Generalized Verma modules, b-functions of semi-invariants and duality for twisted $\mathcal{D}$-modules on generalized flag manifolds

Corrado Marastoni

Dipartimento di Matematica Pura ed Applicata – Università di Padova – Via Belzoni, 7 – I-35131 Padova (Italy) and R.I.M.S. – Kyoto University – Kitashirakawa, Sakyo-ku – Kyoto 606-8502 (Japan)
E-mail: maraston@math.unipd.it

Abstract. Let G be a connected semisimple algebraic Lie group over $\mathbb{C}$, P a parabolic subgroup, $\mathfrak{g}$ and $\mathfrak{p}$ their Lie algebras. We prove a microlocal version of Gyoja's conjectures [2] about a relation between the irreducibility of generalized Verma modules on $\mathfrak{g}$ induced from $\mathfrak{p}$ and the zeroes of b-functions of P-semi-invariants on G. Our method uses a duality for twisted $\mathcal{D}$-modules on generalized flag manifolds.

Modules de Verma généralisés, b-fonctions des quasi-invariants et dualité pour les $\mathcal{D}$-modules tordus sur les variétés de drapeaux généralisées

Résumé. Soient G un groupe algébrique sur $\mathbb{C}$ connexe semi-simple, P un sous-groupe parabolique, $\mathfrak{g}$ et $\mathfrak{p}$ leurs algèbres de Lie. On démontre une version microlocale des conjectures de Gyoja [2] sur une relation entre l'irreductibilité des modules de Verma généralisés sur $\mathfrak{g}$ induits de $\mathfrak{p}$ et les zéros des b-fonctions des quasi-invariants sur G par rapport à P. Notre méthode utilise une dualité pour les $\mathcal{D}$-modules tordus sur les variétés de drapeaux généralisées.

Version française abrégée – Soient G un groupe algébrique sur $\mathbb{C}$ connexe semi-simple avec identité e, B un sous-groupe de Borel, T un tore maximal dans B, B^- le sous-groupe de Borel tel que $B \cap B^- = T$ et $\mathfrak{g}$, $\mathfrak{t}$, $\mathfrak{b}$, $\mathfrak{b}^-$ leurs algèbres de Lie. Soient $\Delta = \Delta^+ \cup \Delta^- \subset \mathfrak{t}^*$ le système des racines, $\{\varpi_i : i \in I_0\} \subset \mathfrak{t}^*$ les poids fondamentaux. Pour un $i \in I_0$, la fonction régulière $f_i : G \to \mathbb{C}$ caractérisée par les propriétés $f_i(e) = 1$ et $f_i(b'gb) = \varpi_i(b')\varpi_i(b)f_i(g)$ pour tout $b' \in B^-$ et $b \in B$, est appelée "i-ème quasi-invariant" de G; en général, si $\nu = \sum_{i \in I_0} \nu_i \varpi_i \in \mathfrak{t}^*$, on définit $f^\nu = \prod_{i \in I_0} f_i^{\nu_i}$. Pour un $I \subsetneq I_0$ fixé, soient $\Delta_I = \sum_{i \in I} \mathbb{Z}\alpha_i \cap \Delta$, $\rho_I = \frac{1}{2}\sum_{\alpha \in \Delta^+ \setminus \Delta_I} \alpha$ et

$$\mathfrak{l}_I = \mathfrak{t} \oplus \textstyle\sum_{\alpha \in \Delta_I} \mathfrak{g}_\alpha, \qquad \mathfrak{n}_I^\pm = \sum_{\alpha \in \Delta^\pm \setminus \Delta_I} \mathfrak{g}_\alpha, \qquad \mathfrak{p}_I^\pm = \mathfrak{l}_I \oplus \mathfrak{n}_I^\pm;$$

considérons les sous-groupes connexes correspondants L_I, $N_I^\pm$ et $P_I^\pm$ de G.
Posons $\mathfrak{t}_I^* = \{\lambda \in \mathfrak{t}^* : \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0} \ \forall \alpha \in \Delta_I^+\}$. Pour un $\lambda \in \mathfrak{t}_I^*$, on notera $V(\lambda)$ la représentation irréductible de $\mathfrak{l}_I$ de plus haut poids λ, étendue à $\mathfrak{p}_I^\pm$ avec l'action triviale de $\mathfrak{n}_I^\pm$. Le $U(\mathfrak{g})$-module $M_I^+(\lambda) = U(\mathfrak{g}) \otimes_{\mathfrak{p}_I^+} V(\lambda)$ est appelé "module de Verma généralisé".

Les conjectures de Gyoja. L'action de $P_I^- \times P_I^+$ sur G donnée par $(x, y)g = xgy^{-1}$ est préhomogène avec orbite ouverte $\Omega_I = G \setminus f_I^{-1}(0)$, où $f_I = \prod_{i \in I_0 \setminus I} f_i$. Soit $\mathcal{N}_I$ le $\mathcal{D}_G$-module holonôme associé avec f_I, et notons Σ_I la variété caractéristique de $\mathcal{N}_I$ (voir [3]). On a l'inclusion $\Sigma_I \subset \bigcup_O \Lambda_O$, où O parcourt l'ensemble fini des orbites de $P_I^- \times P_I^+$ dans G et Λ_O est le fibré conormal à O dans G. Dans l'autre sens, Λ_O est dit "être une bonne Lagrangienne" (voir [8]) si $\Lambda_O \subset \Sigma_I$ et l'action induite de $P_I^- \times P_I^+$ sur Λ_O est préhomogène.

