Radon transforms for quasi-equivariant $\mathcal{D}$-modules on generalized flag manifolds

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Abstract

In this paper we deal with Radon transforms for generalized flag manifolds in the framework of quasi-equivariant $\mathcal{D}$-modules. We shall follow the method employed by Baston-Eastwood and analyze the Radon transform using the Bernstein-Gelfand-Gelfand resolution and the Borel-Weil-Bott theorem. We shall determine the transform completely on the level of the Grothendieck groups. Moreover, we point out a vanishing criterion and give a sufficient condition in order that a $\mathcal{D}$-module associated to an equivariant locally free $\mathcal{O}$-module is transformed into an object of the same type. The case of maximal parabolic subgroups is studied in detail.
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Introduction

Let G be a reductive algebraic group over $\mathbf{C}$, P and Q two parabolic subgroups containing the same Borel subgroup of G. Let $X = G/P$, $Y = G/Q$, and let S be the unique closed G-orbit in $X \times Y$ for the diagonal action. Then we can identify S with $G/P \cap Q$. The natural correspondence

$$X \xleftarrow{f} S \xrightarrow{g} Y,$$

where f and g are the restriction to S of the projections of $X \times Y$ on X and Y, induces an integral transform from X to Y which generalizes the classical Radon-Penrose transform. This subject has been investigated intensively both in the complex and real domains (see e.g. Baston-Eastwood [1], D'Agnolo-Schapira [5], Kakehi [6], Marastoni [10], Oshima [12], Sekiguchi [14], Tanisaki [15]).

Our aim is to study the above transform in the framework of quasi-G-equivariant $\mathcal{D}$-modules (see Kashiwara [7]), i.e. the functor

$$R : \mathbf{D}^{\mathrm{b}}_G(\mathcal{D}_X) \to \mathbf{D}^{\mathrm{b}}_G(\mathcal{D}_Y), \qquad R(\mathcal{M}) = \underline{g}_* \underline{f}^{-1}\mathcal{M}, \tag{0.1}$$

where $\mathbf{D}^{\mathrm{b}}_{G}(\mathcal{D}_{\cdot})$ denotes the derived category of quasi-G-equivariant $\mathcal{D}$-modules with bounded cohomologies, and $\underline{g}_*$ and $\underline{f}^{-1}$ are the operations of direct image (integration) and inverse image (pull-back) for $\mathcal{D}$-modules. More precisely, we consider a $\mathcal{D}_X$-module of type $\mathcal{M} = \mathcal{D}\mathcal{L} = \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{L}$, where $\mathcal{L}$ is an irreducible G-equivariant locally free $\mathcal{O}_X$-module. In this case it is easily seen that

$$H^p(R(\mathcal{D}\mathcal{L})) = 0 \quad \text{for any } p < 0 \tag{0.2}$$

(see Lemma 1.4 below). Note that the Grothendieck group of the category of quasi-G-equivariant $\mathcal{D}_X$-modules of finite length is spanned by elements corresponding to the objects of the form $\mathcal{D}\mathcal{L}$.

As in Baston-Eastwood [1] our analysis relies on the Bernstein-Gelfand-Gelfand resolution and the Borel-Weil-Bott theorem. Using the Bernstein-Gelfand-Gelfand resolution in the parabolic setting (see Bernstein-Gelfand-Gelfand [2], Lepowsky [9], Rocha-Caridi [13]) we obtain a resolution of the quasi-G-equivariant $\mathcal{D}_S$-module $\underline{f}^{-1}(\mathcal{D}\mathcal{L})$ of the form:

$$0 \to \bigoplus_{k=1}^{r_n} \mathcal{D}\mathcal{L}_{nk} \to \cdots \to \bigoplus_{k=1}^{r_0} \mathcal{D}\mathcal{L}_{0k} \to \underline{f}^{-1}(\mathcal{D}\mathcal{L}) \to 0, \tag{0.3}$$

where $\mathcal{L}_{ik}$ are irreducible G-equivariant locally free $\mathcal{O}_S$-modules (see § 2.2 for the explicit description of $\mathcal{L}_{ik}$). Then we have

$$\underline{g}_*(\mathcal{D}\mathcal{L}_{ik}) = \mathcal{D}_Y \otimes_{\mathcal{O}_Y} Rg_*(\mathcal{L}_{ik} \otimes_{\mathcal{O}_S} \Omega_g)$$

