Grassmann duality for $\mathcal{D}$-modules

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Abstract – We generalize the main results on projective duality (see [2], [4], [12]) to the case of the correspondence between "dual" Grassmann manifolds $\mathbb{G}$ and $\mathbb{G}^*$. The new aspect is that the "incidence variety" $S \subset \mathbb{G} \times \mathbb{G}^*$ is no longer smooth, a fact which requires the tools of the theory of b-functions ([7], [17]). In particular, we obtain an equivalence between the categories of sheaves on $\mathbb{G}$ and $\mathbb{G}^*$, as well as between those of $\mathcal{D}$-modules; then, quantizing this equivalence, we explicitly calculate the transform of a $\mathcal{D}$-module associated to a holomorphic line bundle.

Résumé – Nous généralisons les résultats principaux sur la dualité projective (voir [2], [4], [12]) au cas d'une correspondance entre variétés de Grassmann "duales" $\mathbb{G}$ et $\mathbb{G}^*$. Le nouvel aspect est que la "variété d'incidence" $S \subset \mathbb{G} \times \mathbb{G}^*$ n'est plus lisse, ce qui demande de faire appel à la théorie des b-fonctions ([7], [17]). En particulier, nous obtenons une équivalence entre les catégories des faisceaux sur $\mathbb{G}$ et $\mathbb{G}^*$, ainsi que entre celles des $\mathcal{D}$-modules; ensuite, en quantifiant cette équivalence, nous calculons explicitement la transformée d'un $\mathcal{D}$-module associé à un fibré holomorphe en droites.

Contents

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Introduction

The aim of this work is to extend some properties of the *projective duality*, i.e. the natural correspondence between a complex projective space and its dual (see [2], [4]), to the more general setting of "dual" complex Grassmann manifolds $\mathbb{G}$ and $\mathbb{G}^*$. The new aspect in the general case is the non-smoothness of the "incidence variety" in $\mathbb{G} \times \mathbb{G}^*$: an essential tool in treating these singularities will be the theory of b-functions (see ([7], [17]).

Let V be a complex vector space of dimension $n \geq 2$, and $p \in \mathbb{Z}$ such that $1 \leq p \leq \frac{n}{2}$. We denote by $\mathbb{G}$ the Grassmann manifold of p-dimensional linear subspaces of V and by $\mathbb{G}^*$ the dual manifold of $(n-p)$-subspaces; recall that $\mathbb{G}$ and $\mathbb{G}^*$ are complex analytic compact manifolds of complex dimension $N = p(n-p)$, homogeneous under the action of $G = SL(V)$.

Integral transforms. Let q_1 and q_2 be the projections from $\mathbb{G} \times \mathbb{G}^*$ onto $\mathbb{G}$ and $\mathbb{G}^*$. Any object K of the derived category $\mathbf{D}^{\mathrm{b}}(\mathbb{C}_{\mathbb{G}\times\mathbb{G}^*})$ of complexes of sheaves on $\mathbb{G} \times \mathbb{G}^*$ with bounded cohomology is the *kernel* of a *sheaf integral transform*

$$\cdot \circ K : \mathbf{D}^{\mathrm{b}}(\mathbb{C}_{\mathbb{G}}) \to \mathbf{D}^{\mathrm{b}}(\mathbb{C}_{\mathbb{G}^*}), \qquad F \circ K = Rq_{2!}(q_1^{-1} F \otimes K).$$

Similarly, for any object $\mathcal{K}$ of the derived category $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_{\mathbb{G}\times\mathbb{G}^*})$ of complexes of left $\mathcal{D}$-modules on $\mathbb{G} \times \mathbb{G}^*$ with bounded cohomology, one defines a $\mathcal{D}$*-module integral transform* of kernel $\mathcal{K}$

$$\cdot \underline{\circ}\, \mathcal{K} : \mathbf{D}^{\mathrm{b}}(\mathcal{D}_{\mathbb{G}}) \to \mathbf{D}^{\mathrm{b}}(\mathcal{D}_{\mathbb{G}^*}), \qquad \mathcal{M} \,\underline{\circ}\, \mathcal{K} = \underline{q_2}_*(\underline{q_1}^{-1}\mathcal{M} \otimes^L_{\mathcal{O}_{\mathbb{G}\times\mathbb{G}^*}} \mathcal{K}),$$

where $\underline{q_1}^{-1}$ and $\underline{q_2}_*$ are the inverse and direct image in the sense of $\mathcal{D}$-modules. The integral transforms from $\mathbb{G}^*$ to $\mathbb{G}$ are defined in a similar way.

Kernels associated to the transversality relation. Let

$$\Omega = \{(x, y) \in \mathbb{G} \times \mathbb{G}^* : x \cap y = \{0\}\},$$

$j : \Omega \hookrightarrow \mathbb{G} \times \mathbb{G}^*$ the open embedding and S the complex hypersurface complementary to Ω; note that Ω (resp. S) is the open "transversality" (resp. closed "incidence") relation in $\mathbb{G} \times \mathbb{G}^*$. In the case of projective duality ($p = 1$), the hypersurface S is smooth. This is no longer true in the general case, where S admits a Whitney stratification by the locally closed smooth submanifolds of $\mathbb{G} \times \mathbb{G}^*$

$$S_j = \{(x, y) \in \mathbb{G} \times \mathbb{G}^* : \dim(x \cap y) = j\} \qquad (j = 1, \ldots, p).$$

Let us introduce the perverse sheaves

$$K_\Omega = \mathbb{C}_\Omega = j_! j^{-1} \mathbb{C}_{\mathbb{G}\times\mathbb{G}^*} \quad \text{and} \quad K^*_\Omega = R\mathcal{H}om\,(\mathbb{C}_\Omega, \mathbb{C}_{\mathbb{G}\times\mathbb{G}^*}) \simeq Rj_* j^{-1} \mathbb{C}_{\mathbb{G}\times\mathbb{G}^*}.$$

Using the functor $\mathcal{T}hom$ of Kashiwara (see [11]), we may consider the regular holonomic $\mathcal{D}$-modules

$$\mathcal{K}_\Omega = \mathcal{T}hom(K_\Omega, \mathcal{O}_{\mathbb{G}\times\mathbb{G}^*}) \quad \text{and} \quad \mathcal{K}^*_\Omega = \mathcal{T}hom(K^*_\Omega, \mathbb{C}_{\mathbb{G}\times\mathbb{G}^*}).$$

Observe that $\mathcal{K}_\Omega$ is isomorphic to the sheaf $\mathcal{O}_{\mathbb{G}\times\mathbb{G}^*}(*S)$ of meromorphic functions on $\mathbb{G}\times\mathbb{G}^*$ with poles on S, and that $\mathcal{K}^*_\Omega$ is its dual in the sense of $\mathcal{D}$-modules.

Equivalences of categories. Using a geometric criterion, we show that the kernels K_Ω and K^*_Ω (and, through $\mathcal{T}hom$, also $\mathcal{K}_\Omega$ and $\mathcal{K}^*_\Omega$) are "inverse" to each other:

Theorem 1. *The sheaf (resp. $\mathcal{D}$-module) integral transforms defined by the kernels K_Ω and $K^*_{\widetilde{\Omega}}$ (resp. $\mathcal{K}_\Omega$ and $\mathcal{K}^*_{\widetilde{\Omega}}$) are quasi-inverse to each other, and thus define equivalences of categories between $\mathbf{D}^{\mathrm{b}}(\mathbb{C}_{\mathbb{G}})$ and $\mathbf{D}^{\mathrm{b}}(\mathbb{C}_{\mathbb{G}^*})$ (resp. $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_{\mathbb{G}})$ and $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_{\mathbb{G}^*})$), as well as between the full subcategories of objects with $\mathbb{R}$- and $\mathbb{C}$-constructible (resp. good coherent and regular holonomic) cohomology.*

Quantization. Let us study the action of these functors on the family of $\mathcal{D}$-modules associated to holomorphic line bundles. The family of holomorphic line bundles on $\mathbb{G}$ is described (up to isomorphism) by $\{\mathcal{O}_{\mathbb{G}}(\mu) : \mu\in\mathbb{Z}\}$, where $\mathcal{O}_{\mathbb{G}}(\mu)$ is the $-\mu$th tensor power of the determinant of the tautological vector bundle on $\mathbb{G}$ (in particular, the canonical bundle $\Omega_{\mathbb{G}}$ is isomorphic to $\mathcal{O}_{\mathbb{G}}(-n)$). Hence one has the family of locally free $\mathcal{D}$-modules of rank one

$$\{\mathcal{D}_{\mathbb{G}}(\mu) = \mathcal{D}_{\mathbb{G}}\otimes_{\mathcal{O}_{\mathbb{G}}}\mathcal{O}_{\mathbb{G}}(\mu) : \mu\in\mathbb{Z}\}.$$

Let $\lambda\in\mathbb{Z}$, and set $\lambda^* = -n-\lambda$. Following an approach proposed in [4], we will show that *the image of $\mathcal{D}_{\mathbb{G}}(-\lambda)$ by the functors $\cdot\underline{\circ}\,\mathcal{K}_\Omega$ or $\cdot\underline{\circ}\,\mathcal{K}^*_{\widetilde{\Omega}}$* (according to λ) *is isomorphic to $\mathcal{D}_{\mathbb{G}^*}(-\lambda^*)$.* In this direction, we observe that:

(1) The natural isomorphism

$$\alpha_{\mathcal{K}_\Omega} : \Gamma(\mathbb{G}\times\mathbb{G}^*;\mathcal{K}^{(N,0)}_\Omega(-\lambda,\lambda^*)) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\mathcal{D}_{\mathbb{G}^*})}(\mathcal{D}_{\mathbb{G}^*}(-\lambda^*),\mathcal{D}_{\mathbb{G}}(-\lambda)\,\underline{\circ}\,\mathcal{K}_\Omega),$$

where $\mathcal{K}^{(N,0)}_\Omega(-\lambda,\lambda^*) = q_1^{-1}(\mathcal{O}_{\mathbb{G}}(-\lambda)\otimes_{\mathcal{O}_{\mathbb{G}}}\Omega_{\mathbb{G}})\otimes_{q_1^{-1}\mathcal{O}_{\mathbb{G}}}\mathcal{K}_\Omega\otimes_{q_2^{-1}\mathcal{O}_{\mathbb{G}^*}}q_2^{-1}\mathcal{O}_{\mathbb{G}^*}(\lambda^*)$, describes the $\mathcal{D}$-linear morphisms between $\mathcal{D}_{\mathbb{G}^*}(\nu)$ and $\mathcal{D}_{\mathbb{G}}(\mu)\,\underline{\circ}\,\mathcal{K}_\Omega$ in terms of twisted global sections of $\mathcal{K}_\Omega$ (one can argue similarly for $\mathcal{K}^*_\Omega$);

(2) the microlocal correspondence associated to Ω

$$T^*\mathbb{G} \xleftarrow{p_1} SS(\mathbb{C}_\Omega) \xrightarrow{p_2^a} T^*\mathbb{G}^*$$

(here the maps p_1 and p_2 are the natural projections, a is the antipodal map on $T^*\mathbb{G}^*$ and $p_2^a = a\circ p_2$) induces a contact transformation between two open dense subsets $U\subset\dot{T}^*\mathbb{G}$ and $U^*\subset\dot{T}^*\mathbb{G}^*$ with an open dense subset $\Lambda\subset T^*_{S_p}(\mathbb{G}\times\mathbb{G}^*)$ as graph.

The group G acts naturally on $\mathbb{G}\times\mathbb{G}^*$ by the diagonal action, and $(G;\mathbb{G}\times\mathbb{G}^*)$ is a prehomogeneous space with open dense orbit Ω (see [17]) with associated b-function $b(s) = (s+1)\cdots(s+p)$. Using this fact, we find a G-invariant section $s_\lambda\in\Gamma(\mathbb{G}\times\mathbb{G}^*;\mathcal{K}^{(N,0)}_\Omega(-\lambda,\lambda^*))$ (resp. $s^*_\lambda\in\Gamma(\mathbb{G}\times\mathbb{G}^*;\mathcal{K}^{*\,(N,0)}_\Omega(-\lambda,\lambda^*))$) which generates $\mathcal{K}_\Omega$ (resp. $\mathcal{K}^*_\Omega$) microlocally (i.e. as microdifferential module) on Λ for any $\lambda \geq -n+p$ (resp. $\lambda \in \mathbb{Z}$). Then, by Theorem 1 and the theory of [16], we prove that $\alpha_{\mathcal{K}_\Omega}(s_\lambda)$ (resp. $\alpha_{\mathcal{K}^*_\Omega}(s^*_\lambda)$) is an isomorphism for any $\lambda \geq -n+p$ (resp. $\lambda \leq -p$), the inverse morphism being the image of $\alpha_{\mathcal{K}^*_{\tilde{\Omega}}}(\tilde{s}^*_{\lambda^*})$ (resp. $\alpha_{\mathcal{K}_{\tilde{\Omega}}}(\tilde{s}_{\lambda^*})$) by the functor $\cdot \underline{\circ}\, \mathcal{K}_\Omega$ (resp. $\cdot \underline{\circ}\, \mathcal{K}^*_\Omega$), and therefore we obtain:

Theorem 2. *One has $\mathcal{D}$-linear isomorphisms:*

(i) $\mathcal{D}_{\mathbb{G}}(-\lambda) \underline{\circ}\, \mathcal{K}_\Omega \xleftarrow{\sim} \mathcal{D}_{\mathbb{G}^*}(-\lambda^*)$ *for any* $\lambda \geq -n+p$*;*

(ii) $\mathcal{D}_{\mathbb{G}}(-\lambda) \underline{\circ}\, \mathcal{K}^*_\Omega \xleftarrow{\sim} \mathcal{D}_{\mathbb{G}^*}(-\lambda^*)$ *for any* $\lambda \leq -p$*.*

Some applications. In the case of projective duality ($p=1$), let $\mathbb{P} = \mathbb{P}^{n-1}(\mathbb{C})$, $\mathbb{P}^* = \mathbb{P}^{n-1}(\mathbb{C})^*$ and $\mathbb{A} \subset \mathbb{P} \times \mathbb{P}^*$ the smooth incidence relation. Using the kernel

$$\mathcal{K}_{\mathbb{A}} = \mathcal{T}hom(\mathbb{C}_{\mathbb{A}}[-1], \mathcal{O}_{\mathbb{P}\times\mathbb{P}^*}) \simeq \mathcal{B}_{\mathbb{A}|\mathbb{P}\times\mathbb{P}^*},$$

D'Agnolo and Schapira proved in [4] that $\mathcal{D}_{\mathbb{P}}(-\lambda) \underline{\circ}\, \mathcal{K}_{\mathbb{A}} \xleftarrow{\sim} \mathcal{D}_{\mathbb{P}^*}(-\lambda^*)$ for any $-n+1 \leq \lambda \leq -1$. From Theorem 2, we obtain the following generalization with the kernel $\mathcal{K}_S = \mathcal{T}hom(\mathbb{C}_S[-1], \mathcal{O}_{\mathbb{G}\times\mathbb{G}^*})$:

$$\mathcal{D}_{\mathbb{G}}(-\lambda) \underline{\circ}\, \mathcal{K}_S \xleftarrow{\sim} \mathcal{D}_{\mathbb{G}^*}(-\lambda^*) \qquad \text{for any } -n+p \leq \lambda \leq -1.$$

Moreover, using the adjunction formulas of [3] and [11], we get the following isomorphisms for any $F \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_{\mathbb{G}})$ and $-n+p \leq \lambda \leq -p$:

$$\begin{aligned} \mathrm{R}\Gamma(\mathbb{G}; F \otimes \mathcal{O}_{\mathbb{G}}(\lambda)) &\simeq \mathrm{R}\Gamma(\mathbb{G}^*; (F \circ \mathbb{C}_\Omega) \otimes \mathcal{O}_{\mathbb{G}^*}(\lambda^*))[N], \\ \mathrm{R}\Gamma(\mathbb{G}; R\mathcal{H}om\,(F, \mathcal{O}_{\mathbb{G}}(\lambda))) &\simeq \mathrm{R}\Gamma(\mathbb{G}^*; R\mathcal{H}om\,(F \circ \mathbb{C}_\Omega, \mathcal{O}_{\mathbb{G}^*}(\lambda^*)))[-N], \end{aligned}$$

as well as similar isomorphisms when $F \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_{\mathbb{G}})$ with $\otimes$ and $R\mathcal{H}om$ replaced respectively by the functors $\overset{\mathrm{w}}{\otimes}$ and $\mathcal{T}hom$ (see [11]). Here, the calculation of the transform $F \circ \mathbb{C}_\Omega$ is essentially a geometrical problem. For example, let $F = \mathbb{C}_D$ for some $D \subset \mathbb{G}$: then, for any $y \in \mathbb{G}^*$ one has $(\mathbb{C}_D \circ \mathbb{C}_\Omega)_y \simeq \mathrm{R}\Gamma_c(L_D(y); \mathbb{C})$, where $L_D(y) = \{x \in D : x \cap y = \{0\}\}$. We give the following examples.

(1) Let $D \neq \varnothing$ be a compact subset of $\mathbb{G}$, and set

$$D^{\#} = \{y \in \mathbb{G}^* : x \cap y = \{0\} \text{ for any } x \in D\}.$$

Let $\widehat{D} = \mathbb{G}^* \setminus D^{\#}$. We say (cf. [4, Ch. 5.1]) that D is Ω-*trivial* if (a) $\mathrm{R}\Gamma(D; \mathbb{C}) \simeq \mathbb{C}$ and (b) $\mathrm{R}\Gamma(D \setminus L_D(y); \mathbb{C}) \simeq \mathbb{C}$ for any $y \in \widehat{D}$. (E.g. take $D = \{x_0\}$ for some $x_0 \in \mathbb{G}$.) Let D be Ω-trivial, and let $D^{\#} \neq \varnothing$: then D (resp. $D^{\#}$) is contained in an affine chart $E \subset \mathbb{G}$ (resp. $E^* \subset \mathbb{G}^*$). Since $\mathbb{C}_D \circ \mathbb{C}_\Omega \simeq \mathbb{C}_{D^{\#}}$, we get the following isomorphisms:

$$\begin{aligned} \mathrm{R}\Gamma(D; \mathcal{O}_E) &\simeq \mathrm{R}\Gamma_c(D^{\#}; \mathcal{O}_{E^*})[N] \\ \mathrm{R}\Gamma_D(E; \mathcal{O}_E)[N] &\simeq \mathrm{R}\Gamma(D^{\#}; \mathcal{O}_{E^*}), \end{aligned}$$

where all complexes are concentrated in degree zero. This generalizes the results of Martineau [14] (recovered in this language in [4]) on the "linearly convex" compact subsets of the complex projective space.

(2) Let H be an Hermitian form of signature $(p, n-p)$ on V, and let $U = \{x \in \mathbb{G} : H|_x \text{ is positive definite}\}$ and $U^* = \{y \in \mathbb{G}^* : H|_y \text{ is negative definite}\}$; then U (resp. U^*) is a relatively compact open subset of an affine chart $E \subset \mathbb{G}$ (resp. $E^* \subset \mathbb{G}^*$). We prove that $\mathbb{C}_{\overline{U}} \circ \mathbb{C}_\Omega \simeq \mathbb{C}_{U^*}$, and hence we get

$$\begin{aligned} \mathrm{R}\Gamma(\overline{U}; \mathcal{O}_E) &\simeq \mathrm{R}\Gamma_c(U^*; \mathcal{O}_{E^*})[N] \\ \mathrm{R}\Gamma_{\overline{U}}(E; \mathcal{O}_E)[N] &\simeq \mathrm{R}\Gamma(U^*; \mathcal{O}_{E^*}). \end{aligned}$$

Moreover, all these complexes are concentrated in degree zero.

(3) Finally, we give a "non-affine" example. Let z be an hyperplane of V, and consider the embedded Grassmann manifolds $\mathbb{G}_z = \{x \in \mathbb{G} : x \subset z\}$ and $\mathbb{G}^*_z = \{y \in \mathbb{G}^* : y \subset z\}$. We show that $\mathbb{C}_{\mathbb{G}_z} \circ \mathbb{C}_\Omega \simeq \mathbb{C}_{\mathbb{G}^* \setminus \mathbb{G}^*_z}[-2(N-p)]$ and then we get the following isomorphisms for any $-n+p \leq \lambda \leq -p$:

$$\begin{aligned} \mathrm{R}\Gamma(\mathbb{G}_z; \mathcal{O}_{\mathbb{G}}(\lambda)) &\simeq \mathrm{R}\Gamma(\mathbb{G}^*_z; \mathcal{O}_{\mathbb{G}^*}(\lambda^*))[-(N-2p+1)] \\ \mathrm{R}\Gamma_{\mathbb{G}_z}(\mathbb{G}; \mathcal{O}_{\mathbb{G}}(\lambda)) &\simeq \mathrm{R}\Gamma_{\mathbb{G}^*_z}(\mathbb{G}^*, \mathcal{O}_{\mathbb{G}^*}(\lambda^*))[N-2p+1]. \end{aligned}$$

Comments. Let us recall the main results in the case of projective duality, where the classical point of view was to consider the natural geometric correspondence between $\mathbb{P} = \mathbb{P}^n(\mathbb{C})$ and $\mathbb{P}^* = \mathbb{P}^n(\mathbb{C})^*$ given by the smooth hypersurface $S = \mathbb{A} \subset \mathbb{P} \times \mathbb{P}^*$. (In the formalism of kernels, this correspondence is associated to $K_{\mathbb{A}} = \mathbb{C}_{\mathbb{A}}[-1]$ and $\mathcal{K}_{\mathbb{A}} = \mathcal{B}_{\mathbb{A}|\mathbb{P}\times\mathbb{P}^*}$.) Brylinski [2] obtained an equivalence of categories for perverse sheaves on $\mathbb{P}$ and $\mathbb{P}^*$ modulo constant sheaves, as well as for coherent $\mathcal{D}$-modules modulo flat holomorphic connections. D'Agnolo and Schapira [4] quantized the underlying contact transformation using a suitable twisted form due to Leray [13] and proved the isomorphism (no more modulo flat connections) of $\mathcal{D}$-modules recalled above. Finally, Kashiwara and Tanisaki [12] observed that the kernel associated to the open complementary of the incidence relation and its dual give equivalences between some derived categories of sheaves and $\mathcal{D}$-modules on $\mathbb{P}$ and $\mathbb{P}^*$ when $n = 1$.