Considérons $\mathcal{D}_G[\underline{s}] = \mathcal{D}_G \otimes_{\mathbb{C}} \mathbb{C}[\underline{s}]$, faisceau des opérateurs différentiels réguliers sur G avec paramètres complexes $\underline{s} = (s_i)_{i \in I_0 \setminus I}$. En idéntifiant $\underline{s} \simeq \sum_{i \in I_0 \setminus I} s_i \varpi_i$, pour tout $i \in I_0 \setminus I$ l'idéal

$$\mathcal{B}_{I,i} = \{b(\underline{s}) \in \mathbb{C}[\underline{s}] : P(\underline{s}) f^{\underline{s}+\varpi_i} = b(\underline{s}) f^{\underline{s}} \quad \exists P(\underline{s}) \in \mathcal{D}_G[\underline{s}]\}$$

est différent de zéro (voir [7]). D'autre part, si Λ_0 est une bonne Lagrangienne, on peut microlocaliser la définition de $\mathcal{B}_{I,i}$ en Λ_O et obtenir un idéal différent de zéro et principal (voir [8]). Le générateur $b_{I,\Lambda_O,i}(\underline{s})$ de cet idéal, qui divise tout élément de $\mathcal{B}_{I,i}$, est appelé "i-ème b-fonction locale" en Λ_O.

Dans son travail [2] (voir 3.1 et 3.3), A. Gyoja avait formulé les conjectures suivantes:

(C1) il existe une orbite $O \subset G$ telle que Λ_O est une bonne Lagrangienne et $b_{I,\Lambda_O,i}(\underline{s}) \in \mathcal{B}_{I,i}$ pour tout $i \in I_0 \setminus I$ (donc, en particulier, $\mathcal{B}_{I,i}$ est un idéal principal engendré par la "i-ème b-fonction" (globale) $b_{I,i}(\underline{s}) := b_{I,\Lambda_O,i}(\underline{s})$);

(C2) pour $\lambda = \lambda_{\mathsf{c}} \in \sum_{i\in I_0\setminus I} \mathbb{C}\varpi_i$, le module de Verma généralisé $M_I^+(\lambda_{\mathsf{c}})$ est irréductible si et seulement si $b_{I,i}(\lambda_{\mathsf{c}} - \varpi_i - \nu) \neq 0$ pour tout $i \in I_0 \setminus I$ et $\nu \in \sum_{i\in I_0\setminus I} \mathbb{Z}_{\geq 0}\varpi_i$.

Transformations intégrales pour les $\mathcal{D}$-modules tordus. Considérons les variétés de drapeaux généralisées duales $X_I^\pm = G/P_I^\pm$, et notons $\pi_\pm : G \to X_I^\pm$ les projections canoniques.
Soit $\mu \in \mathfrak{t}_I^*$. Pour tout ouvert $U \subset X_I^\pm$ on pose

$$\mathcal{O}_{X_I^\pm}(\mu)(U) = \{\varphi \in \Gamma(\pi_\pm^{-1}(U), \mathcal{O}_G \otimes V(\mu)) : \tfrac{d}{dt}\varphi(ge^{tA})\big|_{t=0} = \mp A \cdot \varphi(g) \ \ \forall\, g \in G,\, A \in \mathfrak{l}_I\}.$$

Si μ est intégral, $\mathcal{O}_{X_I^\pm}(\mu)$ est le faisceau des sections régulières du fibré vectoriel sur $X_I^\pm$ associé à $V(\mu)$; en général, si on écrit $\mu = \mu_{\mathsf{c}} + \mu_{\mathsf{d}}$ avec $\mu_{\mathsf{c}} \in \sum_{i\in I_0\setminus I} \mathbb{C}\varpi_i$ et $\mu_{\mathsf{d}} \in \sum_{i\in I_0} \mathbb{Z}_{\geq 0}\varpi_i$, alors $\mathcal{O}_{X_I^\pm}(\mu)$ a une structure de faisceau sur $X_I^\pm$ tordu par μ_{c} (voir [5]).

Pour un $\lambda = \lambda_{\mathsf{c}} \in \sum_{i\in I_0\setminus I} \mathbb{C}\varpi_i$ donné, soient $\mathcal{D}_{X_I^\pm,\lambda_{\mathsf{c}}}$ l'anneau des opérateurs différentiels sur $\mathcal{O}_{X_I^\pm}(\lambda_{\mathsf{c}})$ (on note que $\mathcal{D}_{X_I^\pm,\lambda_{\mathsf{c}}}$ est un faisceau d'opérateurs différentiels tordus sur $X_I^\pm$, voir [5]) et $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_{X_I^\pm,\lambda_{\mathsf{c}}})$ la catégorie dérivée bornée des $\mathcal{D}_{X_I^\pm,\lambda_{\mathsf{c}}}$-modules à gauche. On considère les opérations d'image inverse $D(\,\cdot\,)^*$, image directe propre $D(\,\cdot\,)_!$, produit tensoriel $\overset{\mathrm{D}}{\otimes}$, produit tensoriel extérieur $\boxtimes^D$, au sens des $\mathcal{D}$-modules tordus (voir [5]). En particulier, on peut construire un faisceau d'opérateurs différentiels tordus $\mathcal{D}_{X_I^+\times X_I^-,\lambda_{\mathsf{c}}} := \mathcal{D}_{X_I^+,\lambda_{\mathsf{c}}} \boxtimes^D \mathcal{D}_{X_I^-,\lambda_{\mathsf{c}}}$ sur $X_I^+ \times X_I^-$.