by the definition of $\underline{g}_*$, where Ω_g denotes the sheaf of relative differential forms with maximal degree along the fibers of g. Moreover, the Borel-Weil-Bott theorem tells us the structure of $Rg_*(\mathcal{L}_{ik} \otimes_{\mathcal{O}_S} \Omega_g)$. In particular, we have either $Rg_*(\mathcal{L}_{ik} \otimes_{\mathcal{O}_S} \Omega_g) = 0$ or there exist a non-negative integer m_{ik} and an irreducible G-equivariant $\mathcal{O}_Y$-module $\mathcal{L}'_{ik}$ such that $Rg_*(\mathcal{L}_{ik} \otimes_{\mathcal{O}_S} \Omega_g) = \mathcal{L}'_{ik}[-m_{ik}]$. Thus setting

$$\mathcal{I} = \{(i,k)\,;\, 0 \le i \le n, 1 \le k \le r_i, Rg_*(\mathcal{L}_{ik} \otimes_{\mathcal{O}_S} \Omega_g) \neq 0\},$$

we have

$$\underline{g}_*(\mathcal{D}\mathcal{L}_{ik}) = \begin{cases} \mathcal{D}\mathcal{L}'_{ik}[-m_{ik}] & ((i,k) \in \mathcal{I}), \\ 0 & ((i,k) \notin \mathcal{I}) \end{cases} \tag{0.4}$$

(see §2.2 below for concrete descriptions of $\mathcal{I}$ and $\mathcal{L}_{ik}, m_{ik}$ for $(i,k) \in \mathcal{I}$).

Then we can study the structure of $R(\mathcal{D}\mathcal{L}) = \underline{g}_*\underline{f}^{-1}(\mathcal{D}\mathcal{L})$ using (0.2), (0.3) and (0.4). For example we have the following result.

Theorem 0.1. *Let the notation be as above.*

(i) *We have*

$$\sum_p (-1)^p [H^p(R(\mathcal{D}\mathcal{L}))] = \sum_{(i,k)\in\mathcal{I}} (-1)^{i-m_{ik}} [\mathcal{D}\mathcal{L}'_{ik}]$$

in the Grothendieck group of the category of quasi-G-equivariant $\mathcal{D}_Y$-modules.

(ii) *If $\mathcal{I} = \emptyset$, then $R(\mathcal{D}\mathcal{L}) = 0$.*

(iii) *If $\mathcal{I}$ consists of a single element (i,k), then* $R(\mathcal{D}\mathcal{L}) = \mathcal{D}\mathcal{L}'_{ik}[i - m_{ik}]$.

(iv) *If $i \geq m_{ik}$ for any $(i,k) \in \mathcal{I}$, then $H^p(R(\mathcal{D}\mathcal{L})) = 0$ unless $p = 0$.*

(v) *If $i > m_{ik}$ for any $(i,k) \in \mathcal{I}$ with $i > 0$ and if $m_{01} = 0$, then there exists an epimorphism $\mathcal{D}_Y\mathcal{L}'_{01} \to H^0(R(\mathcal{D}\mathcal{L}))$ (note that $r_0 = 1$).*

Assume that $\mathcal{L}$ is invertible and that there exists a G-equivariant invertible $\mathcal{O}_Y$-module $\mathcal{L}'$ satisfying $f^*\mathcal{L} \otimes_{\mathcal{O}_S} \Omega_g = g^*\mathcal{L}'$. We call such a pair $(\mathcal{L}, \mathcal{L}')$ an extremal case for the correspondence (if $P \cup Q$ generates the group G and if G is semisimple, then there exists a unique extremal case). In this case there exists a natural nontrivial $\mathcal{D}_Y$-linear morphism

$$\Phi : \mathcal{D}\mathcal{L}' \to H^0(R(\mathcal{D}\mathcal{L})). \tag{0.5}$$

Theorem 0.2. *Let $(\mathcal{L}, \mathcal{L}')$ be an extremal case.*

(i) *We have $H^p(R(\mathcal{D}\mathcal{L})) = 0$ for any $p \neq 0$ if and only if $i \geq m_{ik}$ for any $(i,k) \in \mathcal{I}$.*

(ii) *Assume that $H^p(R(\mathcal{D}\mathcal{L})) = 0$ for any $p \neq 0$. Then Φ is an epimorphism if and only if $i > m_{ik}$ for any $(i,k) \in \mathcal{I}$ with $i > 0$.*

(iii) *Assume that $H^p(R(\mathcal{D}\mathcal{L})) = 0$ for any $p \neq 0$. Then Φ is an isomorphism if and only if $\mathcal{I}$ consists of a single element* $(0,1)$.

