Sheaves and $\mathcal{D}$-modules on dual flag manifolds and generalized Verma modules

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Abstract

Let G be a complex semisimple algebraic group, X and X' be dual generalized flag manifolds of G, and O be the open dense G-orbit in $X \times X'$. We first show that the integral transform from X to X' naturally associated to O provides an equivalence at the level of derived categories of sheaves and $\mathcal{D}$-modules, then we prove that the problem of computing the integral transform of a quasi-G-equivariant locally free $\mathcal{D}_X$-module of finite rank is equivalent to studying the irreducibility of a generalized Verma module. In particular, when $G = SL(n, \mathbb{C})$ and X and X' are dual Grassmann manifolds, we give an alternative proof of our results of [13], and concretely show how the above construction can be applied to the representation theory of real forms of G.

Introduction

In our work [13] we studied the correspondence between dual Grassmann manifolds of a finite-dimensional complex vector space V given by transversal linear subspaces of complementary dimensions in V. In particular we showed that this correspondence defines an equivalence at the level of derived categories of sheaves and $\mathcal{D}$-modules, and computed the induced integral transform of $\mathcal{D}$-modules associated to $SL(V)$-equivariant line bundles. The object of this paper is to consider the same problem in the more general setting of dual generalized flag manifolds of any complex semisimple algebraic group, and of equivariant bundles of any finite rank. Although the problem has an analytic nature, our approach here will be essentially algebraic, and hence quite different from the one of [13].

Let G be a connected and simply connected semisimple algebraic group over $\mathbb{C}$, P and P' a pair of dual parabolic subgroups of G (i.e. $L = P \cap P'$ is reductive), $\mathfrak{g}$, $\mathfrak{p}$ and $\mathfrak{p}'$ their Lie algebras, $X = G/P$ and $X' = G/P'$ the corresponding dual generalized flag manifolds of G. Let $\mathrm{O} \simeq G/L$ be the open dense orbit of G in $X \times X'$ for the diagonal action (note that O is affine), $i : \mathrm{O} \to X \times X'$ be the open embedding, and $p_1 : X \times X' \to X$ and $p_2 : X \times X' \to X'$ be the projections. The relation O provides a natural correspondence $r_{\mathrm{O}} : \mathbf{D}^{\mathrm{b}}(\mathbb{C}_X) \to \mathbf{D}^{\mathrm{b}}(\mathbb{C}_{X'})$ between the bounded derived categories of sheaves on X and X' by defining

$$r_{\mathrm{O}}(F) = Rp_{2!}(p_1^{-1}F \otimes \mathbb{C}_{\mathrm{O}|X\times X'})$$

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where the operations $Rp_{2!}$, p_1^{-1} and $\otimes$ are the proper direct image, inverse image and tensor product in the derived category of sheaves, and $\mathbb{C}_{\mathrm{O}|X\times X'}$ is the constant sheaf with fiber $\mathbb{C}$ on O and zero outside. Similarly, for $\mathcal{D}$-modules one considers the $\mathcal{D}_{X\times X'}$-module $\mathcal{B}_{\mathrm{O}} = i_*\mathcal{O}_{\mathrm{O}}$ of rational functions on $X \times X'$ with poles on the G-invariant closed subset complementary to O (note that the sheaf $K = \mathbb{C}_{\mathrm{O}|X\times X'}$ and the $\mathcal{D}$-module $\mathcal{K} = \mathcal{B}_{\mathrm{O}}$ are related by the Riemann-Hilbert correspondence [8], in the sense that K is the sheaf of homomorphic solutions of $\mathcal{K}$), and defines the functor $\mathcal{R}_{\mathrm{O}} : \mathbf{D}^{\mathrm{b}}(\mathcal{D}_X) \to \mathbf{D}^{\mathrm{b}}(\mathcal{D}_{X'})$ by

$$\mathcal{R}_{\mathrm{O}}(\mathcal{M}) = Dp_{2*}(Dp_1{}^*\mathcal{M} \otimes^D \mathcal{B}_{\mathrm{O}})$$

where the operations Dp_{2*}, $Dp_1{}^*$ and $\otimes^D$ are the direct image, inverse image and tensor product for algebraic $\mathcal{D}$-modules (see 1.2). Our first main result says that the transforms r_{O} and $\mathcal{R}_{\mathrm{O}}$ are nice (see Theorem 2.2 for more details):

Theorem 1. *The functors r_{O} and $\mathcal{R}_{\mathrm{O}}$ are equivalences of categories.*

The next step is to "quantize" the equivalence $\mathcal{R}_{\mathrm{O}}$, by computing the integral transform of a quasi-G-equivariant locally free $\mathcal{D}_X$-module of finite rank.
For a L-dominant integral weight λ let $V(\lambda)$ be the corresponding finite dimensional L-module, and let λ' (see (1.4)) be the L-dominant integral weight associated to λ such that $V(\lambda')$ is the dual L-module of $V(\lambda)$. By considering $V(\lambda)$ as a P-module with the trivial action of the unipotent part of P, one can define the generalized Verma module $M_P(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V(\lambda)$, which is endowed with a natural structure of $(\mathfrak{g}, P)$-module (see 1.3). Let $\mathcal{O}_X(\lambda)$ be the G-equivariant locally free $\mathcal{O}_X$-module of regular sections of the vector bundle $G \times_P V(\lambda)$ on X. The locally free $\mathcal{D}_X$-module $\mathcal{D}_X(\lambda) := \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(\lambda)$ carries a natural structure of quasi-G-equivariant $\mathcal{D}_X$-module (see Section 1.3). The same argument holds for the dual parabolic subgroup P' and the dual manifold X'. The second main result, which relates our quantization problem to the irreducibility of generalized Verma modules, can be stated as follows (see Theorem 2.8 for more details):

Theorem 2. *One has $H^j(\mathcal{R}_{\mathrm{O}}(\mathcal{D}_X(\lambda))) = 0$ for $j \neq 0$, and the following statements are equivalent:*

(i) *there is an isomorphism $\mathcal{D}_{X'}(\lambda') \xrightarrow{\sim} \mathcal{R}_{\mathrm{O}}(\mathcal{D}_X(\lambda))$ in $\mathrm{Mod}(\mathcal{D}_{X'})$;*

(ii) *$M_P(\lambda')$ is an irreducible $\mathfrak{g}$-module.*

This result can also be successfully applied to the representation theory of a real form $G_{\mathbb{R}}$ of G, by letting $G_{\mathbb{R}}$ act on the dual flag manifolds X and X' and then by comparing —by means of the so-called "adjunction formulas", a general family of isomorphisms of cohomology associated to integral transforms, see Section 1.6— different spaces of $G_{\mathbb{R}}$-equivariant holomorphic cohomology on X and X' associated to $G_{\mathbb{R}}$-orbits. A concrete example of this procedure is shown in the final part of the paper, where we work out in detail the above-mentioned case of "Grassmann Duality", with $G = SL(V)$ and X, X' dual Grassmann manifolds of V: after getting an alternative proof of our results of [13], we treat the real form $G_{\mathbb{R}} = SU(Q)$ of G, where Q is a nondegenerate undefinite hermitian form on V.

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1 Preliminary notions

1.1 Geometry and sheaves

We refer to Kashiwara-Schapira [11].

Let Z be a real analytic manifold. We denote by TZ (resp. T^*Z) the tangent (resp. cotangent) bundle; for a submanifold S of Z, the conormal bundle T^*_SZ is defined to be the Zariski closure in T^*Z of $\{(x;\xi) \in T^*Z : x \in S,\ \xi \in (T_xS)^\perp\}$. We denote by $\mathbb{C}_Z$ the constant sheaf on Z with fiber $\mathbb{C}$, by $\mathrm{Mod}(\mathbb{C}_Z)$ the category of sheaves of $\mathbb{C}$-vector spaces on Z and by $\mathbf{D}^{\mathrm{b}}(\mathbb{C}_Z)$ the derived category of $\mathrm{Mod}(\mathbb{C}_Z)$ whose objects have bounded cohomology. We denote by $SS(F) \subset T^*Z$ the microsupport of a sheaf $F \in \mathrm{Mod}(\mathbb{C}_Z)$ (the analogous for sheaves of the notion of characteristic variety for $\mathcal{D}$-modules), and for $F \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_Z)$ we set $SS(F) = \bigcup_{j\in\mathbb{Z}} SS(H^jF)$.

In $\mathbf{D}^{\mathrm{b}}(\mathbb{C}_Z)$ are defined the usual operations (where f a morphism of real analytic manifolds) f^{-1}, Rf_*, $Rf_!$, $\otimes$, $R\mathcal{H}om$ (internal Hom) and RHom (external Hom). We shall also consider the duality functor

$$(\,\cdot\,)^* : \mathbf{D}^{\mathrm{b}}(\mathbb{C}_Z) \to \mathbf{D}^{\mathrm{b}}(\mathbb{C}_Z), \qquad F^* = R\mathcal{H}om\,(F, \mathbb{C}_Z).$$

For a locally closed subset $D \subset Z$, we denote by $\mathbb{C}_{D|Z}$ the sheaf on Z defined to be the constant sheaf with fiber $\mathbb{C}$ over D and zero over $Z \setminus D$ (in other words, $\mathbb{C}_{D|Z} = j_!\mathbb{C}_D$ where $j : D \to Z$ is the natural embedding map).

1.2 Algebraic $\mathcal{D}$-modules

We refer to Borel [3].

Let Z be a smooth algebraic variety over $\mathbb{C}$, of complex dimension d_Z, $f : Z \to Z'$ be a morphism of smooth algebraic varieties over $\mathbb{C}$. We denote by $\mathcal{O}_Z$ the sheaf of regular functions, by Ω_Z the invertible $\mathcal{O}_Z$-module of differential forms of top degree, by Θ_Z the locally free $\mathcal{O}_Z$-module of regular tangent vector fields on Z and by $\mathcal{D}_Z$ the sheaf of algebraic linear differential operators on Z.

All $\mathcal{O}_Z$- and $\mathcal{D}_Z$-modules will be assumed to be quasi-coherent over $\mathcal{O}_Z$. We denote by $\mathrm{Mod}(\mathcal{O}_Z)$ the category of $\mathcal{O}_Z$-modules, by $\mathrm{Mod}(\mathcal{D}_Z)$ the category of left $\mathcal{D}_Z$-modules, by $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_Z)$ the derived category of $\mathrm{Mod}(\mathcal{D}_Z)$ whose objects have bounded cohomology and by $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_Z)$ the full subcategory of objects with coherent cohomologies. We denote by $\mathrm{char}(\mathcal{M}) \subset T^*Z$ the characteristic variety of a coherent $\mathcal{D}_Z$-module $\mathcal{M}$, and for $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_Z)$ we set $\mathrm{char}(\mathcal{M}) = \bigcup_{j\in\mathbb{Z}} \mathrm{char}(H^j\mathcal{M})$.