L'action diagonale de G sur $X_I^+ \times X_I^-$ définit un espace préhomogène qui a le même type de singularité que celle de l'espace préhomogène défini par l'action de $P_I^- \times P_I^+$ sur G. Si O est une orbite de $P_I^- \times P_I^+$ dans G, on note O' l'orbite correspondante de G dans $X_I^+ \times X_I^-$, et $\Lambda_{O'}$ le fibré conormal à O' dans $X_I^+ \times X_I^-$. En particulier, l'orbite ouverte $\Omega_I' \simeq G/L_I$ est une variété affine.
Soit $\mathcal{B}_{\Omega_I',\lambda_{\mathsf{c}}}$ le faisceau sur $X_I^+ \times X_I^-$ des fonctions méromorphes à pôles sur le fermé $(X_I^+ \times X_I^-) \setminus \Omega_I'$ muni d'une structure de module sur $\mathcal{D}_{X_I^+\times X_I^-,\lambda_{\mathsf{c}}}$; le dual (au sens des $\mathcal{D}$-modules à gauche) $\mathcal{B}^*_{\Omega_I',\lambda_{\mathsf{c}}}$ sera donc un module sur $\mathcal{D}_{X_I^+\times X_I^-,-\lambda_{\mathsf{c}}}$. Ces modules donnent deux foncteurs (que l'on appelle "transformations intégrales" d'après [1]) par les formules suivantes, où $X_I^+ \overset{q_1}{\leftarrow} X_I^+ \times X_I^- \overset{q_2}{\rightarrow} X_I^-$ sont les projections:

$$\mathbf{D}^{\mathrm{b}}(\mathcal{D}_{X_I^+,-\lambda_{\mathsf{c}}}) \underset{\mathcal{B}_{\Omega_I',-\lambda_{\mathsf{c}}} \overset{\mathrm{D}}{\circ}\,\cdot}{\overset{\cdot\,\overset{\mathrm{D}}{\circ}\,\mathcal{B}^*_{\Omega_I',-\lambda_{\mathsf{c}}}}{\rightleftarrows}} \mathbf{D}^{\mathrm{b}}(\mathcal{D}_{X_I^-,\lambda_{\mathsf{c}}})\ , \qquad \begin{cases} \cdot\,\overset{\mathrm{D}}{\circ}\,\mathcal{B}^*_{\Omega_I',-\lambda_{\mathsf{c}}} = Dq_{2!}(Dq_1{}^*(\,\cdot\,) \overset{\mathrm{D}}{\otimes} \mathcal{B}^*_{\Omega_I',-\lambda_{\mathsf{c}}}) \\ \mathcal{B}_{\Omega_I',-\lambda_{\mathsf{c}}} \overset{\mathrm{D}}{\circ}\,\cdot = Dq_{1!}(\mathcal{B}_{\Omega_I',-\lambda_{\mathsf{c}}} \overset{\mathrm{D}}{\otimes} Dq_2{}^*(\,\cdot\,)). \end{cases}$$

A l'aide de [6, Lemma 2.2], on démontre que ces foncteurs sont en fait des équivalences de catégories quasi-inverses l'une de l'autre.
Si $\mu \in \mathfrak{t}_I^*$ est intégral, le faisceau $\mathcal{D}_{X_I^\pm,\lambda_{\mathsf{c}}}(\mu) := \mathcal{D}_{X_I^\pm,\lambda_{\mathsf{c}}} \otimes_{\mathcal{O}_{X_I^\pm}} \mathcal{O}_{X_I^\pm}(\mu)$ est un $\mathcal{D}_{X_I^\pm,\lambda_{\mathsf{c}}}$-module à gauche, et on peut donc considérer la transformée $\mathcal{B}_{\Omega_I',-\lambda_{\mathsf{c}}} \overset{\mathrm{D}}{\circ} \mathcal{D}_{X_I^-,\lambda_{\mathsf{c}}}(\mu)$ dans $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_{X_I^+,-\lambda_{\mathsf{c}}})$.

Théorème. *Il existe une et une seule orbite O_I de $P_I^- \times P_I^+$ dans G telle que $\Lambda_{O_I} \subset \Sigma_I$ et les projections naturelles $\Lambda_{O_I'} \to T^*X_I^\pm$ soient surjectives au point générique. Pour une telle orbite, Λ_{O_I} est une bonne Lagrangienne et $\Lambda_{O_I'} \to T^*X_I^\pm$ sont des applications birationnelles.*
En outre, pour $\lambda_{\mathsf{c}} \in \sum_{i\in I_0\setminus I} \mathbb{C}\varpi_i$ les propriétés suivantes sont équivalentes:

(1) $M_I^+(\lambda_c)$ *est un* $\mathfrak{g}$*-module irréductible;*

(2) $b_{I,\Lambda_{O_I},i}(\lambda_c - \varpi_i - \nu) \neq 0$ *pour tout* $i \in I_0 \setminus I$ *et* $\nu \in \sum_{i \in I_0 \setminus I} \mathbb{Z}_{\geq 0} \varpi_i$*;*

(3) $\mathcal{D}_{X_I^+, -\lambda_c} \xrightarrow{\sim} \mathcal{B}_{\Omega'_I, -\lambda_c} \overset{D}{\circ} \mathcal{D}_{X_I^-, \lambda_c}(-2\rho_I)$ *dans* $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_{X_I, -\lambda_c})$.

Ce résultat montre que l'irréductibilité des modules de Verma généralisés est bien contrôlée (voir (C2)) par une famille de b-fonctions, décrites en tant que b-fonctions locales le long du fibré conormal à une orbite "privilégiée" O_I dans G. Cependant, nous ne savons pas si, en général, ces b-fonctions locales sont aussi globales (voir (C1)).