We do not know an example of an extremal case $(\mathcal{L}, \mathcal{L}')$ such that $H^p(R(\mathcal{D}\mathcal{L})) \neq 0$ for some $p \neq 0$. Anyway, we have checked that $H^p(R(\mathcal{D}\mathcal{L})) = 0$ for any $p \neq 0$ by a case-by-case analysis in several situations, e.g. when G is of classical type and P and Q are maximal parabolic subgroups, or when the rank of G is ≤ 6. In general the morphism Φ for an extremal case $(\mathcal{L}, \mathcal{L}')$ is not necessarily an epimorphism nor a monomorphism. It would be an interesting problem to determine the kernel and the cokernel of Φ.

The transform of a $\mathcal{D}$-module, a problem of analytic nature, is not sufficient to cover the general problem of integral geometry. In order to do this, one should couple the transforms in the frameworks of $\mathcal{D}$-modules and sheaves. This is better described in the adjunction formulas (see D'Agnolo-Schapira [5]), and we shall briefly discuss this point with an example in the case of $G = SL_{n+1}(\mathbf{C})$.

1 Preliminaries on $\mathcal{D}$-modules

1.1 Functors for $\mathcal{D}$-modules

Let Z be an algebraic manifold (smooth algebraic variety) over $\mathbf{C}$. We denote by $\mathcal{O}_Z$ the structure sheaf, by Ω_Z the invertible $\mathcal{O}_Z$-module of differential forms of maximal degree, and by $\mathcal{D}_Z$ the sheaf of differential operators. In this paper an $\mathcal{O}_Z$-module means a quasi-coherent $\mathcal{O}_Z$-module and a $\mathcal{D}_Z$-module means a left $\mathcal{D}_Z$-module which is quasi-coherent over $\mathcal{O}_Z$. We denote by $\mathrm{Mod}(\mathcal{D}_Z)$ the category of $\mathcal{D}_Z$-modules and by $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_Z)$ the derived category of $\mathrm{Mod}(\mathcal{D}_Z)$ whose objects have bounded cohomology.

If $f : Z \to Z'$ is a morphism, we set

$$\Omega_f = \Omega_{Z/Z'} = \Omega_Z \otimes_{f^{-1}\mathcal{O}_{Z'}} f^{-1}\Omega_{Z'}^{\otimes -1};$$

and, for an $\mathcal{O}_{Z'}$-module $\mathcal{L}'$, we set

$$f^*\mathcal{L}' = \mathcal{O}_Z \otimes_{f^{-1}\mathcal{O}_{Z'}} f^{-1}\mathcal{L}', \qquad Lf^*\mathcal{L}' = \mathcal{O}_Z \otimes^L_{f^{-1}\mathcal{O}_{Z'}} f^{-1}\mathcal{L}'.$$

We denote by $\underline{f}_*$ and $\underline{f}^{-1}$ the direct and inverse image for left $\mathcal{D}$-modules:

$$\begin{aligned} &\underline{f}_* : \mathbf{D}^{\mathrm{b}}(\mathcal{D}_Z) \to \mathbf{D}^{\mathrm{b}}(\mathcal{D}_{Z'}), \quad && \underline{f}_*\mathcal{M} = Rf_*(\mathcal{D}_{Z' \leftarrow Z} \otimes^L_{\mathcal{D}_Z} \mathcal{M}), \\ &\underline{f}^{-1} : \mathbf{D}^{\mathrm{b}}(\mathcal{D}_{Z'}) \to \mathbf{D}^{\mathrm{b}}(\mathcal{D}_Z), \quad && \underline{f}^{-1}\mathcal{M}' = \mathcal{D}_{Z \to Z'} \otimes^L_{f^{-1}\mathcal{D}_{Z'}} f^{-1}\mathcal{M}', \end{aligned}$$

where a $(\mathcal{D}_Z, f^{-1}\mathcal{D}_{Z'})$-bimodule $\mathcal{D}_{Z\to Z'}$ and an $(f^{-1}\mathcal{D}_{Z'}, \mathcal{D}_Z)$-bimodule $\mathcal{D}_{Z'\leftarrow Z}$ are defined by