The main stimulus in doing this work was to understand the ideas and the results of [4] in the more general situation of Grassmann manifolds. The alternative point of view of [12] suggested to consider the kernels associated to the open transversality relation, a remark that was necessary for the study of the general case, and turned out to be useful also for the basic case of projective duality.

In the real case, similar results in the category of sheaves were obtained using the transversality relation in Sato-Kashiwara-Kawai [16] for projective sphere bundles and in Kashiwara-Schapira [10, Ex. III.15] for real Grassmann manifolds.

The results of this paper have been announced in [15].

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1 Reviews on integral transforms for $\mathcal{D}$- and $\mathcal{E}$-modules

1.1 Notations

We refer to [10] for the theory of sheaves in the framework of derived categories, to [6] for the theory of $\mathcal{D}$-modules and to [16] for the theory of $\mathcal{E}$-modules (see also [19] and [18] for detailed expositions).

Geometry. Given two manifolds X and Y, we denote by $r : X \times Y \to Y \times X$ the canonical map $r(x, y) = (y, x)$, and by q_1 and q_2 the projections from $X \times Y$ onto X and Y. If Z is another manifold, we denote by q_{12} (resp. q_{13}, q_{23}) the projection from $X \times Y \times Z$ onto $X \times Y$ (resp. $X \times Z$, $Y \times Z$). We denote by $\delta : X \to X \times X$ the diagonal embedding $\delta(x) = (x, x)$, and we set $\Delta_X = \delta(X)$. Let $\pi : T^*X \to X$ be the cotangent bundle, $(\,\cdot\,)^a$ the antipodal map of T^*X and $T^*_Y X$ the conormal bundle to a smooth submanifold Y of X; in particular, $T^*_X X$ represents the zero-section of T^*X, and we set $\dot{T}^*X = T^*X \setminus T^*_X X$. To a morphism $f : X \to Y$ of real analytic manifolds one associates the morphisms $T^*X \xleftarrow{{}^t f'} X \times_Y T^*Y \xrightarrow{f_\pi} T^*Y$. In local symplectic coordinates $(x; \xi) \in T^*X$ and $(y; \eta) \in T^*Y$, one has $(x; \xi)^a = (x; -\xi)$, ${}^t f'(x, f(x); \eta) = (x; {}^t f'(x)(\eta))$ and $f_\pi(x, f(x); \eta) = (f(x); \eta)$. We denote by p_1 and p_2 the projections from $T^*(X \times Y)$ onto T^*X and T^*Y, and by p_{12} (resp. p_{13}, p_{23}) the projection from $T^*(X \times Y \times Z)$ onto $T^*(X \times Y)$ (resp. $T^*(X \times Z)$, $T^*(Y \times Z)$).

Sheaves. Let X be a locally compact topological space, and let $\mathrm{Mod}(\mathbb{C}_X)$ be the category of sheaves of $\mathbb{C}$-vector spaces on X. For a locally closed subset A of X, we denote by $\mathbb{C}_A$ the sheaf on X whose restriction to A is the constant sheaf with fiber $\mathbb{C}$ and which is zero on $X \setminus A$. We denote by $\mathbf{D}^{\mathrm{b}}(\mathbb{C}_X)$ the derived category of complexes in $\mathrm{Mod}(\mathbb{C}_X)$ with bounded cohomology, and by $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_X)$ (resp. $\mathbf{D}^{\mathrm{b}}_{\mathbb{C}-c}(\mathbb{C}_X)$) the full triangulated subcategory of $\mathbf{D}^{\mathrm{b}}(\mathbb{C}_X)$ whose objects have $\mathbb{R}$- (resp. $\mathbb{C}$-) constructible cohomology groups. We shall consider the full subcategory $\mathrm{Perv}\,(\mathbb{C}_X)$ of perverse sheaves in $\mathbf{D}^{\mathrm{b}}_{\mathbb{C}-c}(\mathbb{C}_X)$. If $F \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_X)$, we denote by $SS(F)$ the microsupport of F, which is a closed conic involutive subset of T^*X. The six classical operations in the derived category of sheaves are $R\mathcal{H}om\,(\,\cdot\,,\,\cdot\,)$, $\,\cdot \otimes \cdot\,$, Rf_*, f^{-1}, $Rf_!$ and $f^!$, where f is a continuous map. The duality functor $R\mathcal{H}om\,(\,\cdot\,, \mathbb{C}_X) : \mathbf{D}^{\mathrm{b}}(\mathbb{C}_X)^{\mathrm{op}} \to \mathbf{D}^{\mathrm{b}}(\mathbb{C}_X)$ is denoted by $D'(\,\cdot\,)$ for short.

$\mathcal{D}$-modules. Let X be a complex analytic manifold. We write $d_X = \dim_{\mathbb{C}}(X)$. We denote by $\mathcal{O}_X$ the sheaf of holomorphic functions, by Ω_X the canonical line bundle and by $\mathcal{D}_X$ the sheaf of holomorphic linear partial differential operators on X. Let $\mathrm{Mod}(\mathcal{D}_X)$ be the category of left $\mathcal{D}_X$-modules, and let $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_X)$ be the thick subcategory consisting of coherent objects. We shall consider in $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_X)$ the full subcategories $\mathrm{Mod}_{\mathrm{good}}(\mathcal{D}_X)$ of good coherent objects (recall that a coherent $\mathcal{D}_X$-module $\mathcal{M}$ is *good* if it admits, in a neighborhood of any compact subset of X, a finite filtration by coherent $\mathcal{D}_X$-submodules $\mathcal{M}_k$ such that each quotient $\mathcal{M}_k/\mathcal{M}_{k-1}$ can be endowed with a good filtration), and $\mathrm{Mod}_{rh}(\mathcal{D}_X)$ of regular holonomic objects.
We denote by $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$ the derived category of complexes in $\mathrm{Mod}(\mathcal{D}_X)$ with bounded cohomology, and by $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$ (resp. $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X)$, $\mathbf{D}^{\mathrm{b}}_{rh}(\mathcal{D}_X)$) the full triangulated subcategory whose objects have coherent (resp. good, resp. regular holonomic) cohomology groups. Recall the operations in the derived category of left $\mathcal{D}$-modules $R\mathcal{H}om_{\mathcal{D}_X}(\,\cdot\,,\,\cdot\,)$, $\cdot\otimes^L{}_{\mathcal{D}_X}\cdot$, $\underline{f}^{-1}$ and $\underline{f}_*$, where $f : X \to Y$ is a morphism of complex analytic manifolds. In particular, if $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$ and $\mathcal{N} \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_Y)$, then

$$\begin{aligned}\underline{f}^{-1}\mathcal{N} &= \mathcal{D}_{X\to Y} \otimes^L{}_{f^{-1}\mathcal{D}_Y} f^{-1}\mathcal{N},\\ \underline{f}_*\mathcal{M} &= Rf_!(\mathcal{D}_{Y\leftarrow X} \otimes^L{}_{\mathcal{D}_X} \mathcal{M})\end{aligned}$$

where $\mathcal{D}_{X\to Y}$ and $\mathcal{D}_{Y\leftarrow X}$ are the transfer bimodules associated to f. The external product is $\mathcal{M} \,\underline{\boxtimes}\, \mathcal{N} = \mathcal{D}_{X\times Y} \otimes_{\mathcal{D}_X\boxtimes\mathcal{D}_Y} (\mathcal{M}\boxtimes\mathcal{N})$. We denote by $\underline{\mathrm{D}}(\cdot)$ the duality functor $R\mathcal{H}om_{\mathcal{D}_X}(\,\cdot\,,\mathcal{K}_X) : \mathbf{D}^{\mathrm{b}}(\mathcal{D}_X) \to \mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$, where $\mathcal{K}_X = \mathcal{D}_X \otimes_{\mathcal{O}_X} \Omega_X^{\otimes-1}[d_X]$ is the dualizing complex for left $\mathcal{D}_X$-modules, and by $\mathcal{S}ol(\,\cdot\,)$ the functor $R\mathcal{H}om_{\mathcal{D}_X}(\,\cdot\,,\mathcal{O}_X) : \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X) \to \mathbf{D}^{\mathrm{b}}(\mathbb{C}_X)^{\mathrm{op}}$ of holomorphic solutions. Moreover, we shall consider the functors

$$\begin{aligned}\cdot \overset{\mathrm{w}}{\otimes} \mathcal{O}_X &: \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_X) \to \mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)\\ \mathcal{T}hom(\,\cdot\,,\mathcal{O}_X) &: \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_X)^{\mathrm{op}} \to \mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)\end{aligned}$$

of formal and moderate holomorphic cohomology, which allow one to treat $\mathcal{C}^\infty$-functions and distributions on a real analytic manifold in a functorial way (see Kashiwara [7] for $TH(\,\cdot\,) = \mathcal{T}hom(\,\cdot\,,\mathcal{O}_X)$ and Kashiwara-Schapira [11] for $\cdot \overset{\mathrm{w}}{\otimes} \mathcal{O}_X$).
Given $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$, we denote by $\mathrm{char}\,(\mathcal{M})$ the characteristic variety of $\mathcal{M}$, which is a closed conic involutive subvariety of T^*X; recall that $\mathrm{char}\,(\mathcal{M}) = SS(\mathcal{S}ol(\mathcal{M}))$. If Y is a closed smooth complex submanifold of X of codimension d, we denote by $\mathcal{B}_{Y|X} = \mathrm{R}\Gamma_{[Y]}(\mathcal{O}_X)[d] \simeq \mathcal{T}hom(\mathbb{C}_Y[-d],\mathcal{O}_X)$ the regular holonomic $\mathcal{D}_X$-module of holomorphic hyperfunctions along Y. We denote $\mathcal{B}_{\Delta_X|X\times X}$ by $\mathcal{B}_{\Delta_X}$ for short.

$\mathcal{E}$-modules. Let $\mathcal{E}_X$ denote the sheaf (on T^*X) of microdifferential operators of finite order on X, $\mathrm{Mod}(\mathcal{E}_X)$ the category of left $\mathcal{E}_X$-modules, and $\mathbf{D}^{\mathrm{b}}(\mathcal{E}_X)$ the derived category of complexes in $\mathrm{Mod}(\mathcal{E}_X)$ with bounded cohomology. Given $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$, we denote by $\mathcal{E}\mathcal{M} = \mathcal{E}_X \otimes_{\pi^{-1}\mathcal{D}_X} \pi^{-1}\mathcal{M} \in \mathbf{D}^{\mathrm{b}}(\mathcal{E}_X)$ the microlocalization of $\mathcal{M}$. (In particular, if Y is a smooth complex submanifold of X we set $\mathcal{C}_{Y|X} = \mathcal{E}\mathcal{B}_{Y|X}$, the sheaf of microfunctions along Y.) We use the same symbol to denote a section s of $\mathcal{M}$ and the image of s by the canonical morphism $\pi^{-1}\mathcal{M} \to \mathcal{E}\mathcal{M}$. The external product of $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}(\mathcal{E}_X)$ and $\mathcal{N} \in \mathbf{D}^{\mathrm{b}}(\mathcal{E}_Y)$ is $\mathcal{M} \underline{\boxtimes}^{\mu} \mathcal{N} = \mathcal{E}_{X\times Y} \otimes_{\mathcal{E}_X \boxtimes \mathcal{E}_Y} (\mathcal{M} \boxtimes \mathcal{N})$. Moreover, given a morphism $f : X \to Y$, one defines the microlocal inverse and direct images as

$$\begin{aligned} \underline{f}^{\mu}\mathcal{N} &= R^t f'_!(\mathcal{E}_{X\to Y} \otimes^L{}_{f_\pi^{-1}\mathcal{E}_Y} f_\pi^{-1}\mathcal{N}), \\ \underline{f}_{\mu}\mathcal{M} &= Rf_{\pi !}(\mathcal{E}_{Y\leftarrow X} \otimes^L{}_{{}^t f'^{-1}\mathcal{E}_X} {}^t f'^{-1}\mathcal{M}), \end{aligned}$$

where $\mathcal{E}_{X\to Y}$ and $\mathcal{E}_{Y\leftarrow X}$ are the microlocal transfer bimodules associated to f.

1.2 Kernels for sheaves and $\mathcal{D}$-modules

We recall the language of integral transforms as treated in D'Agnolo-Schapira [3], [4], [5] and Kashiwara-Schapira [11]. We also recall some results therein that we shall need in the sequel.

In this section, the manifolds are assumed to be complex analytic and compact.

Kernels. Let X, Y and Z be manifolds, and let $K \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_{X\times Y})$, $K' \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_{Y\times Z})$. One defines the *composition* $K \circ K'$ as

$$K \circ K' = Rq_{13!}\left(q_{12}^{-1}K \otimes q_{23}^{-1}K'\right) \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_{X\times Z}).$$

Similarly, let $\mathcal{K} \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_{X\times Y})$ and $\mathcal{K}' \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_{Y\times Z})$. One defines the *composition* $\mathcal{K} \underline{\circ} \mathcal{K}'$ as

$$\mathcal{K} \underline{\circ} \mathcal{K}' = \underline{q_{13}}_*\left(\underline{q_{12}}^{-1}\mathcal{K} \otimes^L{}_{\mathcal{O}_{X\times Y\times Z}} \underline{q_{23}}^{-1}\mathcal{K}'\right) \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_{X\times Z}).$$

Observe that these operations are associative.

Proposition 1.1. *Let X and Y be manifolds.*

(i) *Let $K \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_{X\times Y})$ and $K' \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_{Y\times Z})$, and assume*

$$(SS(K) \times T^*_Z Z) \cap (T^*_X X \times SS(K')) \subset T^*_{X\times Y\times Z}(X \times Y \times Z). \tag{1.1}$$

Then one has $K \circ K' \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_{X\times Z})$. The same result holds replacing "$\mathbb{R}$-c" by "$\mathbb{C}$-c".

(ii) *Let $\mathcal{K} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_{X\times Y})$ and $\mathcal{K}' \in \mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_{Y\times Z})$, and assume*

$$(\mathrm{char}(\mathcal{K}) \times T^*_Z Z) \cap (T^*_X X \times \mathrm{char}(\mathcal{K}')) \subset T^*_{X\times Y\times Z}(X \times Y \times Z). \tag{1.2}$$

Then one has $\mathcal{K} \underline{\circ} \mathcal{K}' \in \mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_{X\times Z})$. Moreover, the same holds if one replaces "good" by "rh".

The functor of moderate cohomology. Let us recall the *Riemann-Hilbert correspondence* in the formulation of Kashiwara [8], which allows one to associate to any $\mathbb{C}$-constructible sheaf a regular holonomic $\mathcal{D}$-module. Let X be a complex analytic manifold, and consider the functors

$$\begin{aligned} \mathcal{T}hom(\,\cdot\,, \mathcal{O}_X) &: \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_X)^{\mathrm{op}} \to \mathbf{D}^{\mathrm{b}}(\mathcal{D}_X), \\ \mathcal{S}ol(\,\cdot\,) &: \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X) \to \mathbf{D}^{\mathrm{b}}(\mathbb{C}_X)^{\mathrm{op}}. \end{aligned}$$

Theorem 1.2. ([8]) *The functors $\mathcal{T}hom(\,\cdot\,,\mathcal{O}_X)$ and $\mathcal{S}ol(\,\cdot\,)$ are quasi-inverse to each other, and define equivalences between $\mathbf{D}^{\mathrm{b}}_{\mathbb{C}-c}(\mathbb{C}_X)^{\mathrm{op}}$ and $\mathbf{D}^{\mathrm{b}}_{rh}(\mathcal{D}_X)$. Moreover, they induce equivalences between the full subcategories* $\mathrm{Perv}\,(\mathbb{C}_X)^{\mathrm{op}}$ *and* $\mathrm{Mod}_{rh}(\mathcal{D}_X)$.

Hence, to $K \in \mathbf{D}^{\mathrm{b}}_{\mathbb{C}-c}(\mathbb{C}_{X\times Y})$ one naturally associates $\mathcal{K} = \mathcal{T}hom(K, \mathcal{O}_{X\times Y}) \in \mathbf{D}^{\mathrm{b}}_{rh}(\mathcal{D}_{X\times Y})$, and one has $K \simeq \mathcal{S}ol(\mathcal{K})$ and $SS(K) = \mathrm{char}\,(\mathcal{K})$.

Some commutation properties. We observe the following commutation properties for the operations introduced above.

Proposition 1.3. ([5]) *Let X and Y be manifolds.*

(i) *Let $K \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_{X\times Y})$ and $K' \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_{Y\times Z})$, and assume* (1.1). *Then one has*

$$D'(K \circ K') \simeq D'K \circ D'K'[2d_Y].$$

(ii) *Let $\mathcal{K} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_{X\times Y})$ and $\mathcal{K}' \in \mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_{Y\times Z})$, and assume* (1.2). *Then one has*

$$\underline{\mathrm{D}}(\mathcal{K} \underline{\circ} \mathcal{K}') \simeq \underline{\mathrm{D}}\mathcal{K} \underline{\circ} \underline{\mathrm{D}}\mathcal{K}'.$$

Proposition 1.4. ([5]) *Let $K \in \mathbf{D}^{\mathrm{b}}_{\mathbb{C}-c}(\mathbb{C}_{X\times Y})$ and $K' \in \mathbf{D}^{\mathrm{b}}_{\mathbb{C}-c}(\mathbb{C}_{Y\times Z})$. Then, there is a natural isomorphism in $\mathbf{D}^{\mathrm{b}}_{rh}(\mathcal{D}_{X\times Z})$:*

$$\mathcal{T}hom(K, \mathcal{O}_{X\times Y}) \underline{\circ}\, \mathcal{T}hom(K', \mathcal{O}_{Y\times Z}) \simeq \mathcal{T}hom(K \circ K', \mathcal{O}_{X\times Z})[-d_Y].$$

Proposition 1.5. ([5]) *Let $F \in \mathbf{D}^{\mathrm{b}}_{\mathbb{C}-c}(\mathbb{C}_X)$. Then, there is a natural isomorphism in $\mathbf{D}^{\mathrm{b}}_{rh}(\mathcal{D}_X)$:*

$$\mathcal{T}hom(D'F, \mathcal{O}_X) \simeq \underline{\mathrm{D}}\mathcal{T}hom(F, \mathcal{O}_X).$$

Integral transforms. Let X and Y be manifolds. By identifying $\{pt\}\times X$ to X and $\{pt\}\times Y$ to Y above, one associates to any $K \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_{X\times Y})$ a functor

$$\cdot \circ K : \mathbf{D}^{\mathrm{b}}(\mathbb{C}_X) \to \mathbf{D}^{\mathrm{b}}(\mathbb{C}_Y), \qquad F \circ K = Rq_{2!}(q_1^{-1}F \otimes K),$$

called the *sheaf integral transform* from X to Y of *kernel* K. Thus, if Z is another manifold and $K' \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_{Y\times Z})$, then $(F \circ K) \circ K' \simeq F \circ (K \circ K')$ in $\mathbf{D}^{\mathrm{b}}(\mathbb{C}_Z)$.

Similarly, one associates to any $\mathcal{K} \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_{X\times Y})$ a functor

$$\cdot \,\underline{\circ}\, \mathcal{K} : \mathbf{D}^{\mathrm{b}}(\mathcal{D}_X) \to \mathbf{D}^{\mathrm{b}}(\mathcal{D}_Y), \qquad \mathcal{M} \,\underline{\circ}\, \mathcal{K} = \underline{q_2}_*(\underline{q_1}^{-1}\mathcal{M} \otimes^L_{\mathcal{O}_{X\times Y}} \mathcal{K})$$

called the *$\mathcal{D}$-module integral transform* from X to Y of *kernel* $\mathcal{K}$; as above, given $\mathcal{K}' \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_{Y\times Z})$, one has $(\mathcal{M} \,\underline{\circ}\, \mathcal{K}) \,\underline{\circ}\, \mathcal{K}' \simeq \mathcal{M} \,\underline{\circ}\, (\mathcal{K} \,\underline{\circ}\, \mathcal{K}')$ in $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_Z)$.