The inverse image of a $\mathcal{O}_{Z'}$-module $\mathcal{F}'$ is the $\mathcal{O}_Z$-module $\mathcal{F} = \mathcal{O}_Z \otimes_{f^{-1}\mathcal{O}_{Z'}} f^{-1}\mathcal{F}'$. We denote by Df_*, $Df_!$ and Df^* the direct and inverse image for $\mathcal{D}$-modules:

$$\begin{aligned} &Df_{*,!} : \mathbf{D}^{\mathrm{b}}(\mathcal{D}_Z) \to \mathbf{D}^{\mathrm{b}}(\mathcal{D}_{Z'}), \quad Df_{*,!}\mathcal{M} = Rf_*(\mathcal{D}_{Z'\leftarrow Z} \otimes^L_{\mathcal{D}_Z} \mathcal{M}), \\ &Df^* : \mathbf{D}^{\mathrm{b}}(\mathcal{D}_{Z'}) \to \mathbf{D}^{\mathrm{b}}(\mathcal{D}_Z), \quad Df^*\mathcal{M}' = \mathcal{D}_{Z\to Z'} \otimes^L_{f^{-1}\mathcal{D}_{Z'}} f^{-1}\mathcal{M}', \end{aligned}$$

where a $(\mathcal{D}_Z, f^{-1}\mathcal{D}_{Z'})$-bimodule $\mathcal{D}_{Z\to Z'}$ and an $(f^{-1}\mathcal{D}_{Z'}, \mathcal{D}_Z)$-bimodule $\mathcal{D}_{Z'\leftarrow Z}$ are defined by $\mathcal{D}_{Z\to Z'} = \mathcal{O}_Z \otimes_{f^{-1}\mathcal{O}_{Z'}} f^{-1}\mathcal{D}_{Z'}$ and $\mathcal{D}_{Z'\leftarrow Z} = \Omega_Z \otimes_{\mathcal{O}_Z} \mathcal{D}_{Z\to Z'} \otimes_{f^{-1}\mathcal{O}_{Z'}} f^{-1}\Omega_{Z'}^{\otimes -1}$. For $\mathcal{M}, \mathcal{N} \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_Z)$ we write

$$\mathcal{M} \otimes^D \mathcal{N} = \mathcal{M} \otimes^L_{\mathcal{O}_Z} \mathcal{N} \qquad \text{in } \mathbf{D}^{\mathrm{b}}(\mathcal{D}_Z).$$

The duality functor for left $\mathcal{D}_Z$-modules is

$$(\,\cdot\,)^* : \mathbf{D}^{\mathrm{b}}(\mathcal{D}_Z) \to \mathbf{D}^{\mathrm{b}}(\mathcal{D}_Z), \quad \mathcal{M}^* = R\mathcal{H}om_{\mathcal{D}_Z}(\mathcal{M}, \mathcal{K}_Z),$$

where $\mathcal{K}_Z = \mathcal{D}_Z \otimes_{\mathcal{O}_Z} \Omega_Z^{\otimes -1}[d_Z]$ is the dualizing left $\mathcal{D}_Z$-module; we remark that there is an isomorphism of functors from $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_Z)$ to $\mathbf{D}^{\mathrm{b}}(\mathbb{C}_Z)$

$$R\mathcal{H}om_{\mathcal{D}_Z}((\,\cdot\,)^*, \mathcal{O}_Z) \simeq \big(R\mathcal{H}om_{\mathcal{D}_Z}(\,\cdot\,, \mathcal{O}_Z)\big)^* \tag{1.1}$$

(in other words, the functor of holomorphic solutions commutes to duality).

1.3 Quasi-equivariant $\mathcal{D}$-modules

We refer to Kashiwara [9].

Let G be an affine algebraic group over $\mathbb{C}$ with identity element e and Lie algebra $\mathfrak{g}$; we denote by Ad the adjoint action of G on $\mathfrak{g}$. Let Z be a smooth algebraic variety over $\mathbb{C}$ endowed with an algebraic left action $\mu : G \times Z \to Z$ (we write $\mu(g, z) = gz$ for short), and let $L_Z : \mathfrak{g} \to \Theta_Z \subset \mathcal{D}_Z$ be the canonical Lie algebra homomorphism

$$L_Z(A)(f)(z) = \frac{d}{dt} f(\exp(-tA)z)\Big|_{t=0} \qquad (A \in \mathfrak{g},\ f \in \mathcal{O}_Z,\ z \in Z)$$

(in particular, $L_G(A) \in \Theta_G$ is obtained for $Z = G$ and $\mu(g, z) = gz$). We denote by $i : Z \to G \times Z$ the embedding $i(z) = (e, z)$ and by $p : G \times Z \to Z$ and $q : G \times Z \to G$ the projections. Let us define the morphisms $q_j : G \times G \times Z \to G \times Z$ $(j = 1, 2, 3)$ by $q_1(g_1, g_2, z) = (g_1, g_2 z)$, $q_2(g_1, g_2, z) = (g_1 g_2, z)$ and $q_3(g_1, g_2, z) = (g_2, z)$, and observe that $\mu \circ q_1 = \mu \circ q_2$, $p \circ q_2 = p \circ q_3$ and $\mu \circ q_3 = p \circ q_1$.

A G-equivariant $\mathcal{O}_Z$-module is an $\mathcal{O}_Z$-module $\mathcal{M}$ endowed with a $\mathcal{O}_{G\times Z}$-linear isomorphism $\beta : \mu^*\mathcal{M} \xrightarrow{\sim} p^*\mathcal{M}$ such that the following diagram commutes:

$$\begin{tikzcd} {q_2}^*\mu^*\mathcal{M} \arrow[rr, "{q_2}^*\beta"] \arrow[d, equal] & & {q_2}^*p^*\mathcal{M} \arrow[d, equal] \\ {q_1}^*\mu^*\mathcal{M} \arrow[r, "{q_1}^*\beta"] & {q_1}^*p^*\mathcal{M} = {q_3}^*\mu^*\mathcal{M} \arrow[r, "{q_3}^*\beta"] & {q_3}^*p^*\mathcal{M}. \end{tikzcd}$$

To a G-equivariant $\mathcal{O}_Z$-module $\mathcal{M}$ one canonically associates a Lie algebra homomorphism $L_{\mathcal{M},G} : \mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(\mathcal{M})$ by

$$L_{\mathcal{M},G}(A)(u) = i^*((q^* L_G(A))(\beta(\mu^* u))) \qquad (A \in \mathfrak{g},\, u \in \mathcal{M}).$$

Let $\mathcal{O}_G \boxtimes \mathcal{D}_Z$ denote the subalgebra $\mathcal{O}_{G\times Z} \otimes_{p^{-1}\mathcal{O}_Z} p^{-1}\mathcal{D}_Z$ of $\mathcal{D}_{G\times Z}$. A $\mathcal{D}_Z$-module $\mathcal{M}$ is called G-equivariant (resp. quasi-G-equivariant) if it is endowed with a G-equivariant $\mathcal{O}_Z$-module structure such that the isomorphism $\beta : \mu^*\mathcal{M} \xrightarrow{\sim} p^*\mathcal{M}$ is $\mathcal{D}_{G\times Z}$-linear (resp. $\mathcal{O}_G \boxtimes \mathcal{D}_Z$-linear).

We denote by $\mathrm{Mod}_G(\mathcal{O}_Z)$ (resp. $\mathrm{Mod}_G(\mathcal{D}_Z)$) the category of G-equivariant $\mathcal{O}_Z$-modules (resp. quasi-G-equivariant $\mathcal{D}_Z$-modules), and by $\mathbf{D}^{\mathrm{b}}_G(\mathcal{D}_Z)$ the derived category of $\mathcal{D}_Z$-modules with bounded quasi-G-equivariant cohomology.

Let $\mathcal{M}$ be a quasi-G-equivariant $\mathcal{D}_Z$-module. The homomorphism L_Z induces a Lie algebra homomorphism $L_{\mathcal{M},D} : \mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(\mathcal{M})$ by

$$L_{\mathcal{M},D}(A)(u) = L_Z(A)\, u \qquad (A \in \mathfrak{g},\, u \in \mathcal{M}).$$

Set

$$\gamma_{\mathcal{M}} = L_{\mathcal{M},G} - L_{\mathcal{M},D}.$$

Proposition 1.1. [9] *One has $\gamma_{\mathcal{M}}(A) \in \mathrm{End}_{\mathcal{D}_Z}(\mathcal{M})$ for any $A \in \mathfrak{g}$. Moreover, the map $\gamma_{\mathcal{M}} : \mathfrak{g} \to \mathrm{End}_{\mathcal{D}_Z}(\mathcal{M})$ is a Lie algebra homomorphism and $\gamma_{\mathcal{M}} = 0$ if and only if $\mathcal{M}$ is G-equivariant.*

We also denote by

$$\gamma_{\mathcal{M}} : U(\mathfrak{g}) \to \mathrm{End}_{\mathcal{D}_Z}(\mathcal{M})$$

the corresponding homomorphism of associative algebras.

Example 1.2. For a G-equivariant $\mathcal{O}_Z$-module $\mathcal{F}$ the $\mathcal{D}_Z$-module $\mathcal{M} = \mathcal{D}_Z \otimes_{\mathcal{O}_Z} \mathcal{F}$ is endowed with a natural quasi-G-equivariant $\mathcal{D}_Z$-module structure. In this case, for $A \in \mathfrak{g}$, $P \in \mathcal{D}_Z$ and $s \in \mathcal{F}$ one has $L_{\mathcal{M},G}(A)(P \otimes s) = [L_Z(A), P] \otimes s + P \otimes L_{\mathcal{F},G}(A)(s)$ and hence $\gamma_{\mathcal{M}}(A)(P \otimes s) = -PL_Z(A) \otimes s + P \otimes L_{\mathcal{F},G}(A)(s)$.