$$\mathcal{D}_{Z\to Z'} = \mathcal{O}_Z \otimes_{f^{-1}\mathcal{O}_{Z'}} f^{-1}\mathcal{D}_{Z'}, \qquad \mathcal{D}_{Z'\leftarrow Z} = \Omega_Z \otimes_{\mathcal{O}_Z} \mathcal{D}_{Z\to Z'} \otimes_{f^{-1}\mathcal{O}_{Z'}} f^{-1}\Omega_{Z'}^{\otimes -1}.$$

Note that for a $\mathcal{D}_{Z'}$-module $\mathcal{M}$ we have $\underline{f}^{-1}\mathcal{M} \simeq Lf^*\mathcal{M}$ as a complex of $\mathcal{O}_Z$-modules. Note also that we have canonical morphisms $\mathcal{O}_Z \to \mathcal{D}_{Z\to Z'}$ and $\Omega_f \to \mathcal{D}_{Z'\leftarrow Z}$ of $\mathcal{O}_Z$-modules.

The following result is well-known and easy to prove.

Lemma 1.1. *Let $f_1 : Z \to X_1$ and $f_2 : Z \to X_2$ be morphisms of algebraic manifolds.*

(i) *We have*

$$\mathcal{D}_{X_2\leftarrow Z} \otimes^L_{\mathcal{D}_Z} \mathcal{D}_{Z\to X_1} \xrightarrow{\sim} f_1^{-1}\Omega_{X_1} \otimes^L_{f_1^{-1}\mathcal{O}_{X_1}} (\mathcal{D}_{X_1\times X_2 \leftarrow Z} \otimes^L_{\mathcal{D}_Z} \mathcal{O}_Z).$$

(ii) *Assume that $Z \to X_1 \times X_2$ is an embedding. Then we have*

$$\mathcal{D}_{X_2\leftarrow Z} \otimes^L_{\mathcal{D}_Z} \mathcal{D}_{Z\to X_1} = \mathcal{D}_{X_2\leftarrow Z} \otimes_{\mathcal{D}_Z} \mathcal{D}_{Z\to X_1},$$

and the canonical morphism of $(f_2^{-1}\mathcal{O}_{X_2}, f_1^{-1}\mathcal{O}_{X_1})$-bimodules

$$\Omega_{f_2} \to \mathcal{D}_{X_2\leftarrow Z} \otimes_{\mathcal{D}_Z} \mathcal{D}_{Z\to X_1}$$

is a monomorphism.

For a locally free $\mathcal{O}_Z$-module $\mathcal{L}$, we set

$$\mathcal{D}\mathcal{L} = \mathcal{D}_Z \otimes_{\mathcal{O}_Z} \mathcal{L},$$

and for a closed submanifold Z of an algebraic manifold X we define a $\mathcal{D}_X$-module $\mathcal{B}_{Z|X}$ supported on Z by

$$\mathcal{B}_{Z|X} = H^d_{[Z]}(\mathcal{O}_X) = \underline{i}_*\mathcal{O}_Z,$$

where $d = \mathrm{codim}_X Z$ and $i : Z \to X$ denotes the embedding.

1.2 Radon transforms

Let X and Y be algebraic manifolds over $\mathbf{C}$, and denote by q_1 and q_2 the projections of $X \times Y$ onto X and Y respectively. Let S be a locally closed submanifold of $X \times Y$ and let $i : S \to X \times Y$ be the embedding. The geometric correspondence

$$X \xleftarrow{f} S \xrightarrow{g} Y \tag{1.1}$$

where f and g are the restrictions of q_1 and q_2, induces a functor

$$R : \mathbf{D}^{\mathrm{b}}(\mathcal{D}_X) \to \mathbf{D}^{\mathrm{b}}(\mathcal{D}_Y), \qquad R(\mathcal{M}) = \underline{g}_* \underline{f}^{-1}(\mathcal{M}), \tag{1.2}$$

called the Radon transform.