This note is organized as follows. After recalling Gyoja's conjectures (Section 1), we introduce our approach based on an integral transform for twisted $\mathcal{D}$-modules on dual generalized flag manifolds (Section 2) and then we state and comment our main result (Section 3).

1 Generalized Verma modules and b-functions of semi-invariants

Let G be a connected semisimple algebraic group over $\mathbb{C}$ with identity e, B a Borel subgroup of G, T a maximal torus contained in B, B^- the Borel subgroup such that $B \cap B^- = T$, and let $\mathfrak{g}$, $\mathfrak{t}$, $\mathfrak{b}$, $\mathfrak{b}^-$ be their Lie algebras. Let $\Delta = \Delta^+ \cup \Delta^- \subset \mathfrak{t}^*$ be the root system, and denote by r_α the reflection in $\mathfrak{t}^*$ and by $\mathfrak{g}_\alpha$ the root space in $\mathfrak{g}$ associated to $\alpha \in \Delta$. Let $\{\alpha_i : i \in I_0\} \subset \Delta^+$ be the simple roots, $\{\varpi_i : i \in I_0\} \subset \mathfrak{t}^*$ the fundamental weights, $W = \langle r_{\alpha_i} : i \in I_0 \rangle$ the Weyl group. For $i \in I_0$ let v_i be a highest weight vector in the irreducible $\mathfrak{g}$-module with highest weight ϖ_i, and let $v_i^\vee$ be a lowest weight vector in the dual $\mathfrak{g}$-module normalized so that $\langle v_i, v_i^\vee \rangle = 1$. The i-th semi-invariant of G is the regular function $f_i : G \to \mathbb{C}$ defined by $f_i(g) = \langle g v_i, v_i^\vee \rangle$, which is also characterized by its properties $f_i(e) = 1$ and $f_i(b' g b) = \varpi_i(b') \varpi_i(b) f_i(g)$ for any $b' \in B^-$ and $b \in B$. More generally, for $\nu = \sum_{i \in I_0} \nu_i \varpi_i \in \mathfrak{t}^*$ we intend $f^\nu = \prod_{i \in I_0} f_i^{\nu_i}$ as a single valued branch of a multivalued function.

Let us fix $I \subsetneq I_0$, and set $\Delta_I = \sum_{i \in I} \mathbb{Z}\alpha_i \cap \Delta$, $\rho_I = \frac{1}{2} \sum_{\alpha \in \Delta^+ \setminus \Delta_I} \alpha$ and $W_I = \langle r_{\alpha_i} : i \in I \rangle \subset W$; let w_I be the longest element of W_I. We also set

$$\mathfrak{l}_I = \mathfrak{t} \oplus \textstyle\sum_{\alpha \in \Delta_I} \mathfrak{g}_\alpha, \qquad \mathfrak{n}_I^\pm = \sum_{\alpha \in \Delta^\pm \setminus \Delta_I} \mathfrak{g}_\alpha, \qquad \mathfrak{p}_I^\pm = \mathfrak{l}_I \oplus \mathfrak{n}_I^\pm;$$

let L_I, $N_I^\pm$ and $P_I^\pm$ be the associated connected subgroups of G.
The isomorphic classes of finite dimensional irreducible representations of $\mathfrak{l}_I$ are parametrized by

$$\mathfrak{t}_I^* = \{\lambda \in \mathfrak{t}^* : \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0} \ \forall \alpha \in \Delta_I^+\}$$

by associating to a representation its highest weight. For $\lambda \in \mathfrak{t}_I^*$, we denote by $V(\lambda)$ the irreducible representation of $\mathfrak{l}_I$ with heighest weight λ, extended to $\mathfrak{p}_I^\pm$ with the trivial action of $\mathfrak{n}_I^\pm$. The generalized Verma modules associated to the pair (I, λ) are the $U(\mathfrak{g})$-modules with highest weight λ (resp. lowest weight $w_I \lambda$) defined by

$$M_I^\pm(\lambda) = U(\mathfrak{g}) \otimes_{\mathfrak{p}_I^\pm} V(\lambda).$$

The action of $P_I^- \times P_I^+$ on G given by $(x, y)g = x g y^{-1}$ is prehomogeneous, with open orbit $\Omega_I = P_I^- P_I^+$, and one has $\Omega_I = G \setminus f_I^{-1}(0)$, where $f_I = \prod_{i \in I_0 \setminus I} f_i$ (see [2, Lemma 2.6]). To the function f_I one associates a holonomic $\mathcal{D}_G$-module $\mathcal{N}_I$ (see [8]); let Σ_I be the characteristic variety of $\mathcal{N}_I$. One has the estimate $\Sigma_I \subset \bigcup_O \Lambda_O$, where O ranges in the finite set of $(P_I^- \times P_I^+)$-orbits in G and Λ_O is the conormal bundle to O. Conversely, Λ_O is called a "good Lagrangian" (see [8]) if $\Lambda_O \subset \Sigma_I$ and the induced action of $P_I^- \times P_I^+$ on Λ_O is prehomogeneous.