Assume now that the action μ is transitive. Let $z \in Z$, $j : \{z\} \to Z$ the inclusion map and $K = G_z$ (resp. $\mathfrak{k} = \mathrm{Lie}(K) = \mathfrak{g}_z$) the isotropy subgroup (resp. subalgebra) at z: we identify $Z \simeq G/K$. For a G-equivariant $\mathcal{O}_Z$-module $\mathcal{M}$, the fiber

$$\mathcal{M}(z) = Dj^*\mathcal{M} = \mathbb{C} \otimes_{\mathcal{O}_{Z,z}} \mathcal{M}_z$$

of $\mathcal{M}$ at z is endowed with a natural K-module structure. Conversely, for a K-module M one defines a G-equivariant $\mathcal{O}_Z$-module $\mathcal{O}_Z(M)$ as the sheaf of regular sections of the vector bundle $G \times_K M$ on Z: i.e., for $U \subset Z$ open one sets

$$\Gamma(U; \mathcal{O}_Z(M)) = \{s \in \Gamma(\pi^{-1}(U); \mathcal{O}_G \otimes M) : s(xg) = g^{-1}s(x) \text{ for any } x \in G \text{ and } g \in K\},$$

where $\pi : G \to Z \simeq G/K$ is the projection. Let us denote by $\mathrm{Mod}(K)$ the category of algebraic K-modules: the following fact is well-known (see e.g. [14]).

Proposition 1.3. *The categories* $\mathrm{Mod}_G(\mathcal{O}_Z)$ *and* $\mathrm{Mod}(K)$ *are equivalent via* $\mathcal{M} \mapsto \mathcal{M}(z)$ *and* $M \mapsto \mathcal{O}_Z(M)$.

Moreover, if $\mathcal{M}$ is a quasi-G-equivariant $\mathcal{D}_Z$-module, then $\mathcal{M}(z)$ is also endowed with a $\mathfrak{g}$-module structure induced from the $\mathcal{O}_Z$-linear action $\gamma_{\mathcal{M}}$. For $M = \mathcal{M}(z)$ we have the following:

(a) the action of $\mathfrak{k}$ on M given by differentiating the K-module structure coincides with the restriction to $\mathfrak{k}$ of the action of $\mathfrak{g}$,

(b) $gAv = (\mathrm{Ad}(g)A)gv$ for any $g \in K$, $A \in \mathfrak{g}$, $v \in M$.

A vector space M equipped with structures of a K-module and a $\mathfrak{g}$-module is called a $(\mathfrak{g}, K)$-module if it satisfies the conditions (a) and (b) above. Let us denote by $\mathrm{Mod}(\mathfrak{g})$ (resp. $\mathrm{Mod}(\mathfrak{g}, K)$) the category of $\mathfrak{g}$-modules (resp. $(\mathfrak{g}, K)$-modules): observe that the forgetful functor $\mathrm{Mod}(\mathfrak{g}, K) \to \mathrm{Mod}(\mathfrak{g})$ is fully faithful. Conversely, if M is a $(\mathfrak{g}, K)$-module then by (b) one gets a Lie algebra homomorphism $\gamma_{\mathcal{O}_Z(M)} : \mathfrak{g} \to \mathrm{End}_{\mathcal{O}_Z}(\mathcal{O}_Z(M))$, and $L_{\mathcal{O}_Z(M),G} - \gamma_{\mathcal{O}_Z(M)}$ defines a $\mathcal{D}_Z$-module structure on $\mathcal{O}_Z(M)$. One has the following generalization of Proposition 1.3:

Proposition 1.4. [9] *The categories* $\mathrm{Mod}_G(\mathcal{D}_Z)$ *and* $\mathrm{Mod}(\mathfrak{g}, K)$ *are equivalent via* $\mathcal{M} \mapsto \mathcal{M}(z)$ *and* $M \mapsto \mathcal{O}_Z(M)$.

1.4 Generalized Verma modules

From now on let G be a connected and simply connected semisimple algebraic group over $\mathbb{C}$ with identity element e and Lie algebra $\mathfrak{g}$; we denote by $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, $\Delta \subset \mathfrak{h}^*$ the roots of $\mathfrak{g}$, $\{\alpha_i : i \in S\} \subset \Delta$ a family of simple roots, $\{\alpha_i^\vee : i \in S\} \subset \mathfrak{h}$ the simple coroots, $\{\varpi_i : i \in S\} \subset \mathfrak{h}^*$ the fundamental weights (defined by $\langle \varpi_i, \alpha_j^\vee \rangle = \delta_{i,j}$), $\Delta^\pm$ the positive/negative roots, $\rho = \frac{1}{2}\sum_{\alpha \in \Delta^+} \alpha$, $\mathfrak{h}^*_{\mathbb{Z}} = \sum_{i \in S} \mathbb{Z}\varpi_i$ the integral weight lattice; a weight $\lambda = \sum_{i \in S} \lambda_i \varpi_i \in \mathfrak{h}^*_{\mathbb{Z}}$ is called dominant (resp. stricly dominant) when $\lambda_i \geq 0$ (resp. $\lambda_i > 0$) for any $i \in S$, singular if $\langle \lambda, \alpha^\vee \rangle = 0$ for some $\alpha \in \Delta$, regular if non-singular. For $\alpha \in \Delta$ we denote by s_α the reflection in $\mathfrak{h}^*$ w.r.t. α and by $W = \langle s_{\alpha_i} : i \in S \rangle \subset \mathrm{Aut}_{\mathbb{C}}(\mathfrak{h}^*)$ the Weyl group. We denote by $\ell(w)$ the length of $w \in W$. The affine action of $w \in W$ on $\lambda \in \mathfrak{h}^*$ is denoted by $w \cdot \lambda$ (i.e. $w \cdot \lambda = w(\lambda + \rho) - \rho$). If λ is regular, we denote by w_λ the unique element in W such that $w_\lambda \lambda$ is stricly dominant. For $\lambda \in \mathfrak{h}^*$ let $L(\lambda)$ (resp. $L'(\lambda)$) be the irreducible highest (resp. lowest) weight $\mathfrak{g}$-module of highest (resp. lowest) weight λ.

For a $\mathfrak{g}$-module M and $\mu \in \mathfrak{h}^*$, let

$$M_\mu = \{v \in M : Av = \mu(A)v \text{ for any } A \in \mathfrak{h}\}.$$

If $\mathfrak{h}$ acts semisimply on M (i.e. if $M = \bigoplus_{\mu \in \mathfrak{h}^*} M_\mu$) and $\dim_{\mathbb{C}}(M_\mu)$ is finite for any $\mu \in \mathfrak{h}^*$, the character of M is the sum

$$\mathrm{ch}(M) = \sum_{\mu \in \mathfrak{h}^*} \dim_{\mathbb{C}}(M_\mu) \mathrm{e}^\mu;$$

we let $\mathfrak{g}$ act contragradiently on $\mathrm{Hom}\,(M,\mathbb{C})$ by $(Af)(v) = -f(Av)$ for $A \in \mathfrak{g}$, $f \in \mathrm{Hom}\,(M,\mathbb{C})$ and $v \in M$ and we define the dual of M as the $\mathfrak{g}$-module of $\mathfrak{h}$-finite elements of $\mathrm{Hom}\,(M,\mathbb{C})$:

$$M^* = \{f \in \mathrm{Hom}\,(M,\mathbb{C}) : \dim_{\mathbb{C}} U(\mathfrak{h})f \text{ is finite}\}.$$

Note that $(M^*)^* \simeq M$, $M^* = \bigoplus_{\mu\in\mathfrak{h}^*}(M^*)_\mu$ and for any $\mu \in \mathfrak{h}^*$ one has

$$(M^*)_\mu \simeq (M_{-\mu})^*. \tag{1.2}$$

Let us fix once for all a subset $I \subset S$, and let

$$\begin{array}{lll} \Delta_I = \Delta \cap \sum_{i\in I}\mathbb{Z}\alpha_i, & \Delta_I^\pm = \Delta_I \cap \Delta^\pm, & \rho_I = \frac{1}{2}\sum_{\alpha\in\Delta^+\setminus\Delta_I}\alpha, \\ W_I = \langle s_{\alpha_i} : i \in I\rangle, & \mathfrak{l}_I = \mathfrak{h} \oplus \left(\bigoplus_{\alpha\in\Delta_I}\mathfrak{g}_\alpha\right), & \mathfrak{n}_I^\pm = \bigoplus_{\alpha\in\Delta^\pm\setminus\Delta_I}\mathfrak{g}_\alpha, \\ \mathfrak{p}_I = \mathfrak{l}_I \oplus \mathfrak{n}_I^+, & \mathfrak{p}'_I = \mathfrak{l}_I \oplus \mathfrak{n}_I^-. & \end{array}$$

We denote by w_I the longest element of W_I (characterized by $w_I\Delta_I^\pm = \Delta_I^\mp$). Let H be the maximal torus corresponding to $\mathfrak{h}$ and let L_I, $N_I^\pm$, P_I and P'_I be the closed connected subgroups of G corresponding to $\mathfrak{l}_I$, $\mathfrak{n}_I^\pm$, $\mathfrak{p}_I$ and $\mathfrak{p}'_I$. The set of I-dominant integral weights

$$(\mathfrak{h}^*_{\mathbb{Z}})_I = \sum_{i\in I}\mathbb{Z}_{\geq 0}\varpi_i + \sum_{i\in S\setminus I}\mathbb{Z}\varpi_i = \{\lambda \in \mathfrak{h}^*_{\mathbb{Z}} : \langle\lambda, \alpha_i^\vee\rangle \geq 0 \text{ for any } i \in I\} \tag{1.3}$$

is identified to the set of finite dimensional irreducible L_I-modules, by associating to $\lambda \in (\mathfrak{h}^*_{\mathbb{Z}})_I$ the irreducible L_I-module $V(\lambda)$ with highest weight λ.
For $\lambda \in (\mathfrak{h}^*_{\mathbb{Z}})_I$, let

$$\lambda' = -w_I\lambda - 2\rho_I \quad \in (\mathfrak{h}^*_{\mathbb{Z}})_I. \tag{1.4}$$

Note that $V(\lambda')$ is the dual L_I-module to $V(\lambda)$.

We regard $V(\lambda)$ as a P_I-module with the trivial action of N_I^+. The *generalized Verma module* with highest weight λ is the $\mathfrak{g}$-module

$$M_{P_I}(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_I)} V(\lambda). \tag{1.5}$$

The action of P_I defined by $g(A\otimes v) = \mathrm{Ad}(g)(A) \otimes gv$ for $g \in P_I$, $A \in U(\mathfrak{g})$ and $v \in V(\lambda)$ gives to $M_{P_I}(\lambda)$ a structure of $(\mathfrak{g}, P_I)$-module. It is well-known that the dual $M_{P_I}(\lambda)^*$ is locally $\mathfrak{p}'_I$-finite and it has a natural structure of $(\mathfrak{g}, P'_I)$-module with lowest weight $-\lambda$.