From Proposition 1.1 and a well-known result we get

Corollary 1.6. *Let* $K \in \mathbf{D}^{\mathrm{b}}_{\mathbb{C}-c}(\mathbb{C}_{X\times Y})$ *and* $\mathcal{K} = \mathcal{T}hom(K, \mathcal{O}_{X\times Y}) \in \mathbf{D}^{\mathrm{b}}_{rh}(\mathcal{D}_{X\times Y})$*. Assume:*

$$SS(K) \cap (T^*X \times T^*_Y Y) \subset T^*_{X\times Y}(X \times Y). \tag{1.3}$$

Then:

(i) *if* $F \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_X)$*, then* $F \circ K \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_Y)$ *(and also with "*$\mathbb{R}$*-c" replaced by "*$\mathbb{C}$*-c");*

(ii) *if* $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X)$*, then* $\mathcal{M} \underline{\circ} \mathcal{K} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_Y)$ *(and also with "good" replaced by "rh"), and*

$$\mathcal{S}ol(\mathcal{M} \underline{\circ} \mathcal{K}) \simeq \mathcal{S}ol(\mathcal{M}) \circ K[d_X].$$

Invertible kernels. When $X = Y$, the identity transforms are obtained for $K = \mathbb{C}_{\Delta_X}$ and $\mathcal{K} = \mathcal{T}hom(\mathbb{C}_{\Delta_X}[-d_X], \mathcal{O}_{X\times X}) \simeq \mathcal{B}_{\Delta_X}$. This leads immediately to the following *invertibility criterion*, which we shall apply in the next section:

Proposition 1.7. (see [10, Corollary 3.6.5]) *Let* X *and* Y *be manifolds.*

(i) *Let* $K \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_{X\times Y})$*,* $K' \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_{Y\times X})$ *and assume that* $K \circ K' \simeq \mathbb{C}_{\Delta_X}[l]$ *and* $K' \circ K \simeq \mathbb{C}_{\Delta_Y}[l]$ *for some* $l \in \mathbb{Z}$*. Then, the functors* $\cdot \circ K$ *and* $\cdot \circ K'$ *are quasi-inverse to each other and thus they are equivalences of categories between* $\mathbf{D}^{\mathrm{b}}(\mathbb{C}_X)$ *and* $\mathbf{D}^{\mathrm{b}}(\mathbb{C}_Y)$*.*

(ii) *Let* $\mathcal{K} \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_{X\times Y})$*,* $\mathcal{K}' \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_{Y\times X})$ *and assume that* $\mathcal{K} \underline{\circ} \mathcal{K}' \simeq \mathcal{B}_{\Delta_X}$ *and* $\mathcal{K}' \underline{\circ} \mathcal{K} \simeq \mathcal{B}_{\Delta_Y}$*. Then, the functors* $\cdot \underline{\circ} \mathcal{K}$ *and* $\cdot \underline{\circ} \mathcal{K}'$ *are quasi-inverse to each other and thus they are equivalences of categories between* $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$ *and* $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_Y)$*.*

Adjunction formulas. Let X and Y be manifolds, and let $\mathcal{K} \in \mathbf{D}^{\mathrm{b}}_{rh}(\mathcal{D}_{X\times Y})$ and $K = \mathcal{S}ol(\mathcal{K}) \in \mathbf{D}^{\mathrm{b}}_{\mathbb{C}-c}(\mathbb{C}_{X\times Y})$. Set

$$\widetilde{K} = r_* K \in \mathbf{D}^{\mathrm{b}}_{\mathbb{C}-c}(\mathbb{C}_{Y\times X}) \quad \text{and} \quad \widetilde{K}^* = D' r_* K \in \mathbf{D}^{\mathrm{b}}_{\mathbb{C}-c}(\mathbb{C}_{Y\times X}).$$

One has the following adjunction formulas relating the transforms for sheaves and for $\mathcal{D}$-modules.

Proposition 1.8. ([4], [11]) *Assume* (1.3)*. For any* $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$ *and* $G \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_Y)$ *there are isomorphisms*

$$\begin{aligned}
&\mathrm{R}\Gamma(X; R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, (G \circ \widetilde{K}) \otimes \mathcal{O}_X))[d_X] \\
&\qquad \simeq \mathrm{R}\Gamma(Y; R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M} \underline{\circ} \mathcal{K}, G \otimes \mathcal{O}_Y)), \\
&\mathrm{R}\Gamma(X; R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, R\mathcal{H}om(G \circ \widetilde{K}^*, \mathcal{O}_X)))[d_X] \\
&\qquad \simeq \mathrm{R}\Gamma(Y; R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M} \underline{\circ} \mathcal{K}, R\mathcal{H}om(G, \mathcal{O}_Y)))[2d_Y].
\end{aligned}$$

Moreover, if $G \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_Y)$ *there are similar isomorphisms with* $\otimes$ *and* $R\mathcal{H}om$ *replaced respectively by* $\overset{\mathrm{w}}{\otimes}$ *and* $\mathcal{T}hom$*.*

1.3 Twisted sections and integral transforms of line bundles

In this section we still assume that the manifolds are complex analytic and compact.

Twisted sections and $\mathcal{D}$-linear morphisms. Let X and Y be manifolds. Given $\mathcal{K} \in \mathrm{Mod}_{rh}(\mathcal{D}_{X\times Y})$, we set

$$\mathcal{K}^{(d_X,0)} = \mathcal{K} \otimes_{q_1^{-1}\mathcal{O}_X} q_1^{-1}\Omega_X.$$

Let $\mathcal{F}$ and $\mathcal{G}$ be holomorphic line bundles on X and Y respectively, and set

$$\mathcal{K}^{(d_X,0)}(\mathcal{F},\mathcal{G}) = q_1^{-1}\mathcal{F} \otimes_{q_1^{-1}\mathcal{O}_X} \mathcal{K}^{(d_X,0)} \otimes_{q_2^{-1}\mathcal{O}_Y} q_2^{-1}\mathcal{G}.$$

Proposition 1.9. ([4]) *For any $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$ one has a natural isomorphism in $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_Y)$*

$$\mathcal{M} \underline{\circ} \mathcal{K} \ \simeq\ Rq_{2!}(q_1^{-1}\mathcal{M} \otimes^L_{q_1^{-1}\mathcal{D}_X} \mathcal{K}^{(d_X,0)}).$$

The following proposition provides a description of the $\mathcal{D}_Y$-linear morphisms between $\mathcal{D}\mathcal{G}$ and $\mathcal{D}\mathcal{F} \underline{\circ} \mathcal{K}$ in terms of twisted sections:

Proposition 1.10. ([4], [5]) *There is a natural isomorphism*

$$\alpha_{\mathcal{K}} : H^0\mathrm{R}\Gamma(X \times Y; \mathcal{K}^{(d_X,0)}(\mathcal{F},\mathcal{G}^*)) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\mathcal{D}_Y)}(\mathcal{D}\mathcal{G}, \mathcal{D}\mathcal{F} \underline{\circ} \mathcal{K}).$$

Hence to any section $s \in H^0\mathrm{R}\Gamma(X\times Y; \mathcal{K}^{(d_X,0)}(\mathcal{F},\mathcal{G}^*))$ one can associate a morphism $\alpha_{\mathcal{K}}(s) : \mathcal{D}\mathcal{G} \to \mathcal{D}\mathcal{F} \underline{\circ} \mathcal{K}$ in $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_Y)$.

In particular, let $X = Y$, $\mathcal{K} = \mathcal{B}_{\Delta_X}$ and $\mathcal{F}$ a holomorphic line bundle on X. By Proposition 1.10 one gets a natural isomorphism

$$\alpha_{\mathcal{F}} : \Gamma(X \times X; \mathcal{B}^{(d_X,0)}_{\Delta_X}(\mathcal{F},\mathcal{F}^*)) \xrightarrow{\sim} \mathrm{End}_{\mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)}(\mathcal{D}\mathcal{F}). \tag{1.4}$$

We denote by $\delta_{X,\mathcal{F}} \in \Gamma(X \times X; \mathcal{B}^{(d_X,0)}_{\Delta_X}(\mathcal{F},\mathcal{F}^*))$ the canonical section corresponding to $\mathrm{id}_{\mathcal{D}\mathcal{F}}$.

Composition of sections. Let X, Y and Z be manifolds, $\mathcal{K} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_{X\times Y})$ and $\mathcal{K}' \in \mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_{Y\times Z})$. There is a natural $\mathbb{C}$-linear morphism

$$q_{12}^{-1}\mathcal{K} \otimes q_{23}^{-1}\mathcal{K}'^{(d_Y,0)} \to \mathcal{D}_{X\times Z \leftarrow X\times Y\times Z} \otimes^L_{\mathcal{D}_{X\times Y\times Z}} (\underline{q_{12}}^{-1}\mathcal{K} \otimes_{\mathcal{O}_{X\times Y\times Z}} \underline{q_{23}}^{-1}\mathcal{K}'). \tag{1.5}$$

Assume that $\mathcal{K} \in \mathrm{Mod}_{rh}(\mathcal{D}_{X\times Y})$ and $\mathcal{K}' \in \mathrm{Mod}_{rh}(\mathcal{D}_{Y\times Z})$, and let $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ be holomorphic line bundles on X, Y and Z. From (1.5) one gets a *composition* morphism

$$\begin{aligned} \cdot \circ \cdot : H^0\mathrm{R}\Gamma(X \times Y; \mathcal{K}^{(d_X,0)}(\mathcal{F},\mathcal{G}^*)) \otimes H^0\mathrm{R}\Gamma(Y \times Z; \mathcal{K}'^{(d_Y,0)}(\mathcal{G},\mathcal{H}^*)) \\ \to H^0\mathrm{R}\Gamma(X \times Z; (\mathcal{K} \underline{\circ} \mathcal{K}')^{(d_X,0)}(\mathcal{F},\mathcal{H}^*)). \end{aligned} \tag{1.6}$$

Remark 1.11. In the regular holonomic case, one can construct the above morphism also with the functor $\mathcal{T}hom(\cdot,\mathcal{O})$, using Proposition 1.4 and the Leray-Grothendieck *integration* morphism

$$Rq_{13!}\mathcal{O}^{(0,d_Y,0)}_{X\times Y\times Z} \to \mathcal{O}_{X\times Z}[-d_Y].$$

The composition (1.6) is compatible with the isomorphism α:

Proposition 1.12. ([4]) *Let $s \in H^0\mathrm{R}\Gamma(X\times Y;\mathcal{K}^{(d_X,0)}(\mathcal{F},\mathcal{G}^*))$, $t \in H^0\mathrm{R}\Gamma(Y\times Z;\mathcal{K}'^{(d_Y,0)}(\mathcal{G},\mathcal{H}^*))$, and denote by $\alpha_{\mathcal{K}}(s)\,\underline{\circ}\,\mathcal{K}'$ the image of the morphism $\alpha_{\mathcal{K}}(s)$ by the functor $\cdot\,\underline{\circ}\,\mathcal{K}'$. Then one has*

$$\big(\alpha_{\mathcal{K}}(s)\,\underline{\circ}\,\mathcal{K}'\big)\circ\alpha_{\mathcal{K}'}(t)\simeq\alpha_{\mathcal{K}\,\underline{\circ}\,\mathcal{K}'}(s\circ t)$$

as $\mathcal{D}_Z$-linear morphisms from $\mathcal{DH}$ to $\mathcal{DF}\,\underline{\circ}\,(\mathcal{K}\,\underline{\circ}\,\mathcal{K}')$.

In particular, we shall be concerned with the following special situation.

Proposition 1.13. *Let $\mathcal{K}\in\mathrm{Mod}_{rh}(\mathcal{D}_{X\times Y})$, $\mathcal{K}'\in\mathrm{Mod}_{rh}(\mathcal{D}_{Y\times X})$, $\mathcal{F}$, $\mathcal{G}$ holomorphic line bundles on X and Y respectively, $s\in\Gamma(X\times Y;\mathcal{K}^{(d_X,0)}(\mathcal{F},\mathcal{G}^*))$ and $t\in\Gamma(Y\times X;\mathcal{K}'^{(d_Y,0)}(\mathcal{G},\mathcal{F}^*))$. Suppose that:*

(i) $\mathcal{K}\,\underline{\circ}\,\mathcal{K}'\simeq\mathcal{B}_{\Delta_X}$ *and* $\mathcal{K}'\,\underline{\circ}\,\mathcal{K}\simeq\mathcal{B}_{\Delta_Y}$*;*

(ii) $s\circ t=\delta_{X,\mathcal{F}}$ *and* $t\circ s=\delta_{Y,\mathcal{G}}$ *(up to a nonzero multiplicative constant).*

Then the morphisms

$$\alpha_{\mathcal{K}}(s):\mathcal{DG}\to\mathcal{DF}\,\underline{\circ}\,\mathcal{K}\qquad\text{and}\qquad\alpha_{\mathcal{K}'}(t):\mathcal{DF}\to\mathcal{DG}\,\underline{\circ}\,\mathcal{K}'$$

are isomorphisms (in particular, $\mathcal{DF}\,\underline{\circ}\,\mathcal{K}$ and $\mathcal{DG}\,\underline{\circ}\,\mathcal{K}'$ are concentrated in degree zero).

Proof. By (1.4) and Proposition 1.12 we have (up to a nonzero multiplicative constant)

$$\big(\alpha_{\mathcal{K}'}(t)\,\underline{\circ}\,\mathcal{K}\big)\circ\alpha_{\mathcal{K}}(s)=\alpha_{\mathcal{G}}(t\circ s)=\mathrm{id}_{\mathcal{DG}}.$$

For the same reasons, we have $\big(\alpha_{\mathcal{K}}(s)\,\underline{\circ}\,\mathcal{K}'\big)\circ\alpha_{\mathcal{K}'}(t)=\alpha_{\mathcal{F}}(s\circ t)=\mathrm{id}_{\mathcal{DF}}$. Applying the functor $\cdot\,\underline{\circ}\,\mathcal{K}$, we get $\alpha_{\mathcal{K}}(s)\circ\big(\alpha_{\mathcal{K}'}(t)\,\underline{\circ}\,\mathcal{K}\big)=\mathrm{id}_{\mathcal{DF}\,\underline{\circ}\,\mathcal{K}}$, and hence $\alpha_{\mathcal{K}'}(t)\,\underline{\circ}\,\mathcal{K}$ is a two-sided inverse of $\alpha_{\mathcal{K}}(s)$. One argues similarly for $\alpha_{\mathcal{K}'}(t)$. $\square$

1.4 Microlocal sections

Here we microlocalize the preceding construction, using some results of Sato-Kawai-Kashiwara [16] and Kashiwara-Schapira [10].

We still assume that the manifolds are complex analytic and compact.

Let X, Y and Z be manifolds, $\mathcal{K}\in\mathrm{Mod}_{rh}(\mathcal{D}_{X\times Y})$, $\mathcal{K}'\in\mathrm{Mod}_{rh}(\mathcal{D}_{Y\times Z})$, $W=\mathrm{char}\,(\mathcal{K})$ and $W'=\mathrm{char}\,(\mathcal{K}')$. Let $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ be holomorphic line bundles on X, Y and Z, and let $s\in H^0\mathrm{R}\Gamma(X\times Y;\mathcal{K}^{(d_X,0)}(\mathcal{F},\mathcal{G}^*))$ and $t\in H^0\mathrm{R}\Gamma(Y\times Z;\mathcal{K}'^{(d_Y,0)}(\mathcal{G},\mathcal{H}^*))$.

Composition of Lagrangians. Let Λ (resp. Λ') be a Lagrangian submanifold of W (resp. W'). In [10] the composition of Lagrangians is described in a set-theoretical way, as

$$\Lambda\circ\Lambda'=p_{13}(\Lambda\times^a_{T^*Y}\Lambda')\subset T^*(X\times Z),\tag{1.7}$$

where $\cdot \times^a_{T^*Y} \cdot$ denotes the fiber product with respect to the projections $p_2^a : T^*(X \times Y) \to T^*Y$ (i.e. p_2 composed with the antipodal map a on T^*Y) and $p_1 : T^*(Y \times Z) \to T^*Y$. In the smooth case, one has:

Proposition 1.14. ([10, Lemma 7.4.1]) *Let Λ and Λ' be smooth Lagrangians, and suppose that*

$$p_2^a|_\Lambda : \Lambda \to T^*Y \text{ and } p_1|_{\Lambda'} : \Lambda' \to T^*Y \text{ are transversal.} \tag{1.8}$$

Then $\Lambda \circ \Lambda'$ is a smooth Lagrangian.

Microlocal composition. Let $\delta_Y : X \times Y \times Z \to X \times Y \times Y \times Z$ be the diagonal embedding. One defines the *microlocal composition* of $\mathcal{K}$ and $\mathcal{K}'$ as

$$\mathcal{E}\mathcal{K} \underline{\circ}_\mu \mathcal{E}\mathcal{K}' = \underline{q_{13}}_\mu \underline{\delta_Y}^\mu (\mathcal{E}\mathcal{K} \boxtimes^\mu \mathcal{E}\mathcal{K}') \in \mathbf{D}^{\mathrm{b}}(\mathcal{E}_{X \times Z}).$$

Proposition 1.15. ([5]) *Assuming* (1.2), *one has an isomorphism in $\mathbf{D}^{\mathrm{b}}(\mathcal{E}_{X \times Z})$*

$$\mathcal{E}\mathcal{K} \underline{\circ}_\mu \mathcal{E}\mathcal{K}' \simeq \mathcal{E}(\mathcal{K} \underline{\circ} \mathcal{K}').$$

The composition morphism (1.6) can be microlocalized. Set

$$\mathcal{E}\mathcal{K}^{(d_X,0)}(\mathcal{F}, \mathcal{G}^*) = \pi^{-1} q_1^{-1}(\mathcal{F} \otimes_{\mathcal{O}_X} \Omega_X) \otimes_{\pi^{-1}q_1^{-1}\mathcal{O}_X} \mathcal{E}\mathcal{K} \otimes_{\pi^{-1}q_2^{-1}\mathcal{O}_Y} \pi^{-1} q_2^{-1} \mathcal{G}^*.$$

We look at s and t as globally defined microlocal sections, i.e.

$$s \in H^0\mathrm{R}\Gamma(T^*(X \times Y); \mathcal{E}\mathcal{K}^{(d_X,0)}(\mathcal{F}, \mathcal{G}^*)) \text{ and } t \in H^0\mathrm{R}\Gamma(T^*(Y \times Z); \mathcal{E}\mathcal{K}'^{(d_Y,0)}(\mathcal{G}, \mathcal{H}^*)).$$

Let U_X (resp. U_Y, U_Z) be an open subset of T^*X (resp. T^*Y, T^*Z) and set

$$W_U = W \cap (U_X \times T^*Y) \quad \text{and} \quad W'_U = W' \cap (T^*Y \times U_Z).$$

Let us suppose that W_U and W'_U are smooth, and consider the restrictions

$$s|_{W_U} \in H^0\mathrm{R}\Gamma(W_U; \mathcal{E}\mathcal{K}^{(d_X,0)}(\mathcal{F}, \mathcal{G}^*)) \text{ and } t|_{W'_U} \in H^0\mathrm{R}\Gamma(W'_U; \mathcal{E}\mathcal{K}'^{(d_Y,0)}(\mathcal{G}, \mathcal{H}^*)).$$

Proposition 1.16. ([1], [10]) *Assume* (1.2), (1.8) *and*

$$W_U \subset U_X \times U_Y^a, \tag{1.9}$$

$$p_1 : W_U \to U_X \text{ is proper,} \tag{1.10}$$

and the analogous conditions (1.9)' $W'_U \subset U_Y \times U_Z^a$ *and* (1.10)' $p_2^a : W'_U \to U_Z$ *is proper. Then:*

(i) *there is a well-defined* microlocal composition

$$\begin{aligned} \cdot \circ_\mu \cdot : H^0\mathrm{R}\Gamma(W_U; \mathcal{E}\mathcal{K}^{(d_X,0)}(\mathcal{F}, \mathcal{G}^*)) \otimes H^0\mathrm{R}\Gamma(W'_U; \mathcal{E}\mathcal{K}'^{(d_Y,0)}(\mathcal{G}, \mathcal{H}^*)) \\ \to H^0\mathrm{R}\Gamma(W_U \circ W'_U; \mathcal{E}(\mathcal{K} \underline{\circ} \mathcal{K}')^{(d_X,0)}(\mathcal{F}, \mathcal{H}^*)); \end{aligned}$$

(ii) *one has*

$$s|_{W_U} \circ_\mu t|_{W'_U} = s \circ t|_{W_U \circ W'_U},$$

where $s \circ t$ is the composition (1.6).

We will not enter into details about these facts. We just mention that, since our kernels are regular holonomic, then the statements follow from analogous statements for the perverse sheaves $K = \mathcal{S}ol(\mathcal{K})$ and $K' = \mathcal{S}ol(\mathcal{K}')$ by means of the microlocal analogous $\mathcal{T}\mu hom$ of the functor $\mathcal{T}hom$, due to Andronikov [1] (e.g. one has $\mathcal{E}\mathcal{K} \simeq \mathcal{T}\mu hom(K, \mathcal{O}_{X\times Y})$). In particular, using Proposition 1.15, the claim (i) follows from [1, Proposition 3.3.12], and (ii) from an application of [10, Proposition 7.1.2] to the functor $\mathcal{T}\mu hom$.