Let $\mathcal{D}_G$ be the sheaf of rings of linear partial differential operators on G with regular coefficients. Let $\underline{s} = (s_i)_{i\in I_0\setminus I}$ be complex variables, and set $\mathcal{D}_G[\underline{s}] = \mathcal{D}_G \otimes_{\mathbb{C}} \mathbb{C}[\underline{s}]$. By identifying $\underline{s} \simeq \sum_{i\in I_0\setminus I} s_i\varpi_i$, the following ideal of $\mathbb{C}[\underline{s}]$ is nontrivial for any $i \in I_0 \setminus I$ (see [7]):

$$\mathcal{B}_{I,i} = \{b(\underline{s}) \in \mathbb{C}[\underline{s}] : P(\underline{s})f^{\underline{s}+\varpi_i} = b(\underline{s})f^{\underline{s}} \quad \exists\, P(\underline{s}) \in \mathcal{D}_G[\underline{s}]\}.$$

On the other hand, if Λ_O is a good Lagrangian then the following ideal, where $\mathcal{E}_G$ is the sheaf of microdifferential operators of finite order on G, is nontrivial and principal for any $i \in I_0 \setminus I$ (see [8]):

$$\mathcal{B}_{I,\Lambda_O,i} = \{b(\underline{s}) \in \mathbb{C}[\underline{s}] : Q(\underline{s})f^{\underline{s}+\varpi_i} = b(\underline{s})f^{\underline{s}} \quad \exists\, Q(\underline{s}) \in \mathcal{E}_G[\underline{s}] \text{ generically invertible at } \Lambda_O\}.$$

The generator $b_{I,\Lambda_O,i}(\underline{s})$ of $\mathcal{B}_{I,\Lambda_O,i}$ is called the "i-th local b-function" at Λ_O.

In [2], A. Gyoja conjectured that:

Conjecture 1.1. ([2, 3.1]) *There exists a orbit $O \subset G$ such that Λ_O is a good Lagrangian and $b_{I,\Lambda_O,i}(\underline{s}) \in \mathcal{B}_{I,i}$ for any $i \in I_0 \setminus I$.*

Since $b_{I,\Lambda_O,i}(\underline{s})$ divides any element of $\mathcal{B}_{I,i}$, Conjecture 1.1 would imply that $\mathcal{B}_{I,i}$ is a principal ideal generated by the "i-th (global) b-function" $b_{I,i}(\underline{s}) := b_{I,\Lambda_O,i}(\underline{s})$.
Assuming Conjecture 1.1, Gyoja's main conjecture is:

Conjecture 1.2. ([2, 3.3]) *For $\lambda = \lambda_{\mathsf{c}} \in \sum_{i\in I_0\setminus I} \mathbb{C}\varpi_i$, the following statements are equivalent:*

(1) *the generalized Verma module $M_I^+(\lambda_{\mathsf{c}})$ is irreducible;*

(2) $b_{I,i}(\lambda_{\mathsf{c}} - \varpi_i - \nu) \neq 0$ *for any $i \in I_0 \setminus I$ and $\nu \in \sum_{i\in I_0\setminus I} \mathbb{Z}_{\geq 0}\varpi_i$.*

2 Duality for twisted $\mathcal{D}$-modules on generalized flag manifolds

We refer to [5] for the notions of twisted sheaves and twisted $\mathcal{D}$-modules.

Consider the dual generalized flag manifolds $X_I^{\pm} = G/P_I^{\pm}$. The diagonal G-action on $X_I^+ \times X_I^-$ defines a prehomogeneous space whose singularity is of the same type than the one determined by the above prehomogeneous $(P_I^- \times P_I^+)$-action on G. For a $(P_I^- \times P_I^+)$-orbit O in G we denote by O' the corresponding G-orbit in $X_I^+ \times X_I^-$, and by $\Lambda_{O'}$ the conormal bundle to O' in $X_I^+ \times X_I^-$. Observe that the open orbit Ω'_I is an affine variety.

For $\lambda_{\mathsf{c}} \in \sum_{i\in I_0\setminus I} \mathbb{C}\varpi_i$ we denote by $\mathbf{D}^{\mathrm{b}}_{\lambda_{\mathsf{c}}}(\mathbb{C}_{X_I^{\pm}})$ the bounded derived category of λ_{c}-twisted sheaves on $X_I^{\pm}$, and we consider the operations of inverse image, direct image, tensor product etc. in the framework of twisted sheaves. Any $(\mu_{\mathsf{c}}, \nu_{\mathsf{c}})$-twisted sheaf on $X_I^+ \times X_I^-$ induces a $(\mu_{\mathsf{c}} - \nu_{\mathsf{c}})$-twisted sheaf on $\Omega'_I \simeq G/L_I$ via the diagonal embedding $\mathfrak{l}_I \to \mathfrak{p}_I^+ \times \mathfrak{p}_I^-$.

Let us fix $\lambda_{\mathsf{c}} \in \sum_{i\in I_0\setminus I} \mathbb{C}\varpi_i$, and denote by $\Omega'_I \xrightarrow{j} X_I^+ \times X_I^-$ the open embedding and by $k_{\Omega'_I}$ the constant sheaf on Ω'_I with fiber $\mathbb{C}$. To $j_! k_{\Omega'_I}$ we can give a structure of $(\lambda_{\mathsf{c}}, \lambda_{\mathsf{c}})$-twisted sheaf on $X_I^+ \times X_I^-$, and we denote it by $\mathbb{C}_{\Omega'_I,\lambda_{\mathsf{c}}}$; the dual $\mathbb{C}^*_{\Omega'_I,\lambda_{\mathsf{c}}} = R\mathcal{H}om(\mathbb{C}_{\Omega'_I,\lambda_{\mathsf{c}}}, \mathbb{C}_{X_I^+\times X_I^-})$ is in degree zero and has a structure of $(-\lambda_{\mathsf{c}}, -\lambda_{\mathsf{c}})$-twisted sheaf on $X_I^+ \times X_I^-$. By using $\mathbb{C}_{\Omega'_I,\lambda_{\mathsf{c}}}$ and $\mathbb{C}^*_{\Omega'_I,\lambda_{\mathsf{c}}}$ we can define "integral transforms" (see [1]) for twisted sheaves between X_I^+ and X_I^- by the following functors, where $X_I^+ \xleftarrow{q_1} X_I^+ \times X_I^- \xrightarrow{q_2} X_I^-$ are the projections:

$$\mathbf{D}^{\mathrm{b}}_{-\lambda_{\mathsf{c}}}(\mathbb{C}_{X_I^+}) \underset{\mathbb{C}_{\Omega'_I,-\lambda_{\mathsf{c}}}\circ\,\cdot}{\overset{\cdot\,\circ\mathbb{C}^*_{\Omega'_I,-\lambda_{\mathsf{c}}}}{\rightleftarrows}} \mathbf{D}^{\mathrm{b}}_{\lambda_{\mathsf{c}}}(\mathbb{C}_{X_I^-})\,, \qquad \begin{cases} \cdot \circ \mathbb{C}^*_{\Omega'_I,-\lambda_{\mathsf{c}}} = Rq_{2!}(q_1^{-1}(\,\cdot\,) \otimes^L \mathbb{C}^*_{\Omega'_I,-\lambda_{\mathsf{c}}}) \\ \mathbb{C}_{\Omega'_I,-\lambda_{\mathsf{c}}} \circ \cdot = Rq_{1!}(\mathbb{C}^*_{\Omega'_I,-\lambda_{\mathsf{c}}} \otimes^L q_2^{-1}(\,\cdot\,)). \end{cases} \tag{2.1}$$

Let $\pi_\pm : G \to X_I^{\pm}$ be the canonical projections. Given $\mu \in \mathfrak{t}_I^*$, for any open $U \subset X_I^{\pm}$ we define

$$\mathcal{O}_{X_I^{\pm}}(\mu)(U) = \{\varphi \in \Gamma(\pi_\pm^{-1}(U), \mathcal{O}_G \otimes V(\mu)) : \tfrac{d}{dt}\varphi(ge^{tA})\big|_{t=0} = \mp A \cdot \varphi(g) \quad \forall\, g \in G,\, A \in \mathfrak{l}_I\}. \tag{2.2}$$

If μ is integral, $V(\mu)$ can be exponentiated to $P_I^{\pm}$: in this case, $\mathcal{O}_{X_I^{\pm}}(\mu)$ is identified to the sheaf of regular sections of the vector bundle on $X_I^{\pm}$ associated to $V(\mu)$. More generally, if $\mu = \mu_{\mathsf{c}} + \mu_{\mathsf{d}}$ with $\mu_{\mathsf{c}} \in \sum_{i \in I_0 \setminus I} \mathbb{C}\varpi_i$ and $\mu_{\mathsf{d}} \in \sum_{i \in I_0} \mathbb{Z}_{\geq 0}\varpi_i$, then $\mathcal{O}_{X_I^{\pm}}(\mu)$ has a structure μ_{c}-twisted sheaf on $X_I^{\pm}$. The sheaf of rings $\mathcal{D}_{X_I^{\pm},\lambda_{\mathsf{c}}} = \{P \in \mathcal{E}nd_{\mathrm{Mod}_{\lambda_{\mathsf{c}}}(\mathbb{C}_{X_I^{\pm}})}(\mathcal{O}_{X_I^{\pm}}(\lambda_{\mathsf{c}})) : \ P \text{ is locally a differential operator}\}$ is a sheaf of twisted differential operators on $X_I^{\pm}$; let $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_{X_I^{\pm},\lambda_{\mathsf{c}}})$ the bounded derived category of left $\mathcal{D}_{X_I^{\pm},\lambda_{\mathsf{c}}}$-modules. We consider the operations of inverse image $D(\,\cdot\,)^*$, proper direct image $D(\,\cdot\,)_!$, tensor product $\overset{\mathrm{D}}{\otimes}$, external tensor product $\boxtimes^D$ in the framework of twisted $\mathcal{D}$-modules: in particular, we define a sheaf of twisted differential operators on $X_I^+ \times X_I^-$ by $\mathcal{D}_{X_I^+ \times X_I^-,\lambda_{\mathsf{c}}} := \mathcal{D}_{X_I^+,\lambda_{\mathsf{c}}} \boxtimes^D \mathcal{D}_{X_I^-,\lambda_{\mathsf{c}}}$. By using the Riemann-Hilbert correspondence (see [4]), from the twisted sheaf $\mathbb{C}_{\Omega'_I,\lambda_{\mathsf{c}}}$ (resp. $\mathbb{C}^*_{\Omega'_I,\lambda_{\mathsf{c}}}$) we get a regular holonomic $\mathcal{D}_{X_I^+ \times X_I^-,\lambda_{\mathsf{c}}}$-module $\mathcal{B}_{\Omega'_I,\lambda_{\mathsf{c}}}$ (resp. a regular holonomic $\mathcal{D}_{X_I^+ \times X_I^-,-\lambda_{\mathsf{c}}}$-module $\mathcal{B}^*_{\Omega'_I,\lambda_{\mathsf{c}}}$). In fact, $\mathcal{B}_{\Omega'_I,\lambda_{\mathsf{c}}}$ is the sheaf of meromorphic functions with poles on $(X_I^+ \times X_I^-) \setminus \Omega'_I$, endowed with the structure of module over $\mathcal{D}_{X_I^+ \times X_I^-,\lambda_{\mathsf{c}}}$, and $\mathcal{B}^*_{\Omega'_I,\lambda_{\mathsf{c}}}$ is the dual of $\mathcal{B}_{\Omega'_I,\lambda_{\mathsf{c}}}$ in the sense of left $\mathcal{D}$-modules. As in (2.1), we define functors of "integral transforms" for twisted $\mathcal{D}$-modules:

$$\mathbf{D}^{\mathrm{b}}(\mathcal{D}_{X_I^+,-\lambda_{\mathsf{c}}}) \underset{\mathcal{B}_{\Omega'_I,-\lambda_{\mathsf{c}}} \overset{\mathrm{D}}{\circ}\, \cdot}{\overset{\cdot\, \overset{\mathrm{D}}{\circ} \mathcal{B}^*_{\Omega'_I,-\lambda_{\mathsf{c}}}}{\rightleftarrows}} \mathbf{D}^{\mathrm{b}}(\mathcal{D}_{X_I^-,\lambda_{\mathsf{c}}}) \,, \qquad \begin{cases} \cdot\, \overset{\mathrm{D}}{\circ} \mathcal{B}^*_{\Omega'_I,-\lambda_{\mathsf{c}}} = Dq_{2!}(Dq_1{}^*(\,\cdot\,) \overset{\mathrm{D}}{\otimes} \mathcal{B}^*_{\Omega'_I,-\lambda_{\mathsf{c}}}) \\ \mathcal{B}_{\Omega'_I,-\lambda_{\mathsf{c}}} \overset{\mathrm{D}}{\circ} \cdot = Dq_{1!}(\mathcal{B}_{\Omega'_I,-\lambda_{\mathsf{c}}} \overset{\mathrm{D}}{\otimes} Dq_2{}^*(\,\cdot\,)). \end{cases} \tag{2.3}$$

The geometric criterion of [6, Section 2] has a straightforward generalization to the twisted case and can be used to prove the following fact.

Proposition 2.1. *The functors* $\cdot \circ \mathbb{C}^*_{\Omega'_I,-\lambda_{\mathsf{c}}}$ *and* $\mathbb{C}_{\Omega'_I,-\lambda_{\mathsf{c}}} \circ \cdot$ (resp. $\cdot\, \overset{\mathrm{D}}{\circ} \mathcal{B}^*_{\Omega'_I,-\lambda_{\mathsf{c}}}$ and $\mathcal{B}_{\Omega'_I,-\lambda_{\mathsf{c}}} \overset{\mathrm{D}}{\circ} \cdot$) *are quasi-inverse to each other, and thus they are equivalences between* $\mathbf{D}^{\mathrm{b}}_{-\lambda_{\mathsf{c}}}(\mathbb{C}_{X_I^+})$ *and* $\mathbf{D}^{\mathrm{b}}_{\lambda_{\mathsf{c}}}(\mathbb{C}_{X_I^-})$ (resp. $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_{X_I^+,-\lambda_{\mathsf{c}}})$ and $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_{X_I^-,\lambda_{\mathsf{c}}})$).

3 Local b-functions and irreducibility of generalized Verma modules

If $\mu \in \mathfrak{t}_I^*$ is integral, the sheaf

$$\mathcal{D}_{X_I^{\pm},\lambda_{\mathsf{c}}}(\mu) := \mathcal{D}_{X_I^{\pm},\lambda_{\mathsf{c}}} \otimes_{\mathcal{O}_{X_I^{\pm}}} \mathcal{O}_{X_I^{\pm}}(\mu)$$

has a natural structure of quasi-G-equivariant left $\mathcal{D}_{X_I^{\pm},\lambda_{\mathsf{c}}}$-module (see [5]); in particular, we can apply the functor $\mathcal{B}_{\Omega'_I,-\lambda_{\mathsf{c}}} \overset{\mathrm{D}}{\circ} \cdot$ to $\mathcal{D}_{X_I^-,\lambda_{\mathsf{c}}}(\mu)$. We can state our main result.

Theorem 3.1. (i) *There exists a unique* $(P_I^- \times P_I^+)$*-orbit* $O_I \subset G$ *such that* $\Lambda_{O_I} \subset \Sigma_I$ *and the natural projections* $\Lambda_{O'_I} \to T^*X_I^{\pm}$ *are generically surjective. For such an orbit,* Λ_{O_I} *is a good Lagrangian and the maps* $\Lambda_{O'_I} \to T^*X_I^{\pm}$ *are birational.*

(ii) *For* $\lambda_{\mathsf{c}} \in \sum_{i \in I_0 \setminus I} \mathbb{C}\varpi_i$ *the following propositions are equivalent:*

(1) $M_I^+(\lambda_{\mathsf{c}})$ *is an irreducible* $\mathfrak{g}$*-module;*

(2) $b_{I,\Lambda_{O_I},i}(\lambda_{\mathsf{c}} - \varpi_i - \nu) \neq 0$ *for any* $i \in I_0 \setminus I$ *and* $\nu \in \sum_{i \in I_0 \setminus I} \mathbb{Z}_{\geq 0}\varpi_i$*;*

(3) $\mathcal{D}_{X_I^+,-\lambda_{\mathsf{c}}} \xrightarrow{\sim} \mathcal{B}_{\Omega'_I,-\lambda_{\mathsf{c}}} \overset{D}{\circ} \mathcal{D}_{X_I^-,\lambda_{\mathsf{c}}}(-2\rho_I)$ *in* $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_{X_I^+,-\lambda_{\mathsf{c}}})$.