Recall the following property of highest weight modules:

Proposition 1.5. *As a $\mathfrak{g}$-module, $M_{P_I}(\lambda)$ has a unique irreducible quotient* (isomorphic to $L(\lambda)$)*; dually, $M_{P_I}(\lambda)^*$ has a unique irreducible submodule* (isomorphic to $L'(-\lambda)$).

As for the the dual parabolic subgroup P'_I, for $\lambda \in (\mathfrak{h}^*_{\mathbb{Z}})_I$ we set

$$M_{P'_I}(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}'_I)} V(-w_I\lambda) : \tag{1.6}$$

then $M_{P'_I}(\lambda)$ is a $(\mathfrak{g}, P'_I)$-module with lowest weight $w_I(-w_I\lambda) = -\lambda$ and unique irreducible quotient isomorphic to $L'(-\lambda)$. Clearly, $M_{P'_I}(\lambda)$ is irreducible if and only if $M_{P_I}(\lambda)$ is. Let us set also

$$(\mathfrak{h}^*_{\mathbb{Z}})_{I,\mathrm{irr}} = \{\lambda \in (\mathfrak{h}^*_{\mathbb{Z}})_I : M_{P_I}(\lambda) \text{ is an irreducible } \mathfrak{g}\text{-module}\} \quad \subset (\mathfrak{h}^*_{\mathbb{Z}})_I\,. \tag{1.7}$$

Remark 1.6. The following condition is sufficient in order that $\lambda \in (\mathfrak{h}^*_{\mathbb{Z}})_{I,\mathrm{irr}}$ (see Jantzen [7, Corollar 4]):

$$\left\{\begin{array}{l}\text{For any } \alpha \in \Delta^+ \setminus \Delta_I \text{ such that } \langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}_{>0} \\ \text{there exists } \beta \in \Delta \text{ such that } \langle \lambda + \rho, \beta^\vee \rangle = 0 \text{ and } s_\alpha(\beta) \in \Delta_I.\end{array}\right. \tag{1.8}$$

Alternatively, using an approach through $\mathcal{D}$-modules based on the results of Beilinson-Bernstein [1], it is possible to prove that another sufficient condition in order that $\lambda \in (\mathfrak{h}^*_{\mathbb{Z}})_{I,\mathrm{irr}}$ is the following:

$$\langle \lambda, \alpha^\vee \rangle \notin \mathbb{Z}_{\geq 0} \qquad \text{for any } \alpha \in \Delta^+ \setminus \Delta_I. \tag{1.9}$$

Conditions (1.8) and (1.9), even if they are very close to each other, are in fact independent: note that they are both implied by the (in general, strictly) stronger condition

$$\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}_{>0} \qquad \text{for any } \alpha \in \Delta^+ \setminus \Delta_I.$$

The following proposition is well-known.

Proposition 1.7. *There exists a non-zero morphism of $(\mathfrak{g}, P'_I)$-modules*

$$M_{P'_I}(\lambda) \xrightarrow{\phi} M_{P_I}(\lambda)^*$$

which is an isomorphism if and only if $\lambda \in (\mathfrak{h}^*_{\mathbb{Z}})_{I,\mathrm{irr}}$.

1.5 Generalized flag manifolds

Let

$$X_I = G/P_I$$

(i.e. the generalized flag manifold associated to P_I). By the category equivalence given in Proposition 1.4, the isomorphism classes of G-equivariant $\mathcal{O}_{X_I}$-modules (resp. quasi-G-equivariant $\mathcal{D}_{X_I}$-modules) are in one-to-one correspondence with the isomorphism classes of P_I-modules (resp. $(\mathfrak{g}, P_I)$-modules). For $\lambda \in (\mathfrak{h}^*_{\mathbb{Z}})_I$ we denote by $\mathcal{O}_{X_I}(\lambda)$ (corresponding to $\mathcal{O}_{X_I}(V(\lambda))$ in the notation of 1.3) the G-equivariant locally free $\mathcal{O}_{X_I}$-module of regular sections of the homogeneous vector bundle $G \times_{P_I} V(\lambda)$ on X_I. In particular, by using the identification $T^*_{x_0} X_I \simeq (\mathfrak{g}/\mathfrak{p}_I)^* \simeq \mathfrak{p}_I^\perp$, one gets

$$\Omega_{X_I} \simeq \mathcal{O}_{X_I}(2\rho_I). \tag{1.10}$$

We recall the following fact (see also Example 1.2):

Proposition 1.8. [9] *In the equivalence of Proposition 1.4, the quasi-G-equivariant $\mathcal{D}_{X_I}$-module $\mathcal{O}_{X_I}(M_{P_I}(\lambda))$ corresponding to the $(\mathfrak{g}, P_I)$-module $M_{P_I}(\lambda)$ is isomorphic to*

$$\mathcal{D}_{X_I}(\lambda) = \mathcal{D}_{X_I} \otimes_{\mathcal{O}_{X_I}} \mathcal{O}_{X_I}(\lambda). \tag{1.11}$$

Let us also consider the *dual* generalized flag manifold

$$X'_I = G/P'_I\ ;$$

note that it is possible that $X_I = X'_I$ (in fact this happens quite often, and it is equivalent to the fact that P_I and P'_I are conjugated parabolic subgroups). Here $\mathcal{O}_{X'_I}(\lambda)$ is the G-equivariant locally free $\mathcal{O}_{X_I}$-module of regular sections of the homogeneous vector bundle $G \times_{P'_I} V(-w_I\lambda)$ on X'_I; one has $\Omega_{X'_I} \simeq \mathcal{O}_{X'_I}(2\rho_I)$, and $M_{P'_I}(\lambda)$ corresponds to $\mathcal{D}_{X'_I}(\lambda) = \mathcal{D}_{X'_I} \otimes_{\mathcal{O}_{X'_I}} \mathcal{O}_{X'_I}(\lambda)$ in Proposition 1.4.

1.6 Kernels for sheaves and $\mathcal{D}$-modules

We refer to D'Agnolo-Schapira [5].

Let X and Y be complex analytic manifolds, $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ be the projections. Any sheaf $K \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_{X\times Y})$ and any left $\mathcal{D}_{X\times Y}$-module $\mathcal{K} \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_{X\times Y})$ are formally the kernels of "integral transforms" for sheaves or $\mathcal{D}$-modules, i.e. functors

$$\begin{aligned} r_K : \mathbf{D}^{\mathrm{b}}(\mathbb{C}_X) \to \mathbf{D}^{\mathrm{b}}(\mathbb{C}_Y), &\quad r_K(F) = Rp_{2!}(p_1^{-1}F \otimes K) \\ \mathcal{R}_{\mathcal{K}} : \mathbf{D}^{\mathrm{b}}(\mathcal{D}_X) \to \mathbf{D}^{\mathrm{b}}(\mathcal{D}_Y), &\quad \mathcal{R}_{\mathcal{K}}(\mathcal{M}) = Dp_{2!}(Dp_1{}^*\mathcal{M} \otimes^D \mathcal{K}), \end{aligned}$$

as well as the similar ones in the opposite direction (for which we shall use the same notation, for example we also denote $r_K : \mathbf{D}^{\mathrm{b}}(\mathbb{C}_Y) \to \mathbf{D}^{\mathrm{b}}(\mathbb{C}_X)$, $r_K(F) = Rp_{1!}(K \otimes p_2^{-1}F))$; the dual kernels K^* and $\mathcal{K}^*$ will also be considered. In the applications, K and $\mathcal{K}$ are often related by the Riemann-Hilbert correspondence [8] (i.e. $K = R\mathcal{H}om_{\mathcal{D}_Z}(\mathcal{K}, \mathcal{O}_Z)$, in other words K is the sheaf of holomorphic solutions of $\mathcal{K}$); in this case, due to the commutation between duality and holomorphic solutions, it holds also $K^* = R\mathcal{H}om_{\mathcal{D}_Z}(\mathcal{K}^*, \mathcal{O}_Z)$.

These transforms are related by the following general adjunction formulas (where e.g. d_X is the complex dimension of X):

Proposition 1.9. [5] *Let $\mathcal{K} \in \mathbf{D}^{\mathrm{b}}_{coh}(\mathcal{D}_{X\times Y})$, $K = R\mathcal{H}om_{\mathcal{D}_Z}(\mathcal{K}, \mathcal{O}_Z) \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_{X\times Y})$ and assume that* $\mathrm{char}(\mathcal{K}) \cap (T^*X \times T^*_Y Y) \subset T^*(X \times Y)$. *Then for any $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$ and $F \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_Y)$ there are isomorphisms in the derived category of $\mathbb{C}$-vector spaces*

$$\begin{aligned} \mathrm{RHom}_{\mathcal{D}_X}(\mathcal{M}, r_K(F) \otimes \mathcal{O}_X))[d_X] &\simeq \mathrm{RHom}_{\mathcal{D}_Y}(\mathcal{R}_{\mathcal{K}}(\mathcal{M}), F \otimes \mathcal{O}_Y)), \\ \mathrm{RHom}_{\mathcal{D}_X}(\mathcal{M}, R\mathcal{H}om\,(r_K(F), \mathcal{O}_X)))[d_X] &\simeq \mathrm{RHom}_{\mathcal{D}_Y}(\mathcal{R}_{\mathcal{K}^*}(\mathcal{M}), R\mathcal{H}om\,(F, \mathcal{O}_Y)))[2d_Y]. \end{aligned}$$

These formulas are very general and, in this generality, they could not appear to be significant. Anyway, they enclose a possibly infinite amount of applications, up to different choices for F and $\mathcal{M}$: the problem is of course to compute their transforms $r_K(F)$ and $\mathcal{R}_{\mathcal{K}}(\mathcal{M})$. In fact, we shall deal with the following particular cases.