Quantized contact transformations. In the above situation, let $Z = X$, $\mathcal{H} = \mathcal{F}$, $U_Z = U_X$, Λ a smooth Lagrangian submanifold of W, $W' = \widetilde{W} = {}^t r'^{-1}(W)$ and $\Lambda' = \widetilde{\Lambda} = {}^t r'^{-1}(\Lambda)$. Assume the following:

$$(W \times T^*_X X) \cap (T^*_X X \times \widetilde{W}) \subset T^*_{X\times Y\times X}(X \times Y \times X) \tag{1.11}$$

$$\text{the maps } p_1|_\Lambda : \Lambda \to U_X \text{ and } p_2^a|_\Lambda : \Lambda \to U_Y \text{ are isomorphisms} \tag{1.12}$$

$$p_1^{-1}(U_X) = p_2^{a\,-1}(U_Y) = \Lambda. \tag{1.13}$$

(In other words, (1.12) says that Λ is the graph of a contact transformation $p_2^a|_\Lambda \circ p_1|_\Lambda^{-1}$ between U_X and U_Y, and (1.13) means that there is no "interference" of $W \setminus \Lambda$ over U_X and U_Y.)

Proposition 1.17. *Let $\Lambda^0 = \Lambda \circ \widetilde{\Lambda}$, and assume* (1.11), (1.12) *and* (1.13). *Then:*

(i) *Λ^0 is a smooth Lagrangian submanifold of $T^*_{\Delta_X}(X \times X)$. Moreover, the composition $s|_\Lambda \circ_\mu t|_{\widetilde{\Lambda}}$ is a well-defined section of $H^0\mathrm{R}\Gamma(\Lambda^0; \mathcal{E}(\mathcal{K} \underline{\circ} \mathcal{K}')^{(d_X,0)}(\mathcal{F}, \mathcal{F}^*))$ and coincides with $s \circ t|_{\Lambda^0}$;*

(ii) *if $s|_\Lambda$ generates $\mathcal{E}\mathcal{K}$ on Λ and $t|_{\widetilde{\Lambda}}$ generates $\mathcal{E}\mathcal{K}'$ on $\widetilde{\Lambda}$, then the section $s|_\Lambda \circ_\mu t|_{\widetilde{\Lambda}}$ generates $\mathcal{E}(\mathcal{K} \underline{\circ} \mathcal{K}')$ on Λ^0.*

Proof. (i) Λ^0 is a Lagrangian submanifold of $T^*_{\Delta_X}(X \times X)$ since Λ and $\widetilde{\Lambda}$ are graphs of contact transformations, and is smooth by Proposition 1.14, since (1.12) implies (1.8). Conditions (1.9) and (1.10) are satisfied since $W_U = \Lambda$ and $W'_U = \widetilde{\Lambda}$ by (1.12) and (1.13), and (1.11) is nothing but (1.2). Then we may apply Proposition 1.16. The claim (ii) follows from the theory of [16]. □

2 Invertible kernels associated to an open relation

Following an idea of [12] (already introduced in [16] and [10, Ex. III.15] in the real case), we introduce a pair of kernels defined by an open relation between two manifolds, and we study necessary geometrical conditions for their invertibility.

2.1 A geometric criterion

Let X and Y be *real analytic compact orientable* manifolds of the same dimension n, Ω an open subanalytic subset of $X \times Y$. Set $\widetilde{\Omega} = r(\Omega)$, and denote by j (resp. $\tilde{\jmath}$) the embedding of Ω into $X \times Y$ (resp. of $\widetilde{\Omega}$ into $Y \times X$). For any $x \in X$ we set

$$\Omega_x = \{y \in Y : (x, y) \in \Omega\} \subset Y,$$

and similarly for $y \in Y$. Let us consider the kernels

$$K_\Omega = \mathbb{C}_\Omega = j_! j^{-1} \mathbb{C}_{X \times Y} \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_{X \times Y}), \tag{2.1}$$

$$K^*_\Omega = D' \mathbb{C}_\Omega \simeq R j_* j^{-1} \mathbb{C}_{X \times Y} \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_{X \times Y}). \tag{2.2}$$

Following a suggestion of M. Kashiwara, we shall give a geometric criterion which ensures that

$$K_\Omega \circ K^*_{\widetilde{\Omega}} \simeq \mathbb{C}_{\Delta_X}[-n] \quad \text{and} \quad K^*_{\widetilde{\Omega}} \circ K_\Omega \simeq \mathbb{C}_{\Delta_Y}[-n].$$

We shall consider the following geometrical hypotheses.

$$X \text{ is simply connected;} \tag{2.3}$$

$$\mathrm{R}\Gamma(\Omega_x; \mathbb{C}_{\Omega_{x'}}) = \begin{cases} 0 & \text{for } x \neq x' \\ \mathbb{C} & \text{for } x = x' \end{cases}; \tag{2.4}$$

$$SS(\mathbb{C}_\Omega) \cap (T^*_X X \times T^* Y) \subset T^*_{X \times Y}(X \times Y), \tag{2.5}$$

and the similar conditions (2.3)', (2.4)' and (2.5)' obtained from above by interchanging X and Y.

Remark 2.1. We observe the following facts.

(i) Let $x \neq x'$: then, applying the functor $\mathrm{R}\Gamma(\Omega_x; \cdot\,)$ to the exact sequence

$$0 \to \mathbb{C}_{\Omega_x \cap \Omega_{x'}} \to \mathbb{C}_{\Omega_x} \to \mathbb{C}_{\Omega_x \setminus \Omega_{x'}} \to 0,$$

one sees that (2.4) is equivalent to requiring that the natural morphism

$$\mathrm{R}\Gamma(\Omega_x; \mathbb{C}_{\Omega_x}) \to \mathrm{R}\Gamma(\Omega_x \setminus \Omega_{x'}; \mathbb{C}_{\Omega_x \setminus \Omega_{x'}})$$

is an isomorphism.

(ii) (2.5) implies (1.1) for $Z = X$, $K = K_\Omega$ and $K' = K^*_{\widetilde{\Omega}}$, since $SS(K^*_{\widetilde{\Omega}}) = {}^t r' SS(K_\Omega)^a$. (Recall that, if $F \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_X)$, then $SS(D'F) = SS(F)^a$.)

Lemma 2.2. *Assume* (2.3), (2.4), *and* (2.5). *Then*

$$K_\Omega \circ K^*_{\widetilde{\Omega}} \simeq \mathbb{C}_{\Delta_X}[-n].$$

Proof. We set

$$C = K_\Omega \circ K^*_{\widetilde{\Omega}} = Rq_{13!}C_\circ, \qquad C_\circ = D'\mathbb{C}_{X\times\widetilde{\Omega}} \otimes \mathbb{C}_{\Omega\times X}.$$

Our first aim is to prove that $C|_U = 0$, where $U = (X\times X)\setminus\Delta_X$. Let $(x,x')\in U$ (i.e. $x\neq x'$), and consider the diagram

$$\begin{array}{ccccc}
\{x\}\times Y\times\{x'\} & \xrightarrow{\tilde{\imath}_{(x',x)}} & X\times Y\times\{x'\} & \xrightarrow{\tilde{\imath}_{x'}} & X\times Y\times X \\
\downarrow{\scriptstyle q_{13}^{(x',x)}} & & \downarrow{\scriptstyle q_{13}^{x'}} & & \downarrow{\scriptstyle q_{13}} \\
\{x\}\times\{x'\} & \xrightarrow{i_{(x',x)}} & X\times\{x'\} & \xrightarrow{i_{x'}} & X\times X.
\end{array}$$

Sometimes we shall identify $\{x\}\times Y\times\{x'\}$ with Y.

Lemma 2.3. *Let X and Y be real analytic compact manifolds, U a subanalytic open subset of $X\times X$, $C_\circ \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_{X\times Y\times X})$ and $C = Rq_{13!}C_\circ \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_{X\times X})$. Then $C|_U = 0$ if and only if $\mathrm{R}\Gamma(Y;\tilde{\imath}_x^{-1}\tilde{\imath}^!_{x'}C_\circ) = 0$ for any $(x,x')\in U$.*

Proof of the lemma: First note that, if Z is a real analytic manifold and $F\in\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_Z)$, one has $F=0 \Leftrightarrow DF=0$ (where $DF = R\mathcal{H}om(F,\omega_Z)$ and $\omega_Z = or_Z[d_Z]$ is the dualizing complex) $\Leftrightarrow i_z^{-1}DF = Di_z^!F = 0$ for any $z\in Z \Leftrightarrow i_z^!F = 0$ for any $z\in Z$. In our case, let us first suppose that $U = U_1\times U_2$ for some open subsets U_1 and U_2 of X. Then we have that $C|_U = 0 \Leftrightarrow (i_{x'}\circ i_{(x',x)})^!C = i^!_{(x',x)}i^!_{x'}C = 0$ for any $(x,x')\in U \Leftrightarrow i^!_{x'}C = 0$ for any $x'\in U_2 \Leftrightarrow i^{-1}_{(x',x)}i^!_{x'}C = i^{-1}_{(x',x)}i^!_{x'}Rq_{13!}C_\circ = 0$ for any $(x,x')\in U \Leftrightarrow R\big(q_{13}^{(x',x)}\big)_!\tilde{\imath}^{-1}_{(x',x)}\tilde{\imath}^!_{x'}C_\circ = \mathrm{R}\Gamma(Y;\tilde{\imath}^{-1}_{(x',x)}\tilde{\imath}^!_{x'}C_\circ) = 0$ for any $(x,x')\in U$ (for the last equivalence, note that the projections are proper and the square diagrams are cartesian). As for the general case, it suffices to note that U is covered by open subsets of the form $U_1\times U_2$. The lemma is proved.

Therefore, let us verify that $\mathrm{R}\Gamma(Y;\tilde{\imath}^{-1}_{(x',x)}\tilde{\imath}^!_{x'}C_\circ) = 0$. We have:

$$\begin{array}{rcll}
\tilde{\imath}^!_{x'}C_\circ & \simeq & \tilde{\imath}^!_{x'}R\mathcal{H}om(\mathbb{C}_{X\times\widetilde{\Omega}},\mathbb{C}_{\Omega\times X}) & \text{(by (2.5))} \\
& \simeq & R\mathcal{H}om(\tilde{\imath}^{-1}_{x'}\mathbb{C}_{X\times\widetilde{\Omega}}, i^!_{x'}\mathbb{C}_{\Omega\times X}) & \text{(as a sheaf on } X\times Y\times X) \\
& \simeq & R\mathcal{H}om(\tilde{\imath}_{x'!}\tilde{\imath}^{-1}_{x'}\mathbb{C}_{X\times\widetilde{\Omega}},\mathbb{C}_{\Omega\times X}) & \\
& \simeq & R\mathcal{H}om(\mathbb{C}_{X\times\Omega_{x'}},\mathbb{C}_\Omega) & \text{(identifying } X\times Y\times\{x'\} \text{ to } X\times Y) \\
& \simeq & D'\mathbb{C}_{X\times\Omega_{x'}}\otimes\mathbb{C}_\Omega & \text{(by (2.5)).}
\end{array}$$

Then we get

$$\begin{array}{rcll}
\mathrm{R}\Gamma(Y;\tilde{\imath}^{-1}_{(x',x)}\tilde{\imath}^!_{x'}C_\circ) & \simeq & \mathrm{R}\Gamma(Y;D'\mathbb{C}_{\Omega_{x'}}\otimes\mathbb{C}_{\Omega_x}) & \text{(identifying } \{x\}\times Y \text{ to } Y) \\
& \simeq & \mathrm{R}\Gamma_c(\Omega_x;D'\mathbb{C}_{\Omega_{x'}}) & \\
& \simeq & \mathrm{R}\Gamma(\Omega_x;\mathbb{C}_{\Omega_{x'}})^*[-n] & \text{(by Poincaré duality),}
\end{array}$$

where $(\,\cdot\,)^* = \mathrm{Hom}(\,\cdot\,,\mathbb{C})$, and therefore $\mathrm{R}\Gamma(Y;\tilde{\imath}^{-1}_{(x',x)}\tilde{\imath}^!_{x'}C_\circ) = 0$ by (2.4). Thus the support of C

is contained in Δ_X, and hence

$$\begin{aligned} C &\simeq C \otimes \mathbb{C}_{\Delta_X} \\ &= Rq_{13!}\left(D'\mathbb{C}_{X\times\widetilde{\Omega}} \otimes \mathbb{C}_{\Omega\times X}\right) \otimes \mathbb{C}_{\Delta_X} \\ &\simeq Rq_{13!}\left(D'\mathbb{C}_{X\times\widetilde{\Omega}} \otimes \mathbb{C}_{\Omega\times X} \otimes \mathbb{C}_{(X\times Y)\times_X X}\right) \\ &\simeq Rq_{13!}\left(D'\mathbb{C}_{X\times\widetilde{\Omega}} \otimes \mathbb{C}_{\Omega\times_X X}\right). \end{aligned}$$

Now, $SS(\mathbb{C}_{X\times\widetilde{\Omega}}) = T^*_X X \times {}^t r'^{-1} SS(\mathbb{C}_\Omega)$ and $SS(\mathbb{C}_{\Omega\times_X X}) = SS(\mathbb{C}_\Omega) \times^a_{T^*X} T^*X$ (we mean that the fiber product over T^*X is made with respect to the natural projection and the antipodal map): hence, by (2.5) we have $SS(\mathbb{C}_{X\times\widetilde{\Omega}}) \cap SS(\mathbb{C}_{\Omega\times_X X}) \subset T^*_{X\times Y}(X\times Y)$, and we get

$$C \simeq Rq_{13!} R\mathcal{H}om\,(\mathbb{C}_{X\times\widetilde{\Omega}}.\mathbb{C}_{\Omega\times_X X}),$$

Since $\Omega \times_X X$ is a closed subset of $X \times \widetilde{\Omega}$, we obtain

$$C \simeq Rq_{13!}\mathbb{C}_{\Omega\times_X X} \simeq R\delta_! Rq_{1!}\mathbb{C}_\Omega.$$

Since q_1 is proper, we have $SS(Rq_{1!}\mathbb{C}_\Omega) \subset q_{1\pi}{}^t q_1'^{-1} SS(\mathbb{C}_\Omega) = q_{1\pi}{}^t q_1'^{-1}(SS(\mathbb{C}_\Omega) \cap (T^*X \times T^*_Y Y))$, and thus $SS(Rq_{1!}\mathbb{C}_\Omega) \subset T^*_X X$ by (2.5)'. In other words, $Rq_{1!}\mathbb{C}_\Omega$ is locally constant on X, and hence constant by (2.3). Let $x \in X$: since $(Rq_{1!}\mathbb{C}_\Omega)_x \simeq \mathrm{R}\Gamma_c(\Omega_x;\mathbb{C}_{\Omega_x}) \simeq \mathrm{R}\Gamma(\Omega_x;\mathbb{C}_{\Omega_x})^*[-n] \simeq \mathbb{C}[-n]$ by Poincaré duality and (2.4), we get $Rq_{1!}\mathbb{C}_\Omega \simeq \mathbb{C}_X[-n]$, and thus $C \simeq \mathbb{C}_{\Delta_X}[-n]$. □

2.2 The complex case

Let X and Y be complex analytic compact manifolds of the same complex dimension n, and let Ω be an open subset of $X \times Y$ such that the complementary closed set $S = (X\times Y) \setminus \Omega$ is complex analytic (in particular, Ω is subanalytic). Then, the kernels introduced in (2.1)–(2.2) have $\mathbb{C}$-constructible cohomology groups. Therefore, by the Riemann-Hilbert correspondence, we have the associated $\mathcal{D}$-module kernels

$$\mathcal{K}_\Omega = \mathcal{T}hom(K_\Omega, \mathcal{O}_{X\times Y}) \in \mathbf{D}^{\mathrm{b}}_{rh}(\mathcal{D}_{X\times Y}), \tag{2.6}$$

$$\mathcal{K}^*_\Omega = \mathcal{T}hom(K^*_\Omega, \mathcal{O}_{X\times Y}) \in \mathbf{D}^{\mathrm{b}}_{rh}(\mathcal{D}_{X\times Y}). \tag{2.7}$$

Theorem 2.4. *Assume* (2.3), (2.4), (2.5) *and* (2.3)', (2.4)', (2.5)'. *Then:*

(i) *the functors*

$$\cdot \circ K_\Omega : \mathbf{D}^{\mathrm{b}}(\mathbb{C}_X) \to \mathbf{D}^{\mathrm{b}}(\mathbb{C}_Y) \quad \textit{and} \quad \cdot \circ K^*_{\widetilde{\Omega}} : \mathbf{D}^{\mathrm{b}}(\mathbb{C}_Y) \to \mathbf{D}^{\mathrm{b}}(\mathbb{C}_X)$$

are quasi-inverse to each other, and thus they define equivalences of categories between $\mathbf{D}^{\mathrm{b}}(\mathbb{C}_X)$ *and* $\mathbf{D}^{\mathrm{b}}(\mathbb{C}_Y)$, *between* $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_X)$ *and* $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_Y)$ *as well as between* $\mathbf{D}^{\mathrm{b}}_{\mathbb{C}-c}(\mathbb{C}_X)$ *and* $\mathbf{D}^{\mathrm{b}}_{\mathbb{C}-c}(\mathbb{C}_Y)$;

(ii) *the functors*

$$\cdot \underline{\circ}\, \mathcal{K}_\Omega : \mathbf{D}^{\mathrm{b}}(\mathcal{D}_X) \to \mathbf{D}^{\mathrm{b}}(\mathcal{D}_Y) \quad \text{and} \quad \cdot \underline{\circ}\, \mathcal{K}^*_{\widetilde{\Omega}} : \mathbf{D}^{\mathrm{b}}(\mathcal{D}_Y) \to \mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$$

are quasi-inverse to each other, and thus they define equivalences of categories between $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$ and $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_Y)$, between $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X)$ and $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_Y)$ as well as between $\mathbf{D}^{\mathrm{b}}_{rh}(\mathcal{D}_X)$ and $\mathbf{D}^{\mathrm{b}}_{rh}(\mathcal{D}_Y)$.

Proof. We have $K_\Omega \circ K^*_{\widetilde{\Omega}} \simeq \mathbb{C}_{\Delta_X}[-2n]$ and $K^*_{\widetilde{\Omega}} \circ K_\Omega \simeq \mathbb{C}_{\Delta_Y}[-2n]$ by Lemma 2.2. Then, by Proposition 1.4 we obtain $\mathcal{K}_\Omega \underline{\circ}\, \mathcal{K}^*_{\widetilde{\Omega}} \simeq \mathcal{B}_{\Delta_X}$ and $\mathcal{K}^*_{\widetilde{\Omega}} \underline{\circ}\, \mathcal{K}_\Omega \simeq \mathcal{B}_{\Delta_Y}$. Finally, we apply Proposition 1.7 and Corollary 1.6. □

3 Grassmann duality

Let us apply the abstract construction above to the case of a pair of "dual" Grassmann manifolds of a fixed complex vector space.

3.1 Perverse sheaves associated to the transversality relation

Let n and p be positive integers such that $n \geq 2$ and $1 \leq p \leq n/2$, and let V be a n-dimensional complex vector space. We set

$$\begin{aligned} \mathbb{G} &= \{x : x \text{ is a } p\text{-dimensional subspace of } V\}, \\ \mathbb{G}^* &= \{y : y \text{ is a } (n-p)\text{-dimensional subspace of } V\}, \\ \Omega &= \{(x,y) \in \mathbb{G} \times \mathbb{G}^* : x \cap y = \{0\}\}, \\ S &= (\mathbb{G} \times \mathbb{G}^*) \setminus \Omega. \end{aligned}$$

In other words, $\mathbb{G}$ is the Grassmann manifold of p-subspaces of V, $\mathbb{G}^*$ is the "dual" Grassmann manifold of $(n-p)$-subspaces (which is canonically isomorphic to the Grassmann manifold of p-subspaces of the dual vector space V^*), Ω is the open "transversality" relation in $\mathbb{G} \times \mathbb{G}^*$ and S is the "incidence" relation, a closed complex hypersurface of $\mathbb{G} \times \mathbb{G}^*$. We denote by $j : \Omega \hookrightarrow \mathbb{G} \times \mathbb{G}^*$ the embedding, and by $\widetilde{\Omega}$, $\widetilde{S}$ and $\tilde{\jmath}$ the similar objects in $\mathbb{G}^* \times \mathbb{G}$. Recall that $\mathbb{G}$ and $\mathbb{G}^*$ are both complex analytic (in fact, also algebraic) compact manifolds of dimension $N = p(n-p)$.