Let us sketch the proof. Part (i) follows from Proposition 2.1. In Part (ii), we prove that (1) and (2) are equivalent to (3). As for (1), there is an equivalence of categories (see [5]) between quasi-G-equivariant $\mathcal{D}_{X_I^+,-\lambda_{\mathsf{c}}}$-modules and $-\lambda_{\mathsf{c}}$-twisted $(\mathfrak{g}, P_I^+)$-modules, which associates $\mathcal{M} = \mathcal{D}_{X_I^+,-\lambda_{\mathsf{c}}}$ to

$M = M_I^+(\lambda_c)$ and $\mathcal{M}' = \mathcal{B}_{\Omega'_I,-\lambda_c} \overset{D}{\circ} \mathcal{D}_{X_I^-,\lambda_c}(-2\rho_I)$ to $M' = M_I^-(-w_I\lambda_c)^*$ (here $(\cdot)^*$ denotes duality). Now, there exists a canonical nontrivial morphism of $(\mathfrak{g}, P_I^+)$-modules $\phi : M \to M'$ (and hence a nontrivial morphism $\Phi : \mathcal{M} \to \mathcal{M}'$) which is an isomorphism if and only if M is irreducible. The statement (2) is equivalent to the fact that the canonical global section f^{λ_c} of $\mathcal{B}_{\Omega'_I,-\lambda_c}$ is a microlocal generator of $\mathcal{B}_{\Omega'_I,-\lambda_c}$ on $\Lambda_{O'_I}$, and global sections of $\mathcal{B}_{\Omega'_I,-\lambda_c}$ correspond (see [1]) to morphisms $\mathcal{M} \to \mathcal{M}'$; in particular f^{λ_c} corresponds to Φ. Then the proof goes as in [6, Section 3.4].

Remarks 3.2.

(1) Unlike Example 3.3 and several other particular cases, the "privileged" orbit O_I is not always the closed $(P_I^- \times P_I^+)$-orbit in G.

(2) Part (ii)(3) can be viewed as a quantization of the contact transformation between $T^*X_I^+$ and $T^*X_I^-$ induced by $\Lambda_{O'_I}$ (see also [1]).

(3) We do not know whether, in general, the ideals $\mathcal{B}_{I,i}$ $(i \in I_0 \setminus I)$ are principal or not, and in particular whether $b_{I,\Lambda_{O_I},i}(\underline{s}) \in \mathcal{B}_{I,i}$ or not (see Conjecture 1.1).

(4) We have a partial "higher rank version" of Part (ii) (see also [2, p. 398]): for $\lambda = \lambda_c + \lambda_d \in \mathfrak{t}_I^*$ with $\lambda_c \in \sum_{i \in I_0 \setminus I} \mathbb{C}\varpi_i$ and $\lambda_d \in \sum_{i \in I_0} \mathbb{Z}_{\geq 0}\varpi_i$, the properties (1)' $M_I^+(\lambda)$ *is an irreducible* $\mathfrak{g}$*-module*, and (3)' $\mathcal{D}_{X_I^+,-\lambda_c}(\lambda_d) \xrightarrow{\sim} \mathcal{B}_{\Omega'_I,-\lambda_c} \overset{D}{\circ} \mathcal{D}_{X_I^-,\lambda_c}(-w_I\lambda_d - 2\rho_I)$ *in* $\mathbf{D}^b(\mathcal{D}_{X_I^+,-\lambda_c})$ are equivalent. However, we do not have a precise statement for (2)' yet.

Example 3.3. (See [6]) Let $G = SL(n+1, \mathbb{C})$. Using the standard notations of Bourbaki, let $p \in I_0 = \{1, \ldots, n\}$ and set $I = I_p = I_0 \setminus \{p\}$. In this case the b-function, which coincides with the local b-function on the conormal bundle to the closed orbit, is $b_{I_p}(s) = \prod_{j=1}^{p}(s+j)$. Let $\lambda = \lambda_c = r\varpi_p$, where $r \in \mathbb{C}$. Then the above conditions are satisfied if and only if $r \notin \mathbb{Z}_{\geq -(p-1)}$.

Acknowledgments. This work has been prepared during a long-term stay at the Research Institute for Mathematical Sciences of Kyoto University supported by a JSPS fellowship. We are grateful to Masaki Kashiwara for his invaluable help.

References

[1] A. D'Agnolo and P. Schapira, "Leray's quantization of Projective Duality" Duke Math. Journal 84, no. 2 (1996), 453–496.

[2] A. Gyoja, "Highest weight modules and b-functions of semi-invariants". Publications R.I.M.S., Kyoto University 30, no. 3 (1994), 353–400.

[3] M. Kashiwara, "b-functions and holonomic systems". Invent. Math. 38 (1976/77), 33–53.

[4] M. Kashiwara, "The Riemann-Hilbert problem for holonomic systems". Publications R.I.M.S., Kyoto University 20, no. 2 (1984), 319–365.

[5] M. Kashiwara, "Representation theory and $\mathcal{D}$-modules on flag varieties". Astérisque S.M.F. 173-174 (1989), 55–109.

[6] C. Marastoni, "Grassmann duality for $\mathcal{D}$-modules". Ann. Sci. École Norm. Sup. 31 (1998), 459–491.

[7] C. Sabbah, "Proximité évanescente II". Compositio Math. 64 (1987), 213–241.

[8] M. Sato, M. Kashiwara, T. Kimura and T. Oshima, "Microlocal analysis of prehomogeneous vector spaces". Invent. Math. 62 (1980), 117–179.