(1) As for the geometry of the transform, let O be the open complementary to a (non necessarily smooth) closed hypersurface $S \subset X \times Y$, denote by $i : \mathrm{O} \to X_I \times X'_I$ the open embedding, and let $\mathcal{B}_{\mathrm{O}} = Ri_*\mathcal{O}_{\mathrm{O}}$ the regular holonomic $\mathcal{D}_{X\times Y}$-module of rational functions on $X \times Y$ with poles on S. Set $\mathcal{K} = \mathcal{B}_{\mathrm{O}}$: then $K = R\mathcal{H}om_{\mathcal{D}_Z}(\mathcal{K}, \mathcal{O}_Z) = \mathbb{C}_{\mathrm{O}|X\times Y}$, and this amounts to consider O as relation for our transform. We assume also that the characteristic variety $\mathrm{char}(\mathcal{B}_{\mathrm{O}})$ satisfies the transversality condition required in Proposition

1.9: this is true under reasonable geometric hypotheses on O (e.g. in the case where O is a orbit for a group action on $X \times Y$). We shall write for short

$$(1.12) \qquad r_{\mathrm{O}} := r_{\mathbb{C}_{\mathrm{O}|X\times Y}}\,, \qquad r^*_{\mathrm{O}} := r_{\mathbb{C}^*_{\mathrm{O}|X\times Y}}\,, \qquad \mathcal{R}_{\mathrm{O}} := \mathcal{R}_{\mathcal{B}_{\mathrm{O}}}\,, \qquad \mathcal{R}^*_{\mathrm{O}} := \mathcal{R}_{\mathcal{B}^*_{\mathrm{O}}}\,.$$

(2) Let $\mathcal{F}$ be a locally free $\mathcal{O}_X$-module and denote by $\mathcal{DF} := \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F}$ the associated locally free $\mathcal{D}_X$-module. By applying Proposition 1.9 with $\mathcal{M} = \mathcal{DF}$, one gets:

Corollary 1.10. *For any $F \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_Y)$ there are isomorphisms in the derived category of $\mathbb{C}$-vector spaces*

$$\begin{aligned} \mathrm{R}\Gamma(r_{\mathrm{O}}(F), \mathcal{F}^*)) &\simeq \mathrm{RHom}_{\mathcal{D}_Y}(\mathcal{R}_{\mathrm{O}}(\mathcal{DF}), F \otimes \mathcal{O}_Y))[-d_X], \\ \mathrm{RHom}\,(r_{\mathrm{O}}(F), \mathcal{F}^*))) &\simeq \mathrm{RHom}_{\mathcal{D}_Y}(\mathcal{R}_{\mathrm{O}}(\mathcal{DF}), R\mathcal{H}om\,(F, \mathcal{O}_Y)))[2d_Y - d_X]. \end{aligned}$$

(3) The last step is to choose a sheaf F on Y and to compute its transform $r_{\mathrm{O}}(F)$. For example, let $F = \mathbb{C}_{D|Y}$, where D is a locally closed subset of Y: by definition, one has $(r_{\mathrm{O}}(\mathbb{C}_{D|Y}))_x \simeq \mathrm{R}\Gamma_c(L_D(x); \mathbb{C})$ for any $x \in X$, where $L_D(x) = \{y \in D : (x,y) \in \mathrm{O}\}$. let $\widehat{D} = p_2(\mathrm{O} \cap p_1^{-1}(D))$. To simplify our task, assume that D is e.g. an open (resp. compact) subset of Y well-behaving w.r.t. the transform, more precisely assume that $\mathrm{R}\Gamma(L_D(x); \mathbb{C}) \simeq \mathbb{C}$ for any $x \in \widehat{D}$: then one can prove (see [5, Lemma 2.8]) that $r_{\mathrm{O}}(\mathbb{C}_{D|Y}) \simeq \mathbb{C}_{\widehat{D}|X}[-2d_Y]$ (resp. $r_{\mathrm{O}}(\mathbb{C}_{D|Y}) \simeq \mathbb{C}_{\widehat{D}|X}$). By applying Proposition 1.9 with $\mathcal{M} = \mathcal{DF}$ and $F = \mathbb{C}_{D|Y}$ for D open, one then gets

$$\begin{aligned} \mathrm{R}\Gamma_c(\widehat{D}; \mathcal{F}^*) &\simeq \mathrm{R}\Gamma_c(D; R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{R}_{\mathrm{O}}(\mathcal{DF}), \mathcal{O}_Y))[2d_Y - d_X], \\ \mathrm{R}\Gamma(\widehat{D}; \mathcal{F}^*) &\simeq \mathrm{R}\Gamma(D; R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{R}^*_{\mathrm{O}}(\mathcal{DF}), \mathcal{O}_Y))[-d_X], \end{aligned}$$

and with $\mathcal{M} = \mathcal{DF}$ and $F = \mathbb{C}_D$ for D compact,

$$\begin{aligned} \mathrm{R}\Gamma(\widehat{D}; \mathcal{F}^*) &\simeq \mathrm{R}\Gamma(D; R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{R}_{\mathrm{O}}(\mathcal{DF}), \mathcal{O}_Y))[-d_X], \\ \mathrm{R}\Gamma_{\widehat{D}}(X; \mathcal{F}^*) &\simeq \mathrm{R}\Gamma_D(Y; R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{R}^*_{\mathrm{O}}(\mathcal{DF}), \mathcal{O}_Y))[2d_Y - d_X]. \end{aligned}$$

Therefore, the main problem appears —already at this stage— to be the computation of the transforms $\mathcal{R}_{\mathrm{O}}(\mathcal{DF})$ and $\mathcal{R}^*_{\mathrm{O}}(\mathcal{DF})$. This is one of the objects of the present work.

2 Integral transforms on dual generalized flag manifolds

We keep the notations introduced in the previous section.

Let us consider the dual generalized flag manifolds X_I and X'_I, and let $x_0 = eP_I \in X_I$ and $y_0 = eP'_I \in X'_I$. The diagonal action of G on $X_I \times X'_I$ has a unique open dense orbit

$$\mathrm{O} = G(x_0, y_0) \subset X_I \times X'_I :$$

namely, one has $T_{(x_0,y_0)}\mathrm{O} = \mathrm{im}[\mathfrak{g} \to T_{x_0}X_I \times T_{y_0}X'_I] \simeq \mathrm{im}[\mathfrak{g} \to (\mathfrak{g}/\mathfrak{p}_I) \oplus (\mathfrak{g}/\mathfrak{p}'_I)] = (\mathfrak{g}/\mathfrak{p}_I) \oplus (\mathfrak{g}/\mathfrak{p}'_I) \simeq T_{x_0}X_I \times T_{y_0}X'_I$ (since $\mathfrak{g} = \mathfrak{p}_I + \mathfrak{p}'_I$). Therefore one has

$$\mathrm{O} \simeq G/(P_I \cap P'_I) = G/L_I,$$

and hence O is an affine variety because G and L_I are reductive. Similarly, the open dense P'_I-orbit in X_I

$$\mathrm{O}_{y_0} = P'_I x_0 = \{x \in X_I : (x, y_0) \in \mathrm{O}\} \simeq P'_I/L_I$$

is identified to $\mathfrak{n}_I^-$ by $\pi_I \circ \exp$, and hence it is also affine.

The main object of the present paper will be the study of the integral transform defined by O (see Section 1.6 for notations and references), i.e. the functors for sheaves and $\mathcal{D}$-modules

$$(2.1) \qquad r_{\mathrm{O}},\, r^*_{\mathrm{O}} : \mathbf{D}^{\mathrm{b}}(\mathbb{C}_{X_I}) \to \mathbf{D}^{\mathrm{b}}(\mathbb{C}_{X'_I}) \; ; \qquad \mathcal{R}_{\mathrm{O}},\, \mathcal{R}^*_{\mathrm{O}} : \mathbf{D}^{\mathrm{b}}(\mathcal{D}_{X_I}) \to \mathbf{D}^{\mathrm{b}}(\mathcal{D}_{X'_I})$$

defined in (1.12). Recall that we use the same notation for the analogous functors going in the opposite directions.

2.1 Equivalences of derived categories

Our first aim is to show that O defines a nice integral transform, i.e. that the functors of (2.1) are equivalences of categories.

For the notions needed in the following lemma we refer to [9].

Lemma 2.1. *Let X be a G-manifold, $\mu : T^*X \to \mathfrak{g}^*$ the moment map, and let $F \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_X)$ be $\mathbb{R}$-constructible and G-equivariant. Then*

$$SS(F) \subset \mu^{-1}(0) \;\; (= \bigcup \{T^*_O X : \; O \text{ is a } G\text{-orbit in } X\}).$$

In particular, if $X = X' \times X''$ with X' and X'' homogeneous G-manifolds and we let G act diagonally on X, then $SS(F) \cap (T^*_{X'}X' \times T^*X'') \subset T^*_{X' \times X''}(X' \times X'')$.

Proof. Let $f := \sigma(L_A) : T^*X \to \mathbb{C}$ be the principal symbol of L_A. Then $\Lambda := SS(F)$ is a Lagrangian subvariety of T^*X invariant under the action of $H_f = L_A$ for any $A \in \mathfrak{g}$: this implies that $H_f \in T_p\Lambda$ for any $p \in \Lambda$, hence $df|_\Lambda = 0$, i.e. f is constant on Λ, hence $f = 0$ on Λ since f is homogeneous of degree one on T^*X. Therefore we get $SS(F) \subset \sigma(L_A)^{-1}(0)$ for any $A \in \mathfrak{g}$. Since $A \circ \mu = \sigma(L_A)$ (here we consider $A \in \mathfrak{g} \simeq (\mathfrak{g}^*)^*$) and $\bigcap_{A \in \mathfrak{g}} \ker(A : \mathfrak{g}^* \to \mathbb{C}) = 0$, the first part of the statement follows. In particular, if $X = X' \times X''$ where X' and X'' are G-manifolds then we get that $SS(F) \subset (\mu_{X'} \times \mu_{X''})^{-1}(\Delta^a_{\mathfrak{g}^*})$; if moreover X' and X'' are homogeneous then $\mu_{X'} \times \mu_{X''}$ is injective, and the proof is complete. $\square$

Theorem 2.2. *The functors*

$$\mathbf{D}^{\mathrm{b}}(\mathbb{C}_{X_I}) \underset{r^*_{\mathrm{O}}}{\overset{r_{\mathrm{O}}}{\rightleftarrows}} \mathbf{D}^{\mathrm{b}}(\mathbb{C}_{X'_I}) \;, \qquad \mathbf{D}^{\mathrm{b}}(\mathcal{D}_{X_I}) \underset{\mathcal{R}^*_{\mathrm{O}}}{\overset{\mathcal{R}_{\mathrm{O}}}{\rightleftarrows}} \mathbf{D}^{\mathrm{b}}(\mathcal{D}_{X'_I})$$

are quasi-inverse to each other, and thus are equivalences of categories.

Proof. The statement for $\mathcal{D}$-modules follow from the one for sheaves by using Riemann-Hilbert correspondence (see [8]): therefore it is enough to prove that r_{O} and r^*_{O} are quasi-inverse to each other. By using the geometric criterion in [13, Lemma 2.2] it is enough to prove the following statements (where for $g \in G$ we set $S_g := \{xP_I \in X_I : (xP_I, gP'_I) \in O\} = g\mathrm{O}_{y_0}$ for short):

(i) X_I (and X'_I) are simply connected,

(ii) $\begin{cases} \mathrm{R}\Gamma(S_e;\mathbb{C}) \simeq \mathbb{C} \\ \mathrm{R}\Gamma(S_e \setminus S_g;\mathbb{C}) \simeq \mathbb{C} \quad \text{for any } gP'_I \in X'_I \setminus \{eP'_I\} \end{cases}$,

(iii) $SS(\mathbb{C}_{\mathrm{O}}) \cap (T^*_{X_I}X_I \times T^*X'_I) \subset T^*_{X_I \times X'_I}(X_I \times X'_I)$.