Group actions. The grassmannians $\mathbb{G}$ and $\mathbb{G}^*$ are homogeneous manifolds under the natural action of the complex Lie group $G = SL(V)$, whose associated Lie algebra is $\mathfrak{g} = \mathfrak{sl}(V) = \{\alpha \in \mathrm{End}_{\mathbb{C}}(V) : \mathrm{tr}\,(\alpha) = 0\}$. The group G acts naturally also on $\mathbb{G} \times \mathbb{G}^*$ with the diagonal action $g(x,y) = (gx, gy)$, and the set of G-orbits is $\{\Omega, S_1, \ldots, S_p\}$, where

$$S_j = \{(x,y) \in \mathbb{G} \times \mathbb{G}^* : \dim(x \cap y) = j\} \qquad (j = 1, \ldots, p) \tag{3.1}$$

is a locally closed smooth submanifold of $\mathbb{G} \times \mathbb{G}^*$ (with $\mathrm{codim}_{\mathbb{G}\times\mathbb{G}^*} S_j = j^2$): in particular, note that $(G, \mathbb{G} \times \mathbb{G}^*)$ is a prehomogeneous space (see [17]) with open dense orbit Ω. Similar considerations hold for the diagonal action of G on $\mathbb{G}\times\mathbb{G}$, where the G-orbits are $\{(x,x')\in\mathbb{G}\times\mathbb{G} : \dim(x\cap x')=j\}$ for $j=0,\dots,p$ (in particular, for $j=p$, one obtains $\Delta_{\mathbb{G}}$).

Homogeneous coordinates. Let us consider the manifold of *p-frames* in V

$$F_p(V)=\{v=(v_1,\dots,v_p)\in V^p:\ v_1\wedge\cdots\wedge v_p\neq 0\},$$

an open dense subset of $V^p\simeq\mathbb{C}^{np}$ (e.g. $F_1(V)=V\setminus\{0\}$). There are $GL_p(\mathbb{C})$-bundles

$$\begin{array}{ll} q: F_p(V)\to\mathbb{G}, & q(v)=\langle v_1,\dots,v_p\rangle,\\ q^*: F_p(V^*)\to\mathbb{G}^*, & q^*(v^*)=\langle v_1^*,\dots,v_p^*\rangle^{\perp}. \end{array}\tag{3.2}$$

We introduce a system of Stiefel (resp. dual Stiefel) coordinates $[\xi]$ on $\mathbb{G}$ (resp. $[\eta]$ on $\mathbb{G}^*$), i.e. a system of $GL_p(\mathbb{C})$-homogeneous coordinates on $F_p(V)$ (resp. on $F_p(V^*)$). In other words, fixed any basis $\{v_1,\dots,v_n\}$ of V, the matrix $\xi\in M_{p,n}(\mathbb{C})$ is associated to the p-subspace x of V spanned by its p row vectors $(\xi_1,\dots,\xi_p)$, while $\eta\in M_{n,p}(\mathbb{C})$ denotes the $(n-p)$-subspace y of V orthogonal to the p-subspace of V^* spanned by the p column vectors $(\eta_1,\dots,\eta_p)$ of η in the dual basis $\{v_1^*,\dots,v_n^*\}$ of V^*. It is clear that:

(i) these coordinates satisfy the homogeneity conditions $[A\xi]=[\xi]$ and $[\eta A']=[\eta]$ for any $A,A'\in GL_p(\mathbb{C})$;

(ii) If $x=[\xi]$ and $y=[\eta]$, then for any $g\in G$ one has $gx=[\xi g]$ and $gy=[g^{-1}\eta]$ (note that $g^{-1}\eta={}^t({}^t\eta\,{}^tg^{-1})$).

Geometry. We observe the following geometrical facts.

(1) The closed complex hypersurface S is the set of zeros of the homogeneous equation in $\mathbb{G}\times\mathbb{G}^*$

$$f(x(\xi),y(\eta))=\det(\xi\eta),\tag{3.3}$$

where $\xi\eta$ is the usual product of matrices; observe that

$$f(A\xi,\eta A')=(\det A)(\det A')f(\xi,\eta)\qquad\text{for any } A,A'\in GL_p(\mathbb{C}),$$

and that f is G-invariant (for the diagonal action on $\mathbb{G}\times\mathbb{G}^*$), i.e.

$$f(gx,gy)=f(x,y)\qquad\text{for any } x\in\mathbb{G},\ y\in\mathbb{G}^*\text{ and } g\in G.$$

(2) A subanalytic stratification of S is given by $S=\bigcup_{j=1}^{p}S_j$.

(3) The projections

$$q_1|_\Omega:\Omega\to\mathbb{G},\qquad q_2|_\Omega:\Omega\to\mathbb{G}^*$$

have affine fibers. Namely, let $x\in\mathbb{G}$ and fix any basis in V such that $x=[1_p,0]$, where 1_p is the identity matrix of $GL_p(\mathbb{C})$: then, setting $\Omega_x=\{y\in\mathbb{G}^*:(x,y)\in\Omega\}$, one has

$$\Omega_x=\left\{\begin{bmatrix}1_p\\ M\end{bmatrix}: M\in M_{n-p,p}(\mathbb{C})\right\}\simeq\mathbb{C}^N.$$

Kernels associated to Ω. The open subset Ω of $\mathbb{G}\times\mathbb{G}^*$ defines integral transforms between $\mathbb{G}$ and $\mathbb{G}^*$ by means of the kernels introduced in (2.1) and (2.2):

$$K_\Omega = \mathbb{C}_\Omega = j_! j^{-1}\mathbb{C}_{\mathbb{G}\times\mathbb{G}^*} \quad \text{and} \quad K^*_{\Omega} = D'\mathbb{C}_\Omega \simeq Rj_* j^{-1}\mathbb{C}_{\mathbb{G}\times\mathbb{G}^*}.$$

Since Ω is the complementary of a closed complex hypersurface, the kernel K_Ω is a perverse object of $\mathbf{D}^{\mathrm{b}}(\mathbb{C}_{\mathbb{G}\times\mathbb{G}^*})$ (see for example Kashiwara-Schapira [10, ch. X]) and hence so is $K^*_{\widetilde{\Omega}}$ by duality. By the Riemann-Hilbert correspondence, we get the associate regular holonomic $\mathcal{D}$-modules

$$\mathcal{K}_\Omega = \mathcal{T}hom(K_\Omega, \mathcal{O}_{\mathbb{G}\times\mathbb{G}^*}) \quad \text{and} \quad \mathcal{K}^*_{\Omega} = \mathcal{T}hom(K^*_{\Omega}, \mathcal{O}_{\mathbb{G}\times\mathbb{G}^*}).$$

Remark 3.1. By definition of $\mathcal{T}hom$, the sections of $\mathcal{K}_\Omega$ are the meromorphic functions on $\mathbb{G}\times\mathbb{G}^*$ with singularities along S (i.e. $\mathcal{K}_\Omega \simeq \mathcal{O}_{\mathbb{G}\times\mathbb{G}^*}(*S)$ in the more classical notation of Appendix A), and $\mathcal{K}^*_{\Omega} \simeq \underline{\mathrm{D}}\,\mathcal{K}_\Omega$ by Proposition 1.5.

Remark 3.2. For $p=1$ one obtains the *projective duality* between a complex $(n-1)$-dimensional projective space $\mathbb{P}$ and its dual $\mathbb{P}^*$. In this case, S is a smooth hypersurface of $\mathbb{P}\times\mathbb{P}^*$ (see [2], [4]).

3.2 Microlocal geometry

In order to study the microlocal geometry of our correspondence, we use the action of the group G.

Group actions and microlocal geometry. Let X be a complex analytic manifold with a transitive action of a simply connected complex Lie group G with Lie algebra $\mathfrak{g}$. Let $\mathfrak{g}\times TX \to TX$ be the tangent action and $\rho : T^*X \to \mathfrak{g}^*$ be the moment map. Since the G-action on X is transitive, T^*X is identified to a subset of $X\times\mathfrak{g}^*$ by the map (π,ρ). Let Y be another complex analytic manifold with a transitive G-action, and let S be a smooth G-orbit in $X\times Y$ for the diagonal action. One has $T^*_S(X\times Y) \rightarrowtail T^*(X\times Y) \simeq T^*X\times T^*Y$, and $T^*X\times T^*Y$ is embedded in $(X\times Y)\times(\mathfrak{g}^*\times\mathfrak{g}^*)$ by the map $(\pi_X,\pi_Y,\rho_X,\rho_Y)$. Let $p=(x,y;\xi,\eta)\in T^*(X\times Y)$ with $\xi,\eta\in\mathfrak{g}^*$: recalling that the pairing $T(X\times Y)\times T^*(X\times Y)\to\mathbb{C}$ is related to the pairings $TX\times T^*X\to\mathbb{C}$ and $TY\times T^*Y\to\mathbb{C}$ by $\langle(v,w);(\xi,\eta)\rangle = \langle v,\xi\rangle + \langle w,\eta\rangle$, we observe that $p\in T^*_S(X\times Y)$ if and only if $(x,y)\in S$ and $\eta=-\xi$.

In our case (where $G=SL(V)$, $\mathfrak{g}=\mathfrak{sl}(V)$, $X=\mathbb{G}$ and $Y=\mathbb{G}^*$) the above considerations lead to the following useful identifications, where we identify $\mathfrak{g}^*$ with $\mathfrak{g}$ by the Killing form:

$$\begin{aligned}
T^*\mathbb{G} &\simeq \{(x;\xi) : x\in\mathbb{G},\ \xi\in\mathfrak{sl}(V),\ x\subset\ker(\xi),\ \mathrm{im}\,(\xi)\subset x\}\\
&\simeq \{(x;\alpha) : x\in\mathbb{G},\ \alpha\in\mathrm{Hom}_{\mathbb{C}}(\tfrac{V}{x},x)\},\\
T^*\mathbb{G}^* &\simeq \{(y;\eta) : y\in\mathbb{G}^*,\ \eta\in\mathfrak{sl}(V),\ y\subset\ker(\eta),\ \mathrm{im}\,(\eta)\subset y\}\\
&\simeq \{(y;\beta) : y\in\mathbb{G}^*,\ \beta\in\mathrm{Hom}_{\mathbb{C}}(\tfrac{V}{y},y)\},\\
T^*_{S_j}(\mathbb{G}\times\mathbb{G}^*) &\simeq \Big\{(x,y;\alpha,\beta) : (x,y)\in S_j,\ \exists\,\gamma\in\mathrm{Hom}_{\mathbb{C}}(\tfrac{V}{x+y},x\cap y)\\
&\qquad \text{s.t. } \alpha : \tfrac{V}{x}\twoheadrightarrow\tfrac{V}{x+y}\xrightarrow{\gamma}x\cap y\rightarrowtail x,\ \ \beta : \tfrac{V}{y}\twoheadrightarrow\tfrac{V}{x+y}\xrightarrow{-\gamma}x\cap y\rightarrowtail y\Big\}\\
&\simeq \Big\{(x,y;\gamma) : (x,y)\in S_j,\ \gamma\in\mathrm{Hom}_{\mathbb{C}}(\tfrac{V}{x+y},x\cap y)\Big\},
\end{aligned}\tag{3.4}$$

and the projections from $T^*_{S_j}(\mathbb{G}\times\mathbb{G}^*)$ on $T^*\mathbb{G}$ and $T^*\mathbb{G}^*$ are given by

$$\begin{array}{ll} p_1 : T^*_{S_j}(\mathbb{G}\times\mathbb{G}^*) \to T^*\mathbb{G}, & p_1(x,y;\gamma) = (x;\alpha : \frac{V}{x} \twoheadrightarrow \frac{V}{x+y} \xrightarrow{\gamma} x\cap y \rightarrowtail x),\\ p_2 : T^*_{S_j}(\mathbb{G}\times\mathbb{G}^*) \to T^*\mathbb{G}^*, & p_2(x,y;\gamma) = (y;\beta : \frac{V}{y} \twoheadrightarrow \frac{V}{x+y} \xrightarrow{-\gamma} x\cap y \rightarrowtail y),\end{array} \tag{3.5}$$

where $\frac{V}{x} \twoheadrightarrow \frac{V}{x+y}$ and $\frac{V}{y} \twoheadrightarrow \frac{V}{x+y}$ (resp. $x\cap y \rightarrowtail x$ and $x\cap y\rightarrowtail y$) are the natural projection (resp. injection) maps.

Remark 3.3. With these identifications, one can easily prove that the stratification $S = \bigcup_{j=1}^p S_j$ satisfies the μ-condition (see [10, Definition 8.3.19])

$$\left(T^*_{S_j}(\mathbb{G}\times\mathbb{G}^*) \mathbin{\widehat{+}} T^*_{S_k}(\mathbb{G}\times\mathbb{G}^*)\right) \cap \pi^{-1}(S_k) \subset T^*_{S_k}(\mathbb{G}\times\mathbb{G}^*) \quad \text{for any } 1\le j<k\le p,$$

and hence is a Whitney stratification of S.

Microlocal G-actions. The G-actions on $\mathbb{G}$, $\mathbb{G}^*$ and $\mathbb{G}\times\mathbb{G}^*$ induce also natural G-actions on $T^*\mathbb{G}$ and $T^*_{S_j}(\mathbb{G}\times\mathbb{G}^*)$. In the description above, one has e.g. $g(x,\alpha) = (gx, g\alpha)$ where $g\alpha \in \mathrm{Hom}_{\mathbb{C}}(\frac{V}{gx}, gx)$ is defined as follows: the isomorphism g induces natural isomorphisms $g' \in \mathrm{Hom}_{\mathbb{C}}(\frac{V}{x}, \frac{V}{gx})$ and $g'' \in \mathrm{Hom}_{\mathbb{C}}(x, gx)$, and one sets $g\alpha = g''\circ\alpha\circ g'^{-1}$. It follows that the G-orbits in $T^*\mathbb{G}$ are

$$\{(x;\alpha)\in T^*\mathbb{G} : \mathrm{rank}\,(\alpha) = j\} \qquad (j = 0,\dots,p)$$

and the G-orbits in $T^*_{S_j}(\mathbb{G}\times\mathbb{G}^*)$ are

$$\{(x,y;\gamma)\in T^*_{S_j}(\mathbb{G}\times\mathbb{G}^*) : \mathrm{rank}\,(\gamma) = l\} \qquad (l = 0,\dots,j).$$

One argues similarly for $T^*(\mathbb{G}\times\mathbb{G})$: in particular, the G-orbits in $T^*_{\Delta_{\mathbb{G}}}(\mathbb{G}\times\mathbb{G}) \simeq T^*\mathbb{G}$ are $\{(x,x;\alpha,-\alpha) : x\in\mathbb{G},\ \alpha\in\mathrm{Hom}_{\mathbb{C}}(\frac{V}{x}, x),\ \mathrm{rank}\,(\alpha) = j\}$ $(j = 0,\dots,p)$.

The b-function associated to f. We have just observed that $(G, \mathbb{G}\times\mathbb{G}^*)$ is a prehomogeneous space. Let us compute the associated b-function $b_f(s)$ (see Appendix A). Since the problem is local, we choose local coordinates $(1_p, a', a'')$ in $\mathbb{G}$ and $\begin{pmatrix} b'' \\ b' \\ 1_p\end{pmatrix}$ in $\mathbb{G}^*$ (where $a'', b''\in M_p(\mathbb{C})$, $a'\in M_{p,n-2p}(\mathbb{C})$ and $b'\in M_{n-2p,p}(\mathbb{C})$). The function $f(\xi,\eta)$ becomes

$$f(a',a'',b',b'') = \det(b''+a'b'+a''). \tag{3.6}$$

By means of the change of variables

$$(a',a'',b',b'') \mapsto (a',a'',b',\widetilde{b}''), \qquad \widetilde{b}''(a',a'',b',b'') = b''+a'b'+a'',$$

one has $f(a',a'',b',\tilde{b}'') = \det(\widetilde{b}'')$, and hence we are locally in the situation of the determinant function in $X = \mathbb{C}^{p^2}$ (see Proposition A.6). Hence we have

$$b_f(s) = (s+1)\cdots(s+p). \tag{3.7}$$

The characteristic variety. Set

$$W = SS(\mathbb{C}_\Omega) = \mathrm{char}\,(\mathcal{K}_\Omega) \subset T^*(\mathbb{G}\times\mathbb{G}^*).$$

Proposition 3.4. *One has*

$$W = T^*_{\mathbb{G}\times\mathbb{G}^*}(\mathbb{G}\times\mathbb{G}^*) \cup \bigcup_{j=1}^{p} T^*_{S_j}(\mathbb{G}\times\mathbb{G}^*). \tag{3.8}$$

Proof. The inclusion $\subset$ holds since the S_j's form a Whitney stratification of S (see Remark 3.3); on the other hand, by (3.6) the prehomogeneous space $(G, \mathbb{G}\times\mathbb{G}^*)$ is locally isomorphic to $(GL_p(\mathbb{C}), \mathbb{C}^{p^2})$ (where the invariant function is the determinant), and then the opposite inclusion follows from the the theory of [17] since $(GL_p(\mathbb{C}), \mathbb{C}^{p^2})$ is regular and the conormal bundles to the orbits are good Lagrangians (see [9]). $\square$

The irreducible components of W are

$$\begin{aligned}
\Lambda_0 &= T^*_{\mathbb{G}\times\mathbb{G}^*}(\mathbb{G}\times\mathbb{G}^*),\\
\Lambda_j &= \overline{T^*_{S_j}(\mathbb{G}\times\mathbb{G}^*)} = \{(x,y;\gamma) : (x,y) \in \cup_{i=j}^{p} S_i,\ \gamma\in \mathrm{Hom}_{\mathbb{C}}(\tfrac{V}{x+y}, x\cap y),\\
&\qquad\qquad \mathrm{rank}\,\gamma \leq j\} \qquad (j=1,\ldots,p-1),\\
\Lambda_p &= T^*_{S_p}(\mathbb{G}\times\mathbb{G}^*).
\end{aligned}$$

Therefore (see Appendix A) the local b-functions on the Λ_j's are

$$\begin{aligned}
b_{\Lambda_0}(s) &\equiv 1,\\
b_{\Lambda_j}(s) &= (s+1)\cdots(s+j) \qquad (j=1,\ldots,p),.
\end{aligned} \tag{3.9}$$

The contact transformation. The microlocal correspondence associated to our transforms is

$$T^*\mathbb{G} \xleftarrow{p_1} W \xrightarrow{p_2^a} T^*\mathbb{G}^* \tag{3.10}$$

(recall that p_2^a denotes the composition of p_2 with the antipodal map a of $T^*\mathbb{G}^*$). Let us consider the open dense subsets

$$\begin{aligned}
U &= \{(x;\alpha)\in T^*\mathbb{G} : \mathrm{rank}\,\alpha = p\} \subset \dot{T}^*\mathbb{G},\\
U^* &= \{(y;\beta)\in T^*\mathbb{G}^* : \mathrm{rank}\,\beta = p\} \subset \dot{T}^*\mathbb{G}^*\\
\Lambda &= \{(x,y;\gamma)\in T^*_{S_p}(\mathbb{G}\times\mathbb{G}^*) : \mathrm{rank}\,\gamma = p\} \subset \dot{T}^*_{S_p}(\mathbb{G}\times\mathbb{G}^*).
\end{aligned} \tag{3.11}$$

Note that Λ is a G-orbit in $T^*_{S_p}(\mathbb{G}\times\mathbb{G}^*)$.

Proposition 3.5. *Conditions* (1.11), (1.12) *and* (1.13) *are satisfied in our case.*

Proof. This follows easily from (3.8), (3.4) and (3.5). $\square$

Example 3.6. In the case of projective duality, the microlocal correspondence (3.10) induces a globally defined contact transformation between $\dot{T}^*\mathbb{P}$ and $\dot{T}^*\mathbb{P}^*$, since $\Lambda = \dot{T}^*_S(\mathbb{P}\times\mathbb{P}^*)$ (see [4]). The b-function is $b_f(s) = s+1$.

In particular, by Proposition 1.17(i), $\Lambda^0 = \Lambda \circ \widetilde{\Lambda}$ is a smooth Lagrangian submanifold of $T^*_{\Delta_{\mathbb{G}}}(\mathbb{G}\times\mathbb{G}^*)$. In fact, Λ^0 is the open dense G-orbit in $T^*_{\Delta_{\mathbb{G}}}(\mathbb{G}\times\mathbb{G}^*)$:

$$\begin{aligned}\Lambda_0 \;\simeq\; & \Big\{(x,x;\alpha,-\alpha) : x\in\mathbb{G},\ \alpha\in \mathrm{Hom}_{\mathbb{C}}(\tfrac{V}{x},x),\ \exists\, y\in\mathbb{G}^* \\ & \qquad \text{s.t. } x\subset y,\ \exists\,\gamma\in \mathrm{Iso}\,(\tfrac{V}{y},x):\ \alpha:\tfrac{V}{x}\to\tfrac{V}{y}\xrightarrow{\gamma}x\Big\} \\ =\; & \Big\{(x,x;\alpha,-\alpha) : x\in\mathbb{G},\ \alpha\in\mathrm{Hom}_{\mathbb{C}}(\tfrac{V}{x},x),\ \mathrm{rank}\,\alpha = p\Big\}.\end{aligned} \tag{3.12}$$

3.3 Equivalences of derived categories

We show that the geometric hypotheses for the invertibility of K_Ω and $K^*_{\widetilde{\Omega}}$ are fulfilled in this case.

Lemma 3.7. *The triplet* $(\mathbb{G},\mathbb{G}^*;\Omega)$ *satisfies hypotheses* (2.3), (2.4), (2.5) *and* (2.3)', (2.4)', (2.5)'.

Proof. The hypotheses are symmetric in $\mathbb{G}$ and $\mathbb{G}^*$; hence it is enough to check (2.3), (2.4) and (2.5).

Condition (2.3) is clearly verified.

