c(X) \simeq \{\psi dx : \psi \in \mathcal{C}^\infty_{\mathbb{R},c}(X)\}$: hence $H^0_c(X) = 0$, while, given $\psi \in \mathcal{C}^\infty_{\mathbb{R},c}(X)$, one has $\psi dx = d\varphi$ with $\varphi(x) = -\int_x^{+\infty} \psi(t)dt$ (note that $\mathrm{supp}(\varphi) \cap X$ is compact, i.e. $\varphi \in \mathcal{C}^\infty_{\mathbb{R},c}(X)$), and hence also $H^1_c(X) = 0$.

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[80] In fact, $\mathbb{H}^n$ is closed in the paracompact space $\mathbb{R}^n$: see Remark A.4.2.

[81] On the other hand, recall that the dimension of the singular cohomology in degree zero $H^0(U; \mathbb{R})$ is the number of *arcwise* connected components of U.