Statement (i) follows from the exact sequence $0 = \pi_1(G) \to \pi_1(X_I) \to \pi_0(P_I) = 0$, and (iii) from Lemma 2.1. As for (ii), we have $\mathrm{R}\Gamma(S_e;\mathbb{C}) \simeq \mathbb{C}$. Then, if $gP'_I \neq eP'_I$, by the Bruhat decomposition we can write $g = g_1 w g_2$ for some $g_1, g_2 \in P'_I$ and $w \in W \setminus \{e\}$: since $S_g = g_1 S_w$ and $S_e = g_1 S_e$, we may reduce the proof to the case $g = w$. Let $A \in \mathfrak{h}$ be in the positive Weyl chamber (i.e. $\langle A, \alpha\rangle > 0$ for any $\alpha \in \Delta^+$): then the one-parameter subgroup $\{\exp(tA) : t \in \mathbb{R}\}$ acts on $S_e \setminus S_w$ (note that S_e and S_w are H-stable) and this action contracts any compact subset of $S_e \setminus S_w$ into its base point eP_I. This shows that $\mathrm{R}\Gamma(S_e \setminus S_w;\mathbb{C}) \simeq \mathbb{C}$. $\square$

Remark 2.3. Our integral transform is microlocally determined by the microsupport $SS(\mathbb{C}_{\mathrm{O}}) \subset T^*(X_I \times X'_I) \simeq T^*X_I \times T^*X'_I$. By Lemma 2.1 one has

$$SS(\mathbb{C}_{\mathrm{O}}) \subset \bigcup\{T^*_S(X_I \times X'_I) : S \text{ a } G\text{-orbit in } X_I \times X'_I\};$$

on the other hand the converse inclusion is not always true, i.e. there could exist some G-orbit S such that $T^*_S(X_I \times X'_I) \not\subset SS(\mathbb{C}_{\mathrm{O}})$. However, an interesting geometric consequence of Theorem 2.2 is the following:

There exists a unique G-orbit $\tilde{S} \subset X_I \times X'_I$ such that

(a) $T^*_{\tilde{S}}(X_I \times X'_I) \subset SS(\mathbb{C}_{\mathrm{O}})$,

(b) *the natural projections $T^*X_I \leftarrow T^*_{\tilde{S}}(X_I \times X'_I) \to T^*X'_I$ are generically surjective, in particular are birational maps.*

Roughly speaking, the above statement says that the integral transform associated to O is "generically" determined —at the microlocal level— by a unique "privileged" G-orbit $\tilde{S}$ in $X_I \times X'_I$. In order to find the right orbit $\tilde{S}$ (which of course depends on G and I), condition (b) is quite easy to check while (a) is not very. In most situations $\tilde{S}$ turns out to be the closed G-orbit in $X_I \times X'_I$, but this is not always true: for example (in the standard Bourbaki's notation) there are cases where (b) fails, as e.g. G of type A_3 and $I = \{1\}$ or G of type A_5 and $I = \{2\}$; even in the case $X_I = X'_I$ (i.e. P_I and P'_I are conjugated parabolic subgroups), when (b) is obviously true (namely in this case the closed orbit in $X_I \times X'_I$ is just the diagonal, and the projections are the identity), nevertheless (a) fails in some cases, e.g. when G is of type C_2 and $I = \{2\}$.

2.2 Quantizing the integral transforms

Our next aim is to study the transform $\mathcal{R}_{\mathrm{O}}(\mathcal{D}_{X_I}(\lambda))$ in $\mathbf{D}^{\mathrm{b}}_G(\mathcal{D}_{X'_I})$ for $\lambda \in (\mathfrak{h}^*_{\mathbb{Z}})_I$ (see (1.11)).

For $\lambda,\mu\in(\mathfrak{h}^*_{\mathbb{Z}})_I$, set

$$\mathcal{B}_{\mathrm{O}}^{(d_{X_I},0)}(\lambda,\mu)=p_1^{-1}(\Omega_{X_I}\otimes_{\mathcal{O}_{X_I}}\mathcal{O}_{X_I}(\lambda))\otimes_{p_1^{-1}\mathcal{O}_{X_I}}\mathcal{B}_{\mathrm{O}}\otimes_{p_2^{-1}\mathcal{O}_{X'_I}}p_2^{-1}\mathcal{O}_{X'_I}(\mu).$$

One has a natural isomorphism (see [5, Lemma 3.1])

$$\alpha:\Gamma(X_I\times X'_I;\mathcal{B}_{\mathrm{O}}^{(d_{X_I},0)}(\lambda,-w_I\mu))\ \xrightarrow{\ \sim\ }\ \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\mathcal{D}_{X'_I})}(\mathcal{D}_{X'_I}(\mu),\mathcal{R}_{\mathrm{O}}(\mathcal{D}_{X_I}(\lambda)));\tag{2.2}$$

in particular, denoting with the superscript $(\,\cdot\,)^G$ the subspace of G-invariant sections, we get

$$\begin{aligned}\mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}_G(\mathcal{D}_{X'_I})}(\mathcal{D}_{X'_I}(\mu),\mathcal{R}_{\mathrm{O}}(\mathcal{D}_{X_I}(\lambda)))&\simeq\Gamma(X_I\times X'_I;\mathcal{B}_{\mathrm{O}}^{(d_{X_I},0)}(\lambda,-w_I\mu))^G\\&\simeq\Gamma(\mathrm{O};\mathcal{O}_{X_I}(\lambda+2\rho_I)\boxtimes\mathcal{O}_{X'_I}(-w_I\mu))^G\\&\simeq\big[V(\lambda+2\rho_I)\otimes V(\mu)\big]^{L_I}:\end{aligned}$$

here the first isomorphism is induced by α and the second follows from (1.10).

For $\mu=\lambda'$ (see (1.4)) the representations $V(\mu)$ and $V(\lambda+2\rho_I)$ of the group L_I are dual to each other, so there exists a canonical nonzero section

$$s\in\big[V(\lambda+2\rho_I)\otimes V(\lambda')\big]^{L_I}\simeq\Gamma(X_I\times X'_I;\mathcal{B}_{\mathrm{O}}^{(d_{X_I},0)}(\lambda,-w_I\lambda'))^G$$

and hence a non-zero morphism in $\mathbf{D}^{\mathrm{b}}_G(\mathcal{D}_{X'})$

$$\mathcal{D}_{X'_I}(\lambda')\xrightarrow{\alpha(s)}\mathcal{R}_{\mathrm{O}}(\mathcal{D}_{X_I}(\lambda)).\tag{2.3}$$

Let us study the morphism $\alpha(s)$ by means of the equivalence stated in Proposition 1.4. We need to compute $\mathcal{R}_{\mathrm{O}}(\mathcal{D}_{X_I}(\lambda))(y_0)$, where $y_0=eP'_I$.

Proposition 2.4. *There exists an isomorphism of $(\mathfrak{g},P'_I)$-modules*

$$\mathcal{R}_{\mathrm{O}}(\mathcal{D}_{X_I}(\lambda))(y_0)\xrightarrow[\sim]{\varphi}\Gamma(\mathrm{O}_{y_0};\mathcal{O}_{X_I}(\lambda+2\rho_I)).$$

In particular, one has $H^j(\mathcal{R}_{\mathrm{O}}(\mathcal{D}_{X_I}(\lambda))(y_0))=0$ for any $j\neq 0$.

Proof. Let $j:\{y_0\}\to X'_I$ be the inclusion map, $l:X_I\to X_I\times X'_I$ the map $l(x)=(x,y_0)$, $a:X_I\to\{y_0\}$ the constant map: then one has

$$\begin{aligned}\mathcal{R}_{\mathrm{O}}(\mathcal{D}_{X_I}(\lambda))(y_0)&\simeq Dj^*Dp_{2*}(Dp_1{}^*(\mathcal{D}_{X_I}(\lambda))\otimes^D\mathcal{B}_{\mathrm{O}})\\&\simeq Da_*Dl^*(Dp_1{}^*(\mathcal{D}_{X_I}(\lambda))\otimes^D\mathcal{B}_{\mathrm{O}})\\&\simeq Da_*(\mathcal{D}_{X_I}(\lambda)\otimes^D\mathcal{B}_{\mathrm{O}_{y_0}})\\&\simeq \mathrm{R}\Gamma(X_I;\Omega_{X_I}\otimes_{\mathcal{D}_{X_I}}\mathcal{D}_{X_I}(\lambda)\otimes^D\mathcal{B}_{\mathrm{O}_{y_0}})\\&\simeq\Gamma(\mathrm{O}_{y_0};\mathcal{O}_{X_I}(\lambda+2\rho_I)),\end{aligned}$$

where the second step holds by cartesianity, the third since Dl^* commutes to $\otimes^D$ and $Dl^*\mathcal{B}_{\mathrm{O}}\simeq\mathcal{B}_{\mathrm{O}_{y_0}}$ and in the last we used (1.10) and the fact that O_{y_0} is affine. $\square$

Corollary 2.5. *One has $H^j(\mathcal{R}_{\mathrm{O}}(\mathcal{D}_{X_I}(\lambda))) = 0$ for any $j \neq 0$.*

Proposition 2.6. *For any $\mu \in (\mathfrak{h}^*_{\mathbb{Z}})_I$ one has an isomorphism of $(\mathfrak{g}, P'_I)$-modules*

$$\Gamma(\mathrm{O}_{y_0}; \mathcal{O}_{X_I}(\mu)) \xrightarrow[\sim]{\psi} M_{P_I}(-w_I\mu)^*$$