In the above description (3.4) of $T^*_{S_j}(\mathbb{G}\times\mathbb{G}^*)$, notice that if $\alpha=0$, then $\gamma=0$ and then also $\beta=0$. Thus,

$$T^*_{S_j}(\mathbb{G}\times\mathbb{G}^*)\cap(T^*_{\mathbb{G}}\mathbb{G}\times T^*\mathbb{G}^*)\subset T^*_{\mathbb{G}\times\mathbb{G}^*}(\mathbb{G}\times\mathbb{G}^*)$$

for any $j=1,\dots,p$, and hence (2.5) is satisfied thanks to (3.8).

Finally, in order to prove (2.4), let $x,x'\in\mathbb{G}$ with $x\neq x'$; in general, we have $\dim(x\cap x')=j$ with $0\le j\le p-1$, and hence let us choose a basis $\{v_1,\dots,v_n\}$ of V such that $x=\langle v_1,\dots,v_p\rangle$ and $x'=\langle v_1,\dots,v_j,v_{p+1},\dots,v_{2p-j}\rangle$. In Stiefel coordinates, we have

$$x=\begin{pmatrix}1_j & 0 & 0 & 0\\ 0 & 1_{p-j} & 0 & 0\end{pmatrix}\quad\text{and}\quad x'=\begin{pmatrix}1_j & 0 & 0 & 0\\ 0 & 0 & 1_{p-j} & 0\end{pmatrix},$$

where the orders of the row blocks are j and $p-j$, and the orders of the column blocks are j, $p-j$, $p-j$ and $n-2p+j$. On the other hand, Ω_x is an affine chart of $\mathbb{G}^*$ (and hence $\mathrm{R}\Gamma(\Omega_x;\mathbb{C}_{\Omega_x})\simeq\mathbb{C}$): in terms of dual Stiefel coordinates, we have

$$\Omega_x=\left\{y(b)=\begin{pmatrix}1_j & 0\\ 0 & 1_{p-j}\\ b_1 & b_3\\ b_2 & b_4\end{pmatrix}\right\}\simeq\mathbb{C}^N,$$

where the orders of the row blocks are j, $p-j$, $p-j$ and $n-2p+j$, and the orders of the column blocks are j and $p-j$. Thus, we have $\Omega_x\setminus\Omega_{x'}=\{y(b)\in\Omega_x : \det(b_3)=0\}$, a closed conic subset of Ω_x. Therefore we have $\mathrm{R}\Gamma(\Omega_x\setminus\Omega_{x'};\mathbb{C}_{\Omega_x\setminus\Omega_{x'}})\simeq\mathbb{C}$, and hence $\mathrm{R}\Gamma(\Omega_x;\mathbb{C}_{\Omega_{x'}})=0$. The proof is complete. $\square$

Theorem 3.8. (i) *The functors* $\cdot \circ K_\Omega$ *and* $\cdot \circ K^*_{\widetilde{\Omega}}$ *are quasi-inverse to each other, and thus they define equivalences of categories between* $\mathbf{D}^{\mathrm{b}}(\mathbb{C}_{\mathbb{G}})$ *and* $\mathbf{D}^{\mathrm{b}}(\mathbb{C}_{\mathbb{G}^*})$*. Moreover, they induce equivalences between* $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_{\mathbb{G}})$ *and* $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_{\mathbb{G}^*})$ *as well as between* $\mathbf{D}^{\mathrm{b}}_{\mathbb{C}-c}(\mathbb{C}_{\mathbb{G}})$ *and* $\mathbf{D}^{\mathrm{b}}_{\mathbb{C}-c}(\mathbb{C}_{\mathbb{G}^*})$*.*

(ii) *Similarly, the functors* $\cdot \underline{\circ}\, \mathcal{K}_\Omega$ *and* $\cdot \underline{\circ}\, \mathcal{K}^*_{\widetilde{\Omega}}$ *are quasi-inverse to each other, and thus they define equivalences of categories between* $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_{\mathbb{G}})$ *and* $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_{\mathbb{G}^*})$*. Moreover, they induce equivalences between* $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_{\mathbb{G}})$ *and* $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_{\mathbb{G}^*})$ *as well as between* $\mathbf{D}^{\mathrm{b}}_{rh}(\mathcal{D}_{\mathbb{G}})$ *and* $\mathbf{D}^{\mathrm{b}}_{rh}(\mathcal{D}_{\mathbb{G}^*})$*.*

Proof. This is a consequence of Lemma 3.7 and Theorem 2.4. $\square$

Remark 3.9. One has $\mathrm{char}\,(\mathcal{K}_\Omega \underline{\circ}\, \mathcal{K}^*_{\widetilde{\Omega}}) \subset W \circ \widetilde{W}$ (see (1.7)), where $W = \mathrm{char}\,(\mathcal{K}_\Omega)$ and $\widetilde{W} = {}^t r'^{-1}(W) = \mathrm{char}\,(\mathcal{K}^*_{\widetilde{\Omega}})$. In fact, this is a bad estimation in our case (due to the non-smoothness of W), since one can compute that

$$W \circ \widetilde{W} \simeq \Big\{(x, x'; \delta) : x, x' \in \mathbb{G},\ \delta \in \mathrm{Hom}_{\mathbb{C}}(\tfrac{V}{x+x'}, x \cap x')\Big\} \subset T^*(\mathbb{G} \times \mathbb{G})$$

whereas $\mathrm{char}\,(\mathcal{K}_\Omega \underline{\circ}\, \mathcal{K}^*_{\widetilde{\Omega}}) = \mathrm{char}(\mathcal{B}_{\Delta_{\mathbb{G}}}) = T^*_{\Delta_{\mathbb{G}}}(\mathbb{G} \times \mathbb{G})$ by Theorem 3.8.

3.4 Quantization

In this section we want to describe concretely the action of the quasi-inverse functors $\cdot \underline{\circ}\, \mathcal{K}_\Omega$ and $\cdot \underline{\circ}\, \mathcal{K}^*_{\widetilde{\Omega}}$ on a certain class of locally free $\mathcal{D}$-modules. More precisely, we consider the $\mathcal{D}$-modules $\mathcal{D}_{\mathbb{G}} \otimes_{\mathcal{O}_{\mathbb{G}}} \mathcal{F}$ associated to a holomorphic line bundle $\mathcal{F}$ on $\mathbb{G}$.

The holomorphic line bundles on $\mathbb{G}$***.*** It is well-known that the Picard group $\mathrm{Pic}\,(\mathbb{G})$ of $\mathbb{G}$ (i.e., the set of isomorphism classes of holomorphic line bundles on $\mathbb{G}$ endowed with the operation $\otimes_{\mathcal{O}_{\mathbb{G}}}$) is isomorphic to $\mathbb{Z}$. In fact one has

$$\mathrm{Pic}\,(\mathbb{G}) = \{\mathcal{O}_{\mathbb{G}}(\lambda) : \lambda \in \mathbb{Z}\},$$

where $\mathcal{O}_{\mathbb{G}}(\lambda)$ is the holomorphic line bundle on $\mathbb{G}$ whose sections over an open subset U are (see (3.2))

$$\Gamma(U; \mathcal{O}_{\mathbb{G}}(\lambda)) = \{s \in \Gamma(q^{-1}(U); \mathcal{O}_{F_p(V)}) : s(Av) = (\det A)^\lambda s(v)\ \ \forall\, A \in GL_p(\mathbb{C})\}.$$

Remark 3.10. One has $\mathcal{O}_{\mathbb{G}} \simeq \mathcal{O}_{\mathbb{G}}(0)$, $\Omega_{\mathbb{G}} \simeq \mathcal{O}_{\mathbb{G}}(-n)$ and $\mathcal{O}_{\mathbb{G}}(\lambda)^* \simeq \mathcal{O}_{\mathbb{G}}(-\lambda)$ for any $\lambda \in \mathbb{Z}$. Moreover, $\mathcal{O}_{\mathbb{G}}(-1)$ is the determinant of the tautological holomorphic vector bundle of rank p on $\mathbb{G}$, i.e. the subbundle of $\mathbb{G} \times V$ whose fiber over $x \in \mathbb{G}$ is the p-vector space $x \subset V$ itself.

A generalization of Leray's form. Since $\Omega_{\mathbb{G}} \simeq \mathcal{O}_{\mathbb{G}}(-n)$, there is a natural isomorphism of holomorphic line bundles (determined up to a nonzero multiplicative constant) $\Omega_{\mathbb{G}} \otimes_{\mathcal{O}_{\mathbb{G}}} \mathcal{O}_{\mathbb{G}}(n) \xrightarrow{\sim} \mathcal{O}_{\mathbb{G}}$, which holds in particular at the level of global sections:

$$\Gamma(\mathbb{G}, \Omega_{\mathbb{G}} \otimes_{\mathcal{O}_{\mathbb{G}}} \mathcal{O}_{\mathbb{G}}(n)) \xrightarrow{\sim} \Gamma(\mathbb{G}, \mathcal{O}_{\mathbb{G}}) \simeq \mathbb{C}.$$

Hence we get a nowhere vanishing section

$$\omega^*_{\mathbb{G}} \in \Gamma(\mathbb{G}, \Omega_{\mathbb{G}} \otimes_{\mathcal{O}_{\mathbb{G}}} \mathcal{O}_{\mathbb{G}}(n)), \tag{3.13}$$

which generalizes in a natural way the twisted form (see Leray [13])

$$\omega^*_{\mathbb{P}} = \sum_{j=0}^{n-1} (-1)^j \xi_j \, d\xi_0 \wedge \cdots \wedge \widehat{d\xi_j} \wedge \cdots \wedge d\xi_{n-1} \ \in \Gamma(\mathbb{P}, \Omega_{\mathbb{P}} \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{O}_{\mathbb{P}}(n))$$

on the complex projective space $\mathbb{P} \ni [\xi_0, \ldots, \xi_{n-1}]$ when $p = 1$.

Quantization. Let us set

$$\lambda^* = -n - \lambda.$$

To any $\mathcal{O}_{\mathbb{G}}(\lambda)$ one associates the locally free $\mathcal{D}_{\mathbb{G}}$-module of rank one

$$\mathcal{D}_{\mathbb{G}}(\lambda) = \mathcal{D}_{\mathbb{G}} \otimes_{\mathcal{O}_{\mathbb{G}}} \mathcal{O}_{\mathbb{G}}(\lambda),$$

and similarly for $\mathbb{G}^*$. Our aim is *to show that the image of $\mathcal{D}_{\mathbb{G}}(-\lambda)$ by the integral transforms $\cdot \circ \mathcal{K}_\Omega$ or $\cdot \circ \mathcal{K}^*_\Omega$ (according to λ) is isomorphic to $\mathcal{D}_{\mathbb{G}^*}(-\lambda^*)$.* An initial remark is the concentration in degree zero of these transforms (this will also follow later by other methods, using Proposition 1.13).

Proposition 3.11. *For any $\mu \in \mathbb{Z}$ the $\mathcal{D}_{\mathbb{G}^*}$-modules $\mathcal{D}_{\mathbb{G}}(\mu) \circ \mathcal{K}_\Omega$ and $\mathcal{D}_{\mathbb{G}}(\mu) \circ \mathcal{K}^*_\Omega$ are concentrated in degree zero.*

Proof. It is convenient to work in the algebraic setting. Let $\mathbb{G}_{al}$ be the algebraic manifold underlying to $\mathbb{G}$, $\mathcal{O}^{al}_{\mathbb{G}}$ the structural sheaf, $\Omega^{al}_{\mathbb{G}}$ the canonical sheaf and $\mathcal{D}^{al}_{\mathbb{G}}$ the sheaf of linear algebraic differential operators on $\mathbb{G}_{al}$. The canonical morphism of $\mathbb{C}$-ringed spaces $\mathbb{G} \to \mathbb{G}_{al}$ defines a canonical functor $(\cdot)_{an} : \mathbf{D}^{\mathrm{b}}(\mathcal{D}^{al}_{\mathbb{G}}) \to \mathbf{D}^{\mathrm{b}}(\mathcal{D}_{\mathbb{G}})$. Since $\mathcal{D}_{\mathbb{G}}(\mu) \circ \mathcal{K}_\Omega \simeq (\mathcal{D}^{al}_{\mathbb{G}}(\mu) \circ \mathcal{K}^{al}_\Omega)_{an}$, where $\mathcal{K}^{al}_\Omega \simeq j_* j^{-1} \mathcal{O}^{al}_{\mathbb{G}\times\mathbb{G}^*}$ (recall that $\mathcal{K}_\Omega \simeq \mathcal{O}_{\mathbb{G}\times\mathbb{G}^*}(*S)$), it is enough to show that $\mathcal{D}^{al}_{\mathbb{G}}(\mu) \circ \mathcal{K}^{al}_\Omega$ is concentrated in degree zero. Set $\dot{q}_i = q_i|_\Omega$ $(i = 1, 2)$. From an algebraic analogous of Proposition 1.9, recalling that $\Omega^{al}_{\mathbb{G}} \simeq \mathcal{O}^{al}_{\mathbb{G}}(-n)$ and that q_2 is proper we get:

$$\begin{aligned} \mathcal{D}^{al}_{\mathbb{G}}(\mu) \circ \mathcal{K}^{al}_\Omega &\simeq Rq_{2*} Rj_* j^{-1} (q_1^{-1} \mathcal{O}^{al}_{\mathbb{G}}(\mu - n) \otimes_{q_1^{-1}\mathcal{O}^{al}_{\mathbb{G}}} \mathcal{O}^{al}_{\mathbb{G}\times\mathbb{G}^*}) \\ &\simeq R\dot{q}_{2*} (\dot{q}_1^{-1} \mathcal{O}^{al}_{\mathbb{G}}(\mu - n) \otimes_{\dot{q}_1^{-1}\mathcal{O}^{al}_{\mathbb{G}}} \mathcal{O}^{al}_\Omega). \end{aligned}$$

Then the conclusion follows since $\dot{q}_2$ has affine fibers. Using Proposition 1.3, one may argue by duality for $\mathcal{D}_{\mathbb{G}}(\mu) \circ \mathcal{K}^*_\Omega$. □

Let us write for short

$$\mathcal{K}^{(N,0)}_\Omega(\mu, \nu) = \mathcal{K}^{(N,0)}_\Omega(\mathcal{O}_{\mathbb{G}}(\mu), \mathcal{O}_{\mathbb{G}^*}(\nu)).$$

Proposition 3.12. *There is a natural isomorphism*

$$\alpha_\Omega : \Gamma(\mathbb{G} \times \mathbb{G}^*; \mathcal{K}^{(N,0)}_\Omega(-\lambda, \lambda^*)) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\mathcal{D}_{\mathbb{G}^*})}(\mathcal{D}_{\mathbb{G}^*}(-\lambda^*), \mathcal{D}_{\mathbb{G}}(-\lambda) \circ \mathcal{K}_\Omega),$$

and a similar isomorphism α^*_Ω with $\mathcal{K}_\Omega$ replaced by $\mathcal{K}^*_\Omega$.

Proof. Apply Proposition 1.10 for $X = \mathbb{G}$, $Y = \mathbb{G}^*$, $\mathcal{K} = \mathcal{K}_\Omega$ (resp. $\mathcal{K} = \mathcal{K}^*_\Omega$), $\mathcal{F} = \mathcal{O}_\mathbb{G}(-\lambda)$ and $\mathcal{G} = \mathcal{O}_{\mathbb{G}^*}(-\lambda^*)$. $\square$

By (1.6), (1.4) and Theorem 3.8 we get the following composition morphism, where we write $\alpha_{-\lambda} = \alpha_{\mathcal{O}_\mathbb{G}(-\lambda)}$ for short:

$$\Gamma(\mathbb{G}\times\mathbb{G}^*; \mathcal{K}_\Omega^{(N,0)}(-\lambda,\lambda^*)) \otimes \Gamma(\mathbb{G}^*\times\mathbb{G}; \mathcal{K}^{*\,(N,0)}_{\tilde\Omega}(-\lambda^*,\lambda))$$
$$\xrightarrow{\circ} \Gamma(\mathbb{G}\times\mathbb{G}; \mathcal{B}^{(N,0)}_{\Delta_\mathbb{G}}(-\lambda,\lambda)) \xrightarrow{\alpha_{-\lambda}} \mathrm{End}_{\mathbf{D}^{\mathrm{b}}(\mathcal{D}_\mathbb{G})}(\mathcal{D}_\mathbb{G}(-\lambda)),$$

and similarly for $\mathcal{K}^*_\Omega$. Our aims are:

(1) to find some sections

$$s_\lambda \in \Gamma(\mathbb{G}\times\mathbb{G}^*; \mathcal{K}_\Omega^{(N,0)}(-\lambda,\lambda^*)) \quad \text{and} \quad s^*_\lambda \in \Gamma(\mathbb{G}\times\mathbb{G}^*; \mathcal{K}^{*\,(N,0)}_\Omega(-\lambda,\lambda^*))$$

(and, in the other direction, similar sections $\tilde{s}_{\lambda^*} \in \Gamma(\mathbb{G}^*\times\mathbb{G}; \mathcal{K}^{(N,0)}_{\tilde\Omega}(-\lambda^*,\lambda))$ and $\tilde{s}^*_{\lambda^*} \in \Gamma(\mathbb{G}^*\times\mathbb{G}; \mathcal{K}^{*\,(N,0)}_{\tilde\Omega}(-\lambda^*,\lambda))$;

(2) to show that they are microlocal generators of the regular holonomic $\mathcal{E}$-modules $\mathcal{E}\mathcal{K}_\Omega$ and $\mathcal{E}\mathcal{K}^*_\Omega$ on Λ (the graph of the contact transformation in (3.11));

(3) to show that $s_\lambda \circ \tilde{s}^*_{\lambda^*} = \delta_{\mathbb{G},-\lambda}$ and $\tilde{s}^*_{\lambda^*} \circ s_\lambda = \delta_{\mathbb{G}^*,-\lambda^*}$ (up to a nonzero multiplicative constant).

Let $f(\xi,\eta)$ be the function on $\mathbb{G}\times\mathbb{G}^*$ defined in (3.3), and let $P(s) = P(\xi,\eta,\partial_\xi,\partial_\eta; s)$ be a section of $\mathcal{D}_{\mathbb{G}\times\mathbb{G}^*}[s]$ such that $P(s)f^{s+1} = b_f(s)f^s$, where $b_f(s) = (s+1)\cdots(s+p)$ is the b-function associated to f. Recall (see Appendix A) that $\mathcal{K}_\Omega = \mathcal{O}_{\mathbb{G}\times\mathbb{G}^*}(*S)$, and that $\mathcal{K}^*_\Omega = \underline{\mathrm{D}}\,(\mathcal{O}_{\mathbb{G}\times\mathbb{G}^*}(*S))$ has a canonical generator Y_f and a canonical section $\partial^a Y_f = \prod_{j=1}^a P(-j)Y_f$ for any $a \in \mathbb{Z}_{\geq 1}$. We set:

$$\begin{aligned} s_\lambda(\xi,\eta) &= f(\xi,\eta)^{\lambda^*}\omega^*_\mathbb{G}(\xi) && \in \Gamma(\mathbb{G}\times\mathbb{G}^*; \mathcal{K}_\Omega^{(N,0)}(-\lambda,\lambda^*)) \\ s^*_\lambda(\xi,\eta) &= \begin{cases} f(\xi,\eta)^{\lambda^*}\, Y_f\ \omega^*_\mathbb{G}(\xi) & (\lambda^* \geq 0) \\ \partial^{-\lambda^*} Y_f\ \omega^*_\mathbb{G}(\xi) & (\lambda^* < 0) \end{cases} && \in \Gamma(\mathbb{G}\times\mathbb{G}^*; \mathcal{K}^{*\,(N,0)}_\Omega(-\lambda,\lambda^*)), \end{aligned}$$

where $\omega^*_\mathbb{G}(\xi)$ is the twisted form on $\mathbb{G}$ described in (3.13).

Remark 3.13. We have observed that the prehomogeneous space $(G, \mathbb{G}\times\mathbb{G}^*)$ is locally isomorphic to $(GL_p(\mathbb{C}), \mathbb{C}^{p^2})$. Therefore in our case the operator P is locally the determinant of a matrix of partial derivatives (see Remark A.7) and hence it does not depend on s. In particular, one has $\partial^a Y_f = P^a\, Y_f$ for any $a \in \mathbb{Z}_{\geq 1}$.

Lemma 3.14. *The section s_λ (resp. s^*_λ) is a generator of $\mathcal{E}\mathcal{K}_\Omega$ (resp. $\mathcal{E}\mathcal{K}^*_\Omega$) on Λ for any $\lambda \geq -n+p$ (resp. for any $\lambda \in \mathbb{Z}$).*

Proof. Let $b_\Lambda(s)$ be the local b-function on Λ: then one has $b_\Lambda(s) = b_f(s) = (s+1)\cdots(s+p)$ since Λ is contained in the conormal bundle $T^*_{S_p}(\mathbb{G}\times\mathbb{G}^*)$, and S_p is the orbit of minimal dimension (see (3.9)). By Proposition A.5, the section $f(\xi,\eta)^{\lambda^*}$ generates $\mathcal{EK}_\Omega$ on Λ if and only if $b_\Lambda(\lambda^*-\nu)\neq 0$ for any $\nu\in\mathbb{Z}_{\geq 1}$, i.e. if and only if $\lambda \geq -n+p$. On the other hand, the section $f(\xi,\eta)^{\lambda^*}Y_f$ generates $\mathcal{EK}^*_\Omega$ on Λ if and only if $b_\Lambda(\lambda+\nu)\neq 0$ for any $\nu\in\mathbb{Z}_{\geq 0}$, i.e. if and only if $\lambda\geq 0$. Finally, the section $\partial^a Y_f$ is a generator of $\mathcal{EK}^*_\Omega$ on Λ for any $a\in\mathbb{Z}_{>0}$, since the principal symbol $\sigma(P)$ does not vanish on Λ: namely, for any $(x,y;\gamma)\in\Lambda$ (where $\gamma : \frac{V}{y}\xrightarrow{\sim} x$, see (3.11)) one has $\sigma(P)(x,y;\gamma) = \det(\gamma)\neq 0$. □

Proposition 3.15. *For any $\lambda\geq -n+p$, one has $s_\lambda\circ\tilde{s}^*_{\lambda^*} = \delta_{\mathbb{G},-\lambda}$ and $\tilde{s}^*_{\lambda^*}\circ s_\lambda = \delta_{\mathbb{G}^*,-\lambda^*}$ (up to a multiplicative constant).*

Proof. We shall only prove that $s_\lambda\circ\tilde{s}^*_{\lambda^*} = \delta_{\mathbb{G},-\lambda}$ for any $\lambda\geq -n+p$, since the other statement can be verified by similar arguments. By Proposition 3.5, the conditions (1.11), (1.12) and (1.13) are satisfied. Therefore, setting $\Lambda^0 = \Lambda\circ\widetilde{\Lambda}\subset T^*_{\Delta_\mathbb{G}}(\mathbb{G}\times\mathbb{G})$ (see (3.12)) and recalling that $\mathcal{K}_\Omega \underline{\circ}\, \mathcal{K}^*_{\widetilde{\Omega}} \simeq \mathcal{B}_{\Delta_\mathbb{G}}$ (Theorem 3.8), by Proposition 1.17(i) we get that $s_\lambda|_\Lambda\circ_\mu \tilde{s}^*_{\lambda^*}|_{\widetilde{\Lambda}}$ is a well-defined section of $\Gamma(\Lambda^0;\mathcal{C}^{(N,0)}_{\Delta_\mathbb{G}}(-\lambda,\lambda))$ and coincides with $s_\lambda\circ\tilde{s}^*_{\lambda^*}|_{\Lambda^0}$. Moreover, by Lemma 3.14 and Proposition 1.17(ii) the section $s_\lambda\circ\tilde{s}^*_{\lambda^*}|_{\Lambda^0}$ is a generator of $\mathcal{C}_{\Delta_\mathbb{G}}$ on Λ^0 for any $\lambda\geq -n+p$. Since $\mathcal{C}_{\Delta_\mathbb{G}}$ is simple and Λ^0 is a G-orbit of $T^*_{\Delta_\mathbb{G}}(\mathbb{G}\times\mathbb{G})$, there is a unique G-invariant generator (up to a multiplicative constant) of $\Gamma(\Lambda^0;\mathcal{C}^{(N,0)}_{\Delta_\mathbb{G}}(-\lambda,\lambda))$. (Namely, a generator u is univoquely determined by its principal symbol $\sigma(u)$, and if u_1 and u_2 are G-invariant generators on the G-orbit Λ^0, then $\sigma(u_1) = c\sigma(u_2)$ on Λ^0 for some nonzero constant c: this implies $u_1 = cu_2$.) The restriction of the canonical section $\delta_{\mathbb{G},-\lambda}|_{\Lambda^0}$ is obviously G-invariant, and so is $s_\lambda\circ\tilde{s}^*_{\lambda^*}|_{\Lambda^0}$ by construction: therefore we get $s_\lambda\circ\tilde{s}^*_{\lambda^*}|_{\Lambda^0} = \delta_{\mathbb{G},-\lambda}|_{\Lambda^0}$ for any $\lambda\geq -n+p$ (up to a multiplicative constant). Finally, since Λ^0 is a nonempty (in fact, dense) open subset of $T^*_{\Delta_\mathbb{G}}(\mathbb{G}\times\mathbb{G})$ and both $s_\lambda\circ\tilde{s}^*_{\lambda^*}$ and $\delta_{\mathbb{G},-\lambda}$ are globally defined sections of $\mathcal{C}^{(N,0)}_{\Delta_\mathbb{G}}(-\lambda,\lambda)$, they coincide (up to a nonzero multiplicative constant) on all of $T^*_{\Delta_\mathbb{G}}(\mathbb{G}\times\mathbb{G})$ by analytic continuation. □

One proves in a similar way that $s^*_\lambda\circ\tilde{s}_{\lambda^*} = \delta_{\mathbb{G},-\lambda}$ and $\tilde{s}_{\lambda^*}\circ s^*_\lambda = \delta_{\mathbb{G},-\lambda^*}$. By Propositions 3.15 and 1.13 we get that $\alpha_{\mathcal{K}_\Omega}(s_\lambda)$ (resp. $\alpha_{\mathcal{K}^*_\Omega}(s^*_\lambda)$) is invertible for any $\lambda\geq -n+p$ (resp. $\lambda\leq -p$), the inverse morphism being the image of $\alpha_{\mathcal{K}^*_{\widetilde{\Omega}}}(\tilde{s}^*_{\lambda^*})$ (resp. $\alpha_{\mathcal{K}_{\widetilde{\Omega}}}(\tilde{s}_{\lambda^*})$) by the functor $\cdot\,\underline{\circ}\,\mathcal{K}_\Omega$ (resp. $\cdot\,\underline{\circ}\,\mathcal{K}^*_\Omega$) and hence we obtain:

Theorem 3.16. *One has $\mathcal{D}$-linear isomorphisms:*

(i) $\mathcal{D}_\mathbb{G}(-\lambda)\,\underline{\circ}\,\mathcal{K}_\Omega \xleftarrow{\sim} \mathcal{D}_{\mathbb{G}^*}(-\lambda^*)$ *for any $\lambda\geq -n+p$;*

(ii) $\mathcal{D}_\mathbb{G}(-\lambda)\,\underline{\circ}\,\mathcal{K}^*_\Omega \xleftarrow{\sim} \mathcal{D}_{\mathbb{G}^*}(-\lambda^*)$ *for any $\lambda\leq -p$.*

Remark 3.17. (1) In fact, the statement (ii) of Theorem 3.16 is an immediate consequence of (i) and Proposition 1.3, since in our case the dualizing complex $\mathcal{K}_\mathbb{G}$ is isomorphic to $\mathcal{D}_\mathbb{G}(n)[N]$.

(2) Observe that there is an overlap in the ranges of λ in (i) and (ii), which are both valid for $-n+p \leq \lambda \leq -p$ (recall that we assumed $p \leq n/2$).

Applying $\mathcal{S}ol(\,\cdot\,)$ to both sides of (i) and (ii) in Theorem 3.16 and recalling Corollary 1.6, we obtain the analogous results for the complexes of solutions:

Corollary 3.18. *One has $\mathbb{C}$-linear isomorphisms:*

(i) $\mathcal{O}_{\mathbb{G}}(\lambda) \circ K_{\Omega} \xrightarrow{\sim} \mathcal{O}_{\mathbb{G}^*}(\lambda^*)[-N]$ *for any $\lambda \geq -n+p$;*

(ii) $\mathcal{O}_{\mathbb{G}}(\lambda) \circ K^*_{\Omega} \xrightarrow{\sim} \mathcal{O}_{\mathbb{G}^*}(\lambda^*)[-N]$ *for any $\lambda \leq -p$.*

4 Applications

4.1 Integral transforms defined by the incidence relation

Let us treat the integral transform given by the regular holonomic kernel

$$\mathcal{K}_S = \mathcal{T}hom(\mathbb{C}_S[-1], \mathcal{O}_{\mathbb{G}\times\mathbb{G}^*}),$$

which is used in the classical approach to projective duality (see [2], [4]). We recall the following well-known fact (Bott-Cartan-Serre) on the twisted holomorphic cohomology of $\mathbb{G}$:

$$\begin{cases} \Gamma(\mathbb{G}; \mathcal{O}_{\mathbb{G}}(\lambda)) & = & \begin{cases} 0 \text{ for } \lambda < 0, \\ \neq 0 \text{ and finite dimensional for } \lambda \geq 0, \end{cases} \\ H^j(\mathbb{G}; \mathcal{O}_{\mathbb{G}}(\lambda)) & = & 0 \quad \text{for } 0 < j < N \text{ and for every } \lambda, \\ H^N(\mathbb{G}; \mathcal{O}_{\mathbb{G}}(\lambda))' & \simeq & \Gamma(\mathbb{G}; \mathcal{O}_{\mathbb{G}}(\lambda^*)), \end{cases} \tag{4.1}$$

where $N = d_{\mathbb{G}}$ and $(\cdot)'$ denotes the dual of a finite dimensional complex vector space. In particular, from (4.1) one has

$$\mathrm{R}\Gamma(\mathbb{G}; \mathcal{O}_{\mathbb{G}}(\lambda)) = \mathrm{R}\Gamma(\mathbb{G}^*; \mathcal{O}_{\mathbb{G}^*}(\lambda^*)) \equiv 0 \quad \text{for any } -n+1 \leq \lambda \leq -1. \tag{4.2}$$

Applying the functor $\mathcal{D}_{\mathbb{G}}(-\lambda) \underline{\circ}\, \mathcal{T}hom(\,\cdot\,, \mathcal{O}_{\mathbb{G}\times\mathbb{G}^*})$ to the distinguished triangle

$$\mathbb{C}_S[-1] \to \mathbb{C}_{\Omega} \to \mathbb{C}_{\mathbb{G}\times\mathbb{G}^*} \xrightarrow{+1}, \tag{4.3}$$

we get

$$\mathcal{D}_{\mathbb{G}}(-\lambda) \underline{\circ}\, \mathcal{O}_{\mathbb{G}\times\mathbb{G}^*} \to \mathcal{D}_{\mathbb{G}}(-\lambda) \underline{\circ}\, \mathcal{K}_{\Omega} \to \mathcal{D}_{\mathbb{G}}(-\lambda) \underline{\circ}\, \mathcal{K}_S \xrightarrow{+1}. \tag{4.4}$$

On the other hand, we have

$$\begin{aligned} \mathcal{D}_{\mathbb{G}}(-\lambda) \underline{\circ}\, \mathcal{O}_{\mathbb{G}\times\mathbb{G}^*} & \simeq Rq_{2!}(q_1^{-1}\mathcal{D}_{\mathbb{G}}(-\lambda) \otimes_{q_1^{-1}\mathcal{D}_{\mathbb{G}}} \mathcal{O}^{(N,0)}_{\mathbb{G}\times\mathbb{G}^*}) \\ & \simeq Rq_{2!}(q_1^{-1}\mathcal{O}_{\mathbb{G}}(\lambda^*) \otimes_{q_1^{-1}\mathcal{O}_{\mathbb{G}}} \mathcal{O}_{\mathbb{G}\times\mathbb{G}^*}) \\ & \simeq \mathrm{R}\Gamma(\mathbb{G}; \mathcal{O}_{\mathbb{G}}(\lambda^*)) \otimes \mathcal{O}_{\mathbb{G}^*}, \end{aligned} \tag{4.5}$$

where the first isomorphism is in Proposition 1.9, the second holds since $\Omega_{\mathbb{G}} \simeq \mathcal{O}_{\mathbb{G}}(-n)$ and the third follows from the analytic Künneth formula and the finiteness (see (4.1)) of the holomorphic cohomology of $\mathbb{G}$. Hence by (4.4) and (4.5) we may conclude that $\mathcal{D}_{\mathbb{G}}(-\lambda) \underline{\circ} \mathcal{K}_S \simeq \mathcal{D}_{\mathbb{G}}(-\lambda) \underline{\circ} \mathcal{K}_\Omega$ if and only if $\mathrm{R}\Gamma(\mathbb{G}; \mathcal{O}_{\mathbb{G}}(\lambda^*)) = 0$. By (4.2), this is verified when $-n+1 \leq \lambda^* \leq -1$, i.e. $-n+1 \leq \lambda \leq -1$. One can argue similarly for the kernel $\mathcal{K}_S^* = \mathcal{T}hom(D'\mathbb{C}_S[-1], \mathcal{O}_{\mathbb{G}\times\mathbb{G}^*})$ (or again by duality, using Proposition 1.3) and therefore, by Theorem 3.16 we get

Proposition 4.1. *One has $\mathcal{D}$-linear isomorphisms:*

(i) $\mathcal{D}_{\mathbb{G}}(-\lambda) \underline{\circ} \mathcal{K}_S \xleftarrow{\sim} \mathcal{D}_{\mathbb{G}^*}(-\lambda^*)$ *for any* $-n+p \leq \lambda \leq -1$;

(ii) $\mathcal{D}_{\mathbb{G}}(-\lambda) \underline{\circ} \mathcal{K}_S^* \xleftarrow{\sim} \mathcal{D}_{\mathbb{G}^*}(-\lambda^*)$ *for any* $-n+1 \leq \lambda \leq -p$.

Remark 4.2. When $p = 1$ one has $\mathcal{K}_S \simeq \mathcal{K}_S^* \simeq \mathcal{B}_{S|\mathbb{P}\times\mathbb{P}^*}$, and we recover Theorem 4.3 of D'Agnolo-Schapira [4].

4.2 Adjunction formulas and examples

From Proposition 1.8, we get the following adjunction formulas.

Proposition 4.3. *For any $-n+p \leq \lambda \leq -p$ and $F \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_{\mathbb{G}})$ we have isomorphisms*

$$\begin{aligned} \mathrm{R}\Gamma(\mathbb{G}; F \otimes \mathcal{O}_{\mathbb{G}}(\lambda)) &\simeq \mathrm{R}\Gamma(\mathbb{G}^*; (F \circ \mathbb{C}_\Omega) \otimes \mathcal{O}_{\mathbb{G}^*}(\lambda^*))[N], \\ \mathrm{R}\Gamma(\mathbb{G}; R\mathcal{H}om\,(F, \mathcal{O}_{\mathbb{G}}(\lambda))) &\simeq \mathrm{R}\Gamma(\mathbb{G}^*; R\mathcal{H}om\,(F \circ \mathbb{C}_\Omega, \mathcal{O}_{\mathbb{G}^*}(\lambda^*)))[-N], \end{aligned}$$

and similarly for $\otimes$ and $R\mathcal{H}om$ replaced by $\overset{\mathrm{w}}{\otimes}$ and $\mathcal{T}hom$ when $F \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_{\mathbb{G}})$.

Proof. In order to obtain the formulas for $\otimes$ and $\overset{\mathrm{w}}{\otimes}$ (resp. for $R\mathcal{H}om$ and $\mathcal{T}hom$) apply Proposition 1.8 for $X = \mathbb{G}^*$, $Y = \mathbb{G}$, $\mathcal{M} = \mathcal{D}_{\mathbb{G}^*}(-\lambda^*)$, $\mathcal{K} = \mathcal{K}_{\widetilde{\Omega}}$ (resp. $\mathcal{K} = \mathcal{K}^*_{\widetilde{\Omega}}$) and hence $\widetilde{K} = \mathbb{C}_\Omega$ (resp. $\widetilde{K}^* = \mathbb{C}_\Omega$). Finally, recall Theorem 3.16. □

Let us give some applications of these formulas with $F = \mathbb{C}_D$, for D a compact subset of $\mathbb{G}$. Note that for any $y \in \mathbb{G}^*$ one has

$$(\mathbb{C}_D \circ \mathbb{C}_\Omega)_y \simeq \mathrm{R}\Gamma_c\,(L_D(y); \mathbb{C})\,, \qquad L_D(y) = \{x \in D : x \cap y = 0\}. \tag{4.6}$$

(1) Ω-trivial compact subsets. Here we argue in the spirit of [4, Section 5.1]. Let $D \subset \mathbb{G}$ be compact, and set

$$D^\# = \{y \in \mathbb{G}^* : x \cap y = 0 \text{ for any } x \in D\}$$

and $\widehat{D} = \mathbb{G}^* \setminus D^\#$. Observe that for any $y \in D^\#$ one has $L_D(y) = D$. Moreover, it is immediate to verify that D is nonempty (resp. affine) if and only if $D^\#$ is affine (resp. nonempty). (Here "affine" means "contained in an affine chart".)

Definition 4.4. (cf. [4, Definition 5.1]) Let D be a compact nonempty subset of $\mathbb{G}$. We say that D is Ω-*trivial* if (i) $\mathrm{R}\Gamma(D; \mathbb{C}) \simeq \mathbb{C}$ and (ii) $\mathrm{R}\Gamma(D \setminus L_D(y); \mathbb{C}) \simeq \mathbb{C}$ for any $y \in \widehat{D}$.

Remark 4.5. In the case $p = 1$, the Ω-triviality implies the "linear convexity" à la Martineau, i.e. $D^{\#\#} = D$ (see [4, Proposition 5.3])

Lemma 4.6. *Let D be an Ω-trivial compact subset of $\mathbb{G}$, and assume that $D^{\#} \neq \varnothing$. Then $\mathbb{C}_D \circ \mathbb{C}_\Omega \simeq \mathbb{C}_{D^\#}$.*

Proof. Let us compute $\mathbb{C}_D \circ \mathbb{C}_S$. Set $g = q_2|_{q_1^{-1}(D)\cap S}$: since $q_1^{-1}(D) \cap S = q_2^{-1}(\widehat{D})$, from the natural morphism $\mathrm{id} \to Rg_* g^{-1}$ one gets $\mathbb{C}_{\widehat{D}} \to Rg_* \mathbb{C}_{q_1^{-1}(D)\cap S} \simeq \mathbb{C}_D \circ \mathbb{C}_S$, which is an isomorphism by (ii). Applying the functor $\mathbb{C}_D \circ \cdot$ to the triangle $\mathbb{C}_\Omega \to \mathbb{C}_{\mathbb{G}\times\mathbb{G}^*} \to \mathbb{C}_S \xrightarrow{+1}$ and noticing that $\mathbb{C}_D \circ \mathbb{C}_{\mathbb{G}\times\mathbb{G}^*} \simeq \mathrm{R}\Gamma(D; \mathbb{C}_D) \otimes \mathbb{C}_{\mathbb{G}^*} \simeq \mathbb{C}_{\mathbb{G}^*}$ by (i), the lemma follows. $\square$

Remark 4.7. Since $\mathbb{C}_\Omega \circ D'\mathbb{C}_{\widetilde{\Omega}} \simeq \mathbb{C}_{\Delta_\mathbb{G}}[-2N]$, by Lemma 4.6 one gets $\mathbb{C}_{D^\#} \circ D'\mathbb{C}_{\widetilde{\Omega}} \simeq \mathbb{C}_D[-2N]$.

Applying Proposition 4.3 and Lemma 4.6 we get the following result:

Corollary 4.8. *Let D be a compact Ω-trivial subset of $\mathbb{G}$, and assume that $D^\# \neq \varnothing$. Let $x_0 \in D$, $y_0 \in D^\#$ and consider $E = \{x \in \mathbb{G} : x \cap y_0 = \{0\}\} \simeq \mathbb{C}^N$ and $E^* = \{y \in \mathbb{G}^* : x_0 \cap y = \{0\}\} \simeq \mathbb{C}^N$. Then $D \subset E$, $D^\# \subset E^*$ and one has the following isomorphisms:*

$$\begin{aligned} \mathrm{R}\Gamma(D; \mathcal{O}_E) &\simeq \mathrm{R}\Gamma_c(D^\#; \mathcal{O}_{E^*})[N] \\ \mathrm{R}\Gamma_D(E; \mathcal{O}_E)[N] &\simeq \mathrm{R}\Gamma(D^\#; \mathcal{O}_{E^*}). \end{aligned}$$

Moreover, all complexes are in concentrated in degree zero.

In the case $p = 1$, these isomorphisms were firstly obtained by Martineau [14], and reformulated in this language by D'Agnolo-Schapira [4, Theorem 5.5].

Example 4.9. Let $x_0 \in \mathbb{G}$, and set $D = \{x_0\}$: then D is obviously Ω-trivial, and $D^\# = E^* \simeq \mathbb{C}^N$. In this case, Corollary 4.8 gives well-known identifications: e.g., one has $\mathrm{R}\Gamma(\{x_0\}; \mathcal{O}_E) \simeq \mathbb{C}\{z\}$ (the convergent power series in $z = (z_1, \ldots, z_N) \in E \simeq \mathbb{C}^N$) and $\mathrm{R}\Gamma_c(E^*; \mathcal{O}_{E^*})[N] \simeq \Gamma(E^*; \Omega_{E^*})'$ (the analytic functionals of Martineau).

(2) Indefinite Hermitian form. Let H be an Hermitian form of signature $(p, n-p)$ on V, and set

$$\begin{aligned} U &= \{x \in \mathbb{G} : H|_x > 0\}, \\ U^* &= \{y \in \mathbb{G}^* : H|_y < 0\}. \end{aligned}$$

(Here, and in what follows, > 0, ≥ 0, < 0, ≤ 0 mean positive or negative (semi)definiteness.)

Remark 4.10. We observe the following facts.