Proof. Set $V = \Gamma(\mathrm{O}_{y_0}; \mathcal{O}_{X_I}(\mu))$ and $W = M_{P_I}(-w_I\mu)$ for short. Since $\mathfrak{n}_I^- \simeq N_I^- \simeq \mathrm{O}_{y_0}$ via exp and π_I, one has $V \simeq \Gamma(\mathrm{O}_{y_0}; \mathcal{O}_{\mathrm{O}_{y_0}}) \otimes V(\mu) \simeq \Gamma(\mathfrak{n}_I^-; \mathcal{O}_{\mathfrak{n}_I^-}) \otimes V(\mu) \simeq S((\mathfrak{n}_I^-)^*) \otimes V(\mu)$, and therefore $\mathrm{ch}(V) = \mathrm{ch}(V(\mu))\mathrm{ch}(S((\mathfrak{n}_I^-)^*)) = \mathrm{ch}(W^*)$. Thus we get a non-zero morphism $W \to V^*$, and by duality a non-zero morphism $\psi : V \to W^*$: it suffices to prove that ψ is injective. Setting $\nu = w_I\mu$, we have (1) $\psi_\nu : V_\nu \xrightarrow{\sim} (W^*)_\nu$, and (2) $\{u \in V : \mathfrak{n}^- u = 0\} = V_\nu$: (1) is clear ($\psi$ maps the lowest weight vector of V into the lowest weight vector of W^*), and (2) holds since if $\varphi \in \Gamma(\mathrm{O}_{y_0}; \mathcal{O}_{\mathrm{O}_{y_0}})$ and $\mathfrak{n}_I^- \varphi = 0$ then φ is a constant function (recall that the action of $\mathfrak{n}_I^-$ on V is induced by the left action on N_I^- on $\mathrm{O}_{y_0} \simeq N_I^-$) and ν is the lowest weight of $V(\mu)$. If $K = \ker(\psi) \neq 0$, there exists a non-zero vector $u \in K$ such that $\mathfrak{n}^- u = 0$: then $u \in K \cap V_\nu = K_\nu \neq 0$ by (2), but this contradicts (1). $\square$

Now, the following diagram of $(\mathfrak{g}, P'_I)$-modules of lowest weight $-\lambda'$ (where β is the natural isomorphism of Proposition 1.8):

$$\begin{tikzcd}
\mathcal{D}_{X'_I}(\lambda')(y_0) \arrow[r, "\alpha(s)(y_0)"] \arrow[d, "\beta"', "\sim"] & \mathcal{R}_{\mathrm{O}}(\mathcal{D}_{X_I}(\lambda))(y_0) \arrow[d, "\psi\circ\varphi"', "\sim"] \\
M_{P'_I}(\lambda') \arrow[r, "\phi"] & M_{P_I}(\lambda')^*
\end{tikzcd}$$

commutes up to a non-zero constant multiple since $\alpha(s)(y_0) \neq 0$ and $\phi \neq 0$. Using Proposition 1.7 and the equivalence of Proposition 1.4, we have finally proved that:

(2.4) $\quad \alpha(s) : \mathcal{D}_{X'_I}(\lambda') \to \mathcal{R}_{\mathrm{O}}(\mathcal{D}_{X_I}(\lambda))$ is an isomorphism if and only if $\lambda' \in (\mathfrak{h}^*_{\mathbb{Z}})_{I,\mathrm{irr}}$.

It is of interest to study also the dual integral transform given by $\mathcal{R}^*_{\mathrm{O}}$. The first step is:

Lemma 2.7. *Duality commutes with the integral transforms given by $\mathcal{R}_{\mathrm{O}}$ and $\mathcal{R}^*_{\mathrm{O}}$: more precisely, there is an isomorphism of functors $\mathcal{R}_{\mathrm{O}} \circ (\,\cdot\,)^* \simeq (\,\cdot\,)^* \circ \mathcal{R}^*_{\mathrm{O}}$.*

Proof. The duality functor commutes to non-characteristic inverse image, transversal tensor product and proper direct image (see [3] fore more details), and these conditions are easily satisfied in our case. $\square$

By (1.10) one has $\mathcal{K}_{X_I} \simeq \mathcal{D}_{X_I}(-2\rho_I)[d_{X_I}]$, and hence for $\lambda \in (\mathfrak{h}^*_{\mathbb{Z}})_I$ we get

$$(\mathcal{D}_{X_I}(\lambda))^* \simeq \mathcal{D}_{X_I}(\lambda')[d_{X_I}].$$

Therefore, using Lemma 2.7, (2.4) and $(\lambda')' = \lambda$, for any $\lambda \in (\mathfrak{h}^*_{\mathbb{Z}})_{I,\mathrm{irr}}$ we have

$$(\mathcal{R}^*_{\mathrm{O}}(\mathcal{D}_{X_I}(\lambda)))^* \;\simeq\; \mathcal{R}_{\mathrm{O}}((\mathcal{D}_{X_I}(\lambda))^*) \;\simeq\; \mathcal{R}_{\mathrm{O}}(\mathcal{D}_{X_I}(\lambda'))[d_{X_I}] \;\simeq\; \mathcal{D}_{X'_I}(\lambda)[d_{X_I}]$$

and thus, since $(\mathcal{M}^*)^* \simeq \mathcal{M}$ if $\mathcal{M}$ is a coherent $\mathcal{D}_Z$-module, we get

$$\mathcal{R}^*_{\mathrm{O}}(\mathcal{D}_{X_I}(\lambda)) \;\simeq\; (\mathcal{D}_{X'_I}(\lambda))^*[d_{X_I}]) \;\simeq\; \mathcal{D}_{X'_I}(\lambda').$$

We may summarize up our results in the following theorem, which reduces the study of the integral transforms $\mathcal{R}_{\mathrm{O}}$ and $\mathcal{R}^*_{\mathrm{O}}$ on quasi-G-equivariant $\mathcal{D}_{X_I}$-modules (which are — up to isomorphism— of the form $\mathcal{D}_{X_I}(\lambda)$ for some $\lambda \in (\mathfrak{h}^*_{\mathbb{Z}})_I$ by Proposition 1.4) to the study of irreducibility of a generalized Verma module (see Remark 1.6 for more details on irreducibility):

Theorem 2.8. *Let $\lambda \in (\mathfrak{h}^*_{\mathbb{Z}})_I$. Then:*

(i) $\mathcal{R}_{\mathrm{O}}(\mathcal{D}_{X_I}(\lambda)) \simeq \mathcal{D}_{X'_I}(\lambda')$ *if and only if* $\lambda' \in (\mathfrak{h}^*_{\mathbb{Z}})_{I,\mathrm{irr}}$;

(ii) $\mathcal{R}^*_{\mathrm{O}}(\mathcal{D}_{X_I}(\lambda)) \simeq \mathcal{D}_{X'_I}(\lambda')$ *if and only if* $\lambda \in (\mathfrak{h}^*_{\mathbb{Z}})_{I,\mathrm{irr}}$.

Other interesting applications of Theorem 2.8 come from adjunction formulas (see Section 1.6); of course, here we use the underlying complex analytic structures of the dual flag manifolds. By applying Theorem 2.8 to Corollary 1.10 (with $X = X'_I$, $Y = X_I$, $\mathcal{F} = \mathcal{O}_{X'_I}(\lambda)$) we get

Proposition 2.9. *Let $\lambda \in (\mathfrak{h}^*_{\mathbb{Z}})_I$, $F \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_{X_I})$. Then:*

$$\begin{aligned} \mathrm{R}\Gamma(X_I; F \otimes \mathcal{O}_{X_I}(\lambda)) &\simeq \mathrm{R}\Gamma(X'_I; r_{\mathrm{O}}(F) \otimes \mathcal{O}_{X'_I}(\lambda'))[d_{X_I}] && \text{if } \lambda \in (\mathfrak{h}^*_{\mathbb{Z}})_{I,\mathrm{irr}}, \\ \mathrm{RHom}\,(F, \mathcal{O}_{X_I}(\lambda)) &\simeq \mathrm{RHom}\,(r_{\mathrm{O}}(F), \mathcal{O}_{X'_I}(\lambda'))[-d_{X_I}] && \text{if } \lambda' \in (\mathfrak{h}^*_{\mathbb{Z}})_{I,\mathrm{irr}}. \end{aligned}$$

Now we are left with the choice of the sheaf F. For example, to investigate relations with representation theory, let us consider a semisimple real form $G_{\mathbb{R}}$ of G, and let Z be a $G_{\mathbb{R}}$-invariant subset of X_I. The sheaf $F = \mathbb{C}_{Z|X_I}$ and its transform $r_{\mathrm{O}}(\mathbb{C}_{Z|X_I})$ are clearly $G_{\mathbb{R}}$-equivariant, and the adjunction formulas in Proposition 2.9 can be viewed as isomorphisms (in the derived category) of representations of the group $G_{\mathbb{R}}$. If Z has nice geometric properties, as explained in Section 1.6, the transform $r_{\mathrm{O}}(\mathbb{C}_{Z|X_I})$ could be particularly simple to compute: we shall see a concrete example below, while dealing with grassmannians.

3 Example: Grassmann duality and representations of $SU(p,q)$

Let us investigate more closely the case of maximal parabolic subgroups of $G = SL(n,\mathbb{C})$, with $n \geq 2$. In the standard choices of [4], $\mathfrak{h}$ is the subalgebra of diagonal matrices in $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$, i.e. $\mathfrak{h} = \{a \in \bigoplus_{l=1}^n \mathbb{C}E_l : \sum_{l=1}^n a_l = 0\}$ where E_l is the $(n \times n)$-matrix with zero in all entries excepted the (l,l)th which is 1; if $\{\varepsilon_l : l = 1, \dots, n\}$ is the basis dual to $\{E_l : l = 1, \dots, n\}$, one has $\mathfrak{h}^* = (\bigoplus_{l=1}^n \mathbb{C}\varepsilon_l)\,/\mathbb{C}\,(\sum_{l=1}^n \varepsilon_l)$, and so we always argue modulo the subspace $\mathbb{C}\,(\sum_{l=1}^n \varepsilon_l)$. The roots of $\mathfrak{g}$ are $\pm\{\varepsilon_i - \varepsilon_j : 1 \leq i < j \leq n\}$, the set of simple roots is $\{\alpha_i = \varepsilon_i - \varepsilon_{i+1} : i \in S\}$ with $S = \{1, \dots, n-1\}$, the fundamental weights are $\{\varpi_i = \sum_{l=1}^i \varepsilon_l : i \in S\}$ and $\rho = \sum_{l=1}^{n-1}(n-l)\varepsilon_l$.