(i) Let $H = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, where the orders of the diagonal blocks are p and $n-p$, and consider the affine charts $E = \{x = [1_p, A] \in \mathbb{G} : A \in M_{p,n-p}(\mathbb{C})\}$ and $E^* = \{y = \left[\begin{smallmatrix} 1_p \\ B \end{smallmatrix}\right] \in \mathbb{G}^* : B \in M_{n-p,p}(\mathbb{C})\}$. Then U (resp. U^*) is a relatively compact subset of E (resp. E^*). More precisely, one has $U = \{x = [1_p, A] : 1_p - AA^* > 0\}$ and $U^* = \{y = \left[\begin{smallmatrix} 1_p \\ B \end{smallmatrix}\right] : 1_p - BB^* > 0\}$, where $(\,\cdot\,)^* = {}^t\overline{(\,\cdot\,)}$.

(ii) The real Lie group $SU(p,n-p)$ is a real form of the complex semisimple Lie group $SL(n,\mathbb{C})$, which acts transitively on $\mathbb{G}$ and $\mathbb{G}^*$. The $SU(p,n-p)$-orbits in $\mathbb{G}$ and $\mathbb{G}^*$ are

$$\begin{aligned} U_{p',q'} &= \{x \in \mathbb{G} : H|_x \text{ has signature } (p',q')\}, \\ U^*_{p'',q''} &= \{y \in \mathbb{G}^* : H|_y \text{ has signature } (p'',q'')\} \end{aligned}$$

for $p'+q' \le p$ and $p'' \le p$, $q'' \ge n-2p$ and $p''+q'' \le n-p$. In particular, one has $U = U_{p,0}$ and $U^* = U^*_{0,n-p}$.

Lemma 4.11. *One has $\mathbb{C}_{\overline{U}} \circ \mathbb{C}_\Omega \simeq \mathbb{C}_{U^*}$.*

Proof. (M. Kashiwara) We argue on each $SU(p,n-p)$-orbit in $\mathbb{G}^*$. Let $y \in U^*_{p'',q''}$, and let us calculate $\left(\mathbb{C}_{\overline{U}} \circ \mathbb{C}_\Omega\right)_y \simeq \mathrm{R}\Gamma_c\left(L_{\overline{U}}(y);\mathbb{C}\right)$. We may suppose that the dual Stiefel coordinates of y are $\left[\begin{smallmatrix} 0 \\ 1_p \end{smallmatrix}\right]$, and that H is associated to the hermitian $(n \times n)$-matrix

$$M_H = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

where the orders of the diagonal blocks are p'', q'', $n-p-p''-q''$, $n-p-p''-q''$, $q''-n+2p$ and p''. The generic element of $E_y = \{x \in \mathbb{G} : x \cap y = 0\} \simeq \mathbb{C}^N$ has Stiefel coordinates

$$X = \begin{pmatrix} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{pmatrix},$$

where the orders of the row blocks are $n-p-p''-q''$, $q''-n+2p$ and p'', and the orders of the column blocks are p'', q'', $n-p-p''-q''$, $n-p-p''-q''$, $q''-n+2p$ and p''. The condition $x \in \overline{U}$ is expressible as the positive semidefiniteness of the hermitian $(p \times p)$-matrix $XM_HX^* =$

$$\begin{pmatrix} a_{11}{a_{11}}^* - a_{12}{a_{12}}^* + {a_{13}}^* + a_{13} & a_{11}{a_{21}}^* - a_{12}{a_{22}}^* + {a_{23}}^* & a_{11}{a_{31}}^* - a_{12}{a_{32}}^* + {a_{33}}^* \\ a_{21}{a_{11}}^* - a_{22}{a_{12}}^* + a_{23} & a_{21}{a_{21}}^* - a_{22}{a_{22}}^* + 1 & a_{21}{a_{31}}^* - a_{22}{a_{32}}^* \\ a_{31}{a_{11}}^* - a_{32}{a_{12}}^* + a_{33} & a_{31}{a_{21}}^* - a_{32}{a_{22}}^* & a_{31}{a_{31}}^* - a_{32}{a_{32}}^* - 1 \end{pmatrix},$$

where the orders of the diagonal blocks are $n-p-p''-q''$, $q''-n+2p$ and p''.

Let $p''+q'' < n-p$, i.e. suppose that $H|_y$ is degenerate. Then, up to a change of coordinates, it is not restrictive to suppose $XM_HX^* = \left(\begin{smallmatrix} x & y \\ y^* & A \end{smallmatrix}\right)$ with $x \in \mathbb{R}$, $y \in \mathbb{C}^{p-1}$ and A a positive semidefinite hermitian $(p-1)\times(p-1)$-matrix. Since for any fixed y and A the set $\{x \in \mathbb{R} : \left(\begin{smallmatrix} x & y \\ y^* & A \end{smallmatrix}\right) \ge 0\}$ is either empty or a closed half real line, and $\mathrm{R}\Gamma_c(\overline{\mathbb{R}^+};\mathbb{C}) \equiv 0$, we get $\mathrm{R}\Gamma_c(L_{\overline{U}}(y);\mathbb{C}) = 0$. Therefore, we may suppose that $n-p-p''-q'' = 0$, and hence we write

$$X = \begin{pmatrix} a_{11} & a_{12} & 1 & 0 \\ a_{21} & a_{22} & 0 & 1 \end{pmatrix},$$

where the orders of the row blocks are $p-p''$ and p'', and the orders of the column blocks are p'', $n-p-p''$, $p-p''$ and p''. Set $u = \left(\begin{smallmatrix} a_{11} \\ a_{21} \end{smallmatrix}\right)$ and $v = \left(\begin{smallmatrix} a_{12} \\ a_{22} \end{smallmatrix}\right)$. One has

$$L_{\overline{U}}(y) = \left\{ a = (u,v) \in E_y : uu^* + \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) - vv^* \ge 0 \right\}.$$

For fixed u_0, $L_{\overline{U}}(y) \cap \{u = u_0\}$ is a compact subset in the space of v stable under multiplication by $c \in \mathbb{C}$ with $|c| < 1$. Moreover, $L_{\overline{U}}(y) \cap \{u = u_0\} \neq \varnothing$ if and only if $uu^* + \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \geq 0$. Therefore, it is not restrictive to set $v = 0$, i.e. we have $\mathrm{R}\Gamma_c(L_{\overline{U}}(y); \mathbb{C}) \simeq \mathrm{R}\Gamma_c(Z; \mathbb{C})$, where

$$Z = \left\{ u = \left(\begin{smallmatrix} a_{11} \\ a_{21} \end{smallmatrix}\right) : uu^* + \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \geq 0 \right\}.$$

Observe that Z is closed. If $p'' \neq 0$, then clearly $0 \notin Z$. In addition, Z is stable under multiplication by $t \in \mathbb{R}^+$ with $t > 1$. Therefore the fibers of the natural map $Z \to Z/\mathbb{R}^+$ are closed half real lines, and one has again $\mathrm{R}\Gamma_c(Z; \mathbb{C}) = 0$. Finally, if $p'' = 0$ (and hence $y \in U^*_{0,n-p} = U^*$) one has $Z = \{a \in M_{p,n-p}(\mathbb{C}) : 1 - aa^* \geq 0\}$, and thus $\mathrm{R}\Gamma_c(Z; \mathbb{C}) \simeq \mathbb{C}$. Since $\mathbb{C}_{\overline{U}} \circ \mathbb{C}_\Omega$ is locally constant on the $SU(p, n-p)$-orbits, the proof is complete. □

By Proposition 4.3 and Lemma 4.11 we get:

Corollary 4.12. *One has the following isomorphisms:*

$$\begin{aligned} \mathrm{R}\Gamma(\overline{U}; \mathcal{O}_E) &\simeq \mathrm{R}\Gamma_c(U^*; \mathcal{O}_{E^*})[N] \\ \mathrm{R}\Gamma_{\overline{U}}(E; \mathcal{O}_E)[N] &\simeq \mathrm{R}\Gamma(U^*; \mathcal{O}_{E^*}). \end{aligned}$$

Moreover, all these complexes are concentrated in degree zero.

(3) Embedded Grassmann manifolds. Let us give a "non-affine" example. Fix any hyperplane $z \subset V$ and set

$$\begin{aligned} \mathbb{G}_z &= \{x \in \mathbb{G} : x \subset z\} \\ \mathbb{G}^*_z &= \{y \in \mathbb{G}^* : y \subset z\}. \end{aligned}$$

Then $\mathbb{G}_z$ (resp. $\mathbb{G}^*_z$) is the Grassmann manifold of p- (resp. $(n-p)$-)subspaces of z, and hence its complex dimension is $N - p$ (resp. $N - (n-p)$). It is easy to verify that $L_{\mathbb{G}_z}(y) = \varnothing$ if $y \in \mathbb{G}^*_z$ and $L_{\mathbb{G}_z}(y) \simeq \mathbb{C}^{N-p}$ otherwise. Since $\mathbb{C}_{\mathbb{G}_z} \circ \mathbb{C}_\Omega$ is locally constant on $\mathbb{G}^* \setminus \mathbb{G}^*_z$, which is simply connected, we get

Lemma 4.13. *One has $\mathbb{C}_{\mathbb{G}_z} \circ \mathbb{C}_\Omega \simeq \mathbb{C}_{\mathbb{G}^* \setminus \mathbb{G}^*_z}[-2(N-p)]$.*

We then obtain

Corollary 4.14. *For any $-n + p \leq \lambda \leq -p$ one has the following isomorphisms:*

$$\begin{aligned} \mathrm{R}\Gamma(\mathbb{G}_z; \mathcal{O}_\mathbb{G}(\lambda)) &\simeq \mathrm{R}\Gamma(\mathbb{G}^*_z; \mathcal{O}_{\mathbb{G}^*}(\lambda^*))[-(N - 2p + 1)] \\ \mathrm{R}\Gamma_{\mathbb{G}_z}(\mathbb{G}; \mathcal{O}_\mathbb{G}(\lambda)) &\simeq \mathrm{R}\Gamma_{\mathbb{G}^*_z}(\mathbb{G}^*, \mathcal{O}_{\mathbb{G}^*}(\lambda^*))[N - 2p + 1]. \end{aligned}$$

Proof. One has the distinguished triangle $\mathbb{C}_{\mathbb{G}^* \setminus \mathbb{G}^*_z} \to \mathbb{C}_{\mathbb{G}^*} \to \mathbb{C}_{\mathbb{G}^*_z} \xrightarrow{+1}$. Applying the functor $\mathrm{R}\Gamma_c(\,\cdot\,; \mathcal{O}_{\mathbb{G}^*}(\lambda^*))$ and recalling (4.2), the first isomorphism follows from Proposition 4.3 and Lemma 4.13, and the second is proved similarly by using the functor $\mathrm{R}\Gamma(\,\cdot\,; \mathcal{O}_{\mathbb{G}^*}(\lambda^*))$. □

Example 4.15. Let $\mathbb{P}$ be a m-dimensional projective space and let $\mathbb{P}'$ be a $(m-1)$-dimensional projective space embedded in $\mathbb{P}$. Applying Corollary 4.14 for $n = m+1$ and $p = 1$, we recover that for any $-m \leq \lambda \leq -1$ the complex $\mathrm{R}\Gamma(\mathbb{P}'; \mathcal{O}_{\mathbb{P}}(\lambda))$ (resp. $\mathrm{R}\Gamma_{\mathbb{P}'}(\mathbb{P}; \mathcal{O}_{\mathbb{P}}(\lambda))$) is concentrated in degree $m-1$ (resp. 1) and is infinite dimensional.

A b-functions

In this appendix we recall the results on the theory of Bernstein-Sato's b-functions which are used here. We refer to the works of Kashiwara [7] and Sato-Kashiwara-Kimura-Oshima [17] for the proofs of the statements below, and to Kashiwara [9] for an introductory exposition.

Let X be a complex analytic manifold, $x_\circ \in X$, and let $f \in (\mathcal{O}_X)_{x_\circ}$ be a germ of holomorphic function at $x_\circ$ such that $f(x_\circ) = 0$. Set $S = f^{-1}(0)$ and $\Omega = X \setminus S$.

Let $\mathcal{J}_f = (f) \subset \mathcal{O}_X$, and let

$$\mathcal{O}_X(*S) = \varinjlim_{n \in \mathbb{N}} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{J}_f^n, \mathcal{O}_X)$$

be the sheaf of meromorphic functions on X with singularities on S. Recall that $\mathcal{O}_X(*S)$, and its dual $\underline{\mathrm{D}}\mathcal{O}_X(*S)$, are regular holonomic left $\mathcal{D}_X$-modules. There is a natural injective morphism $\mathcal{O}_X \to \mathcal{O}_X(*S)$.

The b-function. Let s be an indeterminate on $\mathcal{D}_X$, and set $\mathcal{D}_X[s] = \mathcal{D}_X \otimes_{\mathbb{C}} \mathbb{C}[s]$. We define the ideal

$$\mathcal{I} = \{P(x, \partial_x; s) \in \mathcal{D}_X[s] : P(s) f(x)^s = 0 \text{ for } s \in \mathbb{Z}_{\geq 0},\ \ x \in \Omega\}$$

and we set

$$\mathcal{N} = \mathcal{D}_X[s]/\mathcal{I} = \mathcal{D}_X[s]\, f^s,$$

where f^s is the canonical generator $1 + \mathcal{I}$.

Definition A.1. The *b-function* $b_f(s)$ associated to f is the monic generator of the ideal of polynomials $b(s)$ in $\mathbb{C}[s]$ such that

$$P(x, \partial_x; s) f(x)^{s+1} = b(s) f(x)^s, \qquad s \in \mathbb{Z},\ x \in \Omega \tag{A.1}$$

for some $P(x, \partial_x; s) \in \mathcal{D}_X[s]$.

For $a \in \mathbb{C}$, set

$$\mathcal{I}(a) = \{R(x, \partial_x) \in \mathcal{D}_X; \exists\, Q(x, \partial_x; s) \in \mathcal{I} \text{ s.t. } R(x, \partial_x) = Q(x, \partial_x; a)\}$$

and define

$$\mathcal{M}_a = \mathcal{D}_X/\mathcal{I}(a) = \mathcal{D}_X\, u_a,$$

where u_a is the canonical generator $1 + \mathcal{I}(a)$ of $\mathcal{M}_a$. Observe that in general one has $\mathcal{I}(a) \neq \mathcal{I}'(a) = \{R(x, \partial_x) \subset \mathcal{D}_X : R(x, \partial_x)\, f(x)^a = 0,\ x \in \Omega\}$, and hence the natural morphism $\mathcal{M}_a \to \mathcal{D}_X/\mathcal{I}'(a)$ is not necessarily an isomorphism.

Proposition A.2. *Let $a \in \mathbb{Z}$.*

(i) *If $b_f(a-\nu) \neq 0$ for any $\nu \in \mathbb{Z}_{\geq 1}$, then $\mathcal{M}_a \simeq \mathcal{O}_X(*S)$;*

(ii) *If $b_f(a+\nu) \neq 0$ for any $\nu \in \mathbb{Z}_{\geq 0}$, then $\mathcal{M}_a \simeq \underline{\mathrm{D}}\mathcal{O}_X(*S)$.*

Remark A.3. Let us note some consequences of Proposition A.2.

(i) By Kashiwara [7], the roots of $b_f(s)$ are negative and rational. Therefore, one has

$$\mathcal{M}_0 \simeq \underline{\mathrm{D}}\mathcal{O}_X(*S).$$

The image of u^0 by this natural isomorphism provides a canonical generator of $\underline{\mathrm{D}}\mathcal{O}_X(*S)$, which is usually denoted by Y_f by analogy with the smooth case.

(ii) Moreover, for any $a \in \mathbb{Z}_{\geq 1}$ one gets a canonical section $\partial^a Y_f$ of $\underline{\mathrm{D}}\mathcal{O}_X(*S)$ as follows. Let $P(s) = P(x, \partial_x; s) \in \mathcal{D}_X[s]$ be an operator satisfying eq:bfctn). Since

$$P(s-a)\cdots P(s-1)\, f^s = b_f(s-a)\cdots b_f(s-1)\, f^{s-a},$$

the section $\partial^a f^s = \prod_{j=1}^a P(s-j)\, f^s \in \mathcal{N}$ does not depend on P. Hence, one obtains the desired section as the image of $\partial^a f^s$ by the canonical morphism $\mathcal{N} \to \underline{\mathrm{D}}\mathcal{O}_X(*S)$, $R(s)f^s \mapsto R(0)\, Y_f$, i.e.

$$\partial^a Y_f = \prod_{j=1}^{a} P(-j)\, Y_f. \tag{A.2}$$

Observe that $\partial^a Y_f$ is not necessarily a generator of $\underline{\mathrm{D}}\mathcal{O}_X(*S)$, even on a single irreducible component of $W = \mathrm{char}\, \underline{\mathrm{D}}\mathcal{O}_X(*S)$ (see below).

Local b-functions. The above considerations can be refined microlocally. In other words, let Λ be a *good* irreducible component of $W = \mathrm{char}\, \mathcal{O}_X(*S)$: then one can ask only whether

$$\mathcal{E}\mathcal{O}_X(*S) \simeq \mathcal{E}\mathcal{M}_a \qquad \text{on } \Lambda.$$

(Recall that we set $\mathcal{E}\mathcal{M} = \mathcal{E}_X \otimes_{\pi^{-1}\mathcal{D}_X} \pi^{-1}\mathcal{M}$ for a $\mathcal{D}_X$-module $\mathcal{M}$.)

Proposition A.4. *If Λ is a good Lagrangian, there exists a monic polynomial $b_\Lambda(s)$ of degree m_Λ (where m_Λ is the order of zero of $f \circ \pi|_W$ along Λ) and an* invertible *microdifferential operator P_Λ of order m_Λ such that $P_\Lambda f^{s+1} = b_\Lambda(s) f^s$ on Λ.*

This polynomial, which in fact divides $b_f(s)$, is called the *local b-function* of f along Λ. One has a microlocal analogue of Proposition A.2.

Proposition A.5. *Let $a \in \mathbb{Z}$.*

(i) *If $b_\Lambda(a-\nu) \neq 0$ for any $\nu \in \mathbb{Z}_{\geq 1}$, then $\mathcal{E}\mathcal{M}_a \simeq \mathcal{E}\mathcal{O}_X(*S)$ on Λ;*

(ii) *If* $b_\Lambda(a+\nu) \neq 0$ *for any* $\nu \in \mathbb{Z}_{\geq 0}$*, then* $\mathcal{E}\mathcal{M}_a \simeq \mathcal{E}\underline{\mathrm{D}}\mathcal{O}_X(*S)$ *on* Λ*.*

When all irreducible components $\{\Lambda_1, \ldots, \Lambda_r\}$ of W are good Lagrangians, then $b(s)$ is the least common multiple of the b_Λ's.

If two components Λ_i and Λ_j $(1 \leq i,j \leq r)$ have *good intersection* and $m_{\Lambda_i} > m_{\Lambda_j}$, then it is possible to calculate the ratio $b_{\Lambda_i}(s)/b_{\Lambda_j}(s)$ (see e.g. [17]). This gives an useful algorithm to compute the local b-functions, as well as the b-function itself.

We refer to Kashiwara [9] for some examples. In particular, let us recall one of these results, which is useful for our purposes:

Proposition A.6. *Let* $X = M_n(\mathbb{C}) = \mathbb{C}^{n^2}$ *and* $f(x) = \det(x)$*. Then the* b*-function associated to* f *is* $b_f(s) = \prod_{j=1}^n (s+j)$*.*

Remark A.7. In this case, observe that:

(i) as an operator satisfying (A.1) one can choose $P(x;\partial_x) = \det(\partial_{x_{ij}})_{i,j=1,\ldots,n}$. In particular, P does not depend on s, and therefore $\partial^a Y_f = \det(\partial_x)^a Y_f$ for any $a \in \mathbb{Z}_{\geq 1}$.

(ii) One has a natural action of $G = SL_n(\mathbb{C})$ on X, and (G,X) is a prehomogeneous space with open dense orbit $\Omega = \{x \in X : x \text{ is nonsingular}\} = X \setminus f^{-1}(0)$. The other G-orbits in X are the locally closed submanifolds $S_j = \{x \in X : \mathrm{rank}\,(x) = n-j\}$ $(j = 1,\ldots,n)$.

(iii) Moreover, (G,X) is a regular prehomogeneous space (see [17]), and this implies the equality

$$W = \mathrm{char}\ \mathcal{O}_X(*S) = T^*_X X \cup \bigcup_{j=1}^n T^*_{S_j} X.$$

The irreducible components of W are $\Lambda_0 = T^*_X X$, $\Lambda_j = \overline{T^*_{S_j} X}$ $(j = 1,\ldots,n-1)$ and $\Lambda_n = T^*_{\{0\}} X$. One can check that the Λ_0, Λ_1, $\ldots$, Λ_{n-1}, Λ_n are good Lagrangeans, that the multiplicity of zero of $f \circ \pi$ on Λ_j is $m_{\Lambda_j} = j$ and that the pairs $(\Lambda_{j-1}, \Lambda_j)$ have good intersection for $j = 1,\ldots,n$. The local b-functions are $b_{\Lambda_0}(s) = 1$ and $b_{\Lambda_j}(s) = \prod_{i=1}^j (s+i)$ $(j = 1,\ldots,n)$.

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