Let us fix $p \in \mathbb{Z}$, $1 \leq p \leq n/2$. We set $I = S \setminus \{p\}$: since $\Delta_I = \pm\{\varepsilon_i - \varepsilon_j : 1 \leq i < j \leq p \text{ or } p+1 \leq i < j \leq n\}$, one has $2\rho_I = n\varpi_p$. If we write $g \in G$ as $g = \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix}$ with $g_1 \in M_p(\mathbb{C})$ and $g_4 \in M_{n-p}(\mathbb{C})$, one has $P_I = \{g \in G : g_3 = 0\}$ and $P'_I = \{g \in G : g_2 = 0\}$: therefore, X_I (resp. X'_I) is identified with the Grassmann manifold $\mathbb{G}$ (resp. $\mathbb{G}'$) of p- (resp. $(n-p)$-)dimensional complex linear subspaces of $\mathbb{C}^n$; one has $d_{\mathbb{G}} = d_{\mathbb{G}'} = p(n-p)$. The open dense G-orbit in $\mathbb{G} \times \mathbb{G}'$ is the transversality relation

$$\mathrm{O} = \{(x, y) \in \mathbb{G} \times \mathbb{G}' : x \cap y = \{0\}\}.$$

The weights $\lambda \in (\mathfrak{h}^*_{\mathbb{Z}})_I$ are of the form $\lambda = \sum_{i=1}^{n-1} \lambda_i \varpi_i$ with $\lambda_p \in \mathbb{Z}$ and $\lambda_i \in \mathbb{Z}_{\geq 0}$ for any $i \neq p$. Let us fix such a λ. One computes that

$$\lambda' = \sum\nolimits_{i=1}^{p-1} \lambda_{p-i}\varpi_i - (\lambda_1 + \cdots + \lambda_{n-1} + n)\varpi_p + \sum\nolimits_{j=p+1}^{n-1} \lambda_{n+p-j}\varpi_j.$$

Notation 3.1. In $\mathcal{P}(\mathbb{Z})$ we introduce the relation "$\dot{\leq}$": for $A, B \subset \mathbb{Z}$, we say that $A \,\dot{\leq}\, B$ if and only if $x \leq y$ for any $x \in A \setminus B$ and any $y \in B \setminus A$. (Observe that $\dot{\leq}$ is trivially reflexive, but neither antisymmetric nor transitive.)

Lemma 3.2. *Let $\lambda = \sum_{i=1}^{n-1} \lambda_i \varpi_i \in (\mathfrak{h}^*_{\mathbb{Z}})_I$, and consider the increasing subsets of $\mathbb{Z}$*

$$\begin{aligned} A_\lambda &= \{i + (\lambda_{p-i+1} + \cdots) + \lambda_p : i = 1, \ldots, p\} \subset \mathbb{Z}_{\geq 1} + \lambda_p, \\ B_\lambda &= \{-[n - j + (\lambda_{p+1} + \cdots + \lambda_{n+p-j})] : j = p+1, \ldots, n-1\} \cup \{0\} \subset \mathbb{Z}_{\leq 0}. \end{aligned}$$

Then $\lambda \in (\mathfrak{h}^*_{\mathbb{Z}})_{I,\mathrm{irr}}$ if and only if $A_\lambda \,\dot{\leq}\, B_\lambda$.

Proof. In this case, where $\mathfrak{g}$ is of type A_{n-1}, the sufficient criterion (1.8) for $\lambda \in (\mathfrak{h}^*_{\mathbb{Z}})_{I,\mathrm{irr}}$ is also necessary (see [7, Satz 4]), and it is not too difficult to check that the above condition is equivalent to it: just remark that, by writing $\lambda + \rho = \sum_{\ell=1}^n a_\ell \varepsilon_l$, one has —up to $\mathbb{Z}(\varepsilon_1 + \cdots + \varepsilon_n)$— $A_\lambda = \{a_p, \ldots, a_1\}$ and $B_\lambda = \{a_n, \ldots, a_{p+1}\}$. □

Hence, from Theorem 2.8 and Lemma 3.2 we get the following proposition (for (i) note that, up to a common integral translation, one has $A_{\lambda'} = -A_\lambda$ and $B_{\lambda'} = -B_\lambda$):

Proposition 3.3. *Let $\lambda = \sum_{i=1}^{n-1} \lambda_i \varpi_i \in (\mathfrak{h}^*_{\mathbb{Z}})_I$. Then:*

(i) $\mathcal{R}_{\mathrm{O}}(\mathcal{D}_{\mathbb{G}}(\lambda)) \simeq \mathcal{D}_{\mathbb{G}'}(\lambda')$ *if and only if* $A_\lambda \,\dot{\geq}\, B_\lambda$;

(ii) $\mathcal{R}^*_{\mathrm{O}}(\mathcal{D}_{\mathbb{G}}(\lambda)) \simeq \mathcal{D}_{\mathbb{G}'}(\lambda')$ *if and only if* $A_\lambda \,\dot{\leq}\, B_\lambda$.

In particular, observe that $A_\lambda \,\dot{\geq}\, B_\lambda$ (resp. $\dot{\leq}$) is trivially true (resp. false) when $\lambda_p \geq -1$ and trivially false (resp. true) when $\lambda_p \leq -(n-1) - \sum_{i \neq p} \lambda_i$.

Let us put in evidence some particular cases.

(1) *Projective duality.* When $p = 1$, one has $A_\lambda = \{1 + \lambda_1\}$ and $B_\lambda = \{-(n - 2 + \lambda_2 + \cdots + \lambda_{n-1}), \ldots, -(1 + \lambda_2), 0\}$ so that $A_\lambda \,\dot{\geq}\, B_\lambda$ (resp $\dot{\leq}$) if and only if $\lambda_1 \geq -1$ (resp. $\lambda_1 \leq -(n-1) - \sum_{i=2}^{n-1} \lambda_i$) or —for $n \geq 4$— in the singular cases $\lambda_1 = -(n+1-j) - (\lambda_2 + \cdots + \lambda_{n+1-j})$ with $j = 3, \ldots, n-1$ (resp. the same). For example, when $n = 4$ (the "twistor transform" between $X_I = \mathbb{C}\mathbf{P}^3$ and $X'_I = (\mathbb{C}\mathbf{P}^3)^*$, see [2, Chapter 10]) one has $\lambda_1 \geq -1$ (resp. $\lambda_1 \leq -3 - \lambda_2 - \lambda_3$) and the singular case $\lambda_1 = -2 - \lambda_2$.

(2) *Grassmann duality for line bundles.* When $\lambda_i = 0$ for $i \neq p$, one has $A_\lambda = \{1, \dots, p\} + \lambda_p$ and $B_\lambda = \{-(n-p-1), \dots, -1, 0\}$, thus $A_\lambda \dot{\geq} B_\lambda$ (resp. $\dot{\leq}$) if and only if $1 + \lambda_p \geq -(n-p-1)$, i.e. $\lambda_p \geq -(n-p)$ (resp. $p + \lambda_p \leq 0$, i.e. $\lambda_p \leq -p$). We therefore get an alternative proof of [13, Theorem 3.16(i)].

(3) *The conformal compactification of complexified space-time.* In the case $n = 4$ and $p = 2$, $\mathbb{G} = \mathbb{G}'$ may be viewed as the conformal compactification of complexified Minkowski space in Penrose's twistor approach to massless fields (see e.g. [2]). Here, one has $A_\lambda = \{1 + \lambda_2, 2 + \lambda_1 + \lambda_2\}$ and $B_\lambda = \{-1 - \lambda_3, 0\}$: thus $A_\lambda \dot{\geq} B_\lambda$ (resp. $\dot{\leq}$) if and only if $\lambda_2 = -2 - \min\{\lambda_1, \lambda_3\}$ or $\lambda_2 \geq -1$ (resp. $\lambda_2 \leq -3 - \lambda_1 - \lambda_3$ or $\lambda_2 = -2 - \max\{\lambda_1, \lambda_3\}$).

Now let us show a concrete example of how adjunction formulas provide informations about representations of real forms of G.

Let $k \in \mathbb{Z}$, $1 \leq p \leq k \leq n/2$, and let Q be a nondegenerate hermitian form on $\mathbb{C}^n$ of signature $(k, n-k)$. The real semisimple Lie group $G_\mathbb{R} = SU(Q) \simeq SU(k, n-k)$ of elements of $G = SL(n, \mathbb{C})$ preserving Q is a real form of G, and the $G_\mathbb{R}$-orbits in $\mathbb{G}$ are

$$U_{k',k''} = \{x \in \mathbb{G} : Q|_x \text{ has signature } (k', k'')\} \qquad (k', k'' \geq 0, \quad 0 \leq k' + k'' \leq p).$$

The open $G_\mathbb{R}$-orbits are $U_{k',p-k'}$ (with $0 \leq k' \leq p$) and the only closed one is $U_{0,0}$ (the Q-isotropic p-subspaces). In general, one has $\overline{U_{k',k''}} = \bigcup_{i,j} U_{i,j}$ (with $i = 0, \dots k'$, $j = 0, \dots k''$). Similarly, the $G_\mathbb{R}$-orbits in $\mathbb{G}'$ are

$$U'_{\ell',\ell''} = \{y \in \mathbb{G}' : Q|_y \text{ has signature } (\ell', \ell'')\}$$
$$(k - p \leq \ell' \leq k, \quad n-k-p \leq \ell'' \leq n-k, \quad n - 2p \leq \ell' + \ell'' \leq n-p).$$

Let $U_+ = U_{p,0}$ and $U'_- = U'_{k-p,n-k}$. We are going to consider the constant sheaf $F = \mathbb{C}_{\overline{U_+}|\mathbb{G}}$ (note that the subset $Z = \overline{U_+}$ of $\mathbb{G}$ is clearly $G_\mathbb{R}$-invariant, since it is the union of the $G_\mathbb{R}$-orbits $U_{k',0}$ for $0 \leq k' \leq p$).

Lemma 3.4. *One has* $r_\circ(\mathbb{C}_{\overline{U_+}|\mathbb{G}}) \simeq \mathbb{C}_{U'_-|\mathbb{G}'}[-2p(k-p)]$.

Proof. Is a natural generalization of the proof of Lemma 4.11 in [13], where $k = p$. □

Hence, by choosing $F = \mathbb{C}_{\overline{U_+}|\mathbb{G}}$ into Proposition 2.9, thanks to Lemma 3.4 we get the following isomorphisms in the derived category of representations of $G_\mathbb{R}$:

$$\begin{aligned} \mathrm{R}\Gamma(\overline{U_+}; \mathcal{O}_\mathbb{G}(\lambda)) &\simeq \mathrm{R}\Gamma_c(U'_-; \mathcal{O}_{\mathbb{G}'}(\lambda'))[p(n-2k+p)] && \text{if } A_\lambda \dot{\leq} B_\lambda, \\ \mathrm{R}\Gamma_{\overline{U_+}}(\mathbb{G}; \mathcal{O}_\mathbb{G}(\lambda)) &\simeq \mathrm{R}\Gamma(U'_-; \mathcal{O}_{\mathbb{G}'}(\lambda'))[-p(n-2k+p)] && \text{if } A_\lambda \dot{\geq} B_\lambda. \end{aligned}$$

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