Lemma 1.2. *Let $\mathcal{M}$ be a $\mathcal{D}_X$-module.*

(i) *We have*

$$\begin{aligned} R(\mathcal{M}) &= Rg_*((\mathcal{D}_{Y \leftarrow S} \otimes_{\mathcal{D}_S} \mathcal{D}_{S \to X}) \otimes^L_{f^{-1}\mathcal{D}_X} f^{-1}\mathcal{M}) \\ &= Rg_*(f^{-1}(\Omega_X \otimes_{\mathcal{O}_X} \mathcal{M}) \otimes^L_{f^{-1}\mathcal{D}_X} (\mathcal{D}_{X \times Y \leftarrow S} \otimes_{\mathcal{D}_S} \mathcal{O}_S)). \end{aligned}$$

(ii) *If S is closed in $X \times Y$, then we have*

$$R(\mathcal{M}) = \underline{q_2}_*(\underline{q_1}^{-1}\mathcal{M} \otimes^L_{\mathcal{O}_{X \times Y}} \mathcal{B}_{S|X \times Y}).$$

Proof. (i) follows from the definition and Lemma 1.1, and (ii) is a consequence of the projection formula for $\mathcal{D}$-modules. □

Let us consider the special case where $\mathcal{M} = \mathcal{D}\mathcal{L} = \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{L}$. By Lemma 1.2 we have the following.

Lemma 1.3. *Let $\mathcal{L}$ be a locally free $\mathcal{O}_X$-module.*

(i) *We have*

$$\begin{aligned} R(\mathcal{D}\mathcal{L}) &= Rg_*((\mathcal{D}_{Y \leftarrow S} \otimes_{\mathcal{D}_S} \mathcal{D}_{S \to X}) \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{L}) \\ &= Rg_*(f^{-1}(\Omega_X \otimes_{\mathcal{O}_X} \mathcal{L}) \otimes_{f^{-1}\mathcal{O}_X} (\mathcal{D}_{X \times Y \leftarrow S} \otimes_{\mathcal{D}_S} \mathcal{O}_S)). \end{aligned}$$

(ii) *If S is closed in $X \times Y$, then we have*

$$R(\mathcal{D}\mathcal{L}) = Rq_{2*}(q_1^{-1}(\Omega_X \otimes_{\mathcal{O}_X} \mathcal{L}) \otimes_{q_1^{-1}\mathcal{O}_X} \mathcal{B}_{S|X \times Y}).$$

An immediate consequence of Lemma 1.3(i) is:

Lemma 1.4. *For any locally free $\mathcal{O}_X$-module $\mathcal{L}$ we have $H^p(R(\mathcal{D}\mathcal{L})) = 0$ for any $p < 0$.*

Definition 1.5. Let $\mathcal{L}$ (resp. $\mathcal{L}'$) be a locally free $\mathcal{O}_X$- (resp. $\mathcal{O}_Y$-)module. We say that the pair $(\mathcal{L}, \mathcal{L}')$ is an extremal case for the correspondence (1.1) if there is an $\mathcal{O}_S$-linear isomorphism

$$\Omega_g \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{L} \simeq g^*\mathcal{L}'.$$

Proposition 1.6. *Let $(\mathcal{L}, \mathcal{L}')$ be an extremal case for (1.1). Then there exists a natural nontrivial $\mathcal{D}_Y$-linear morphism*

$$\mathcal{D}\mathcal{L}' \to H^0(R(\mathcal{D}\mathcal{L})). \tag{1.3}$$

Proof. The canonical morphism $\Omega_g \to \mathcal{D}_{Y\leftarrow S} \otimes_{\mathcal{D}_S} \mathcal{D}_{S\to X}$ induces a monomorphism

$$g^*\mathcal{L}' \simeq \Omega_g \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{L} \to \mathcal{D}_{Y\leftarrow S} \otimes_{\mathcal{D}_S} \mathcal{D}_{S\to X} \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{L}$$

of $g^{-1}\mathcal{O}_Y$-modules. Applying g_* we obtain a sequence of morphisms

$$\begin{array}{rcl} \mathcal{L}' & \to & \mathcal{L}' \otimes_{\mathcal{O}_Y} g_*\mathcal{O}_S \simeq g_*(g^*\mathcal{L}') \\ & \to & g_*(\mathcal{D}_{Y\leftarrow S} \otimes_{\mathcal{D}_S} \mathcal{D}_{S\to X} \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{L}) = H^0(R(\mathcal{D}\mathcal{L})) \end{array}$$

of $\mathcal{O}_Y$-modules. The morphism $\mathcal{L}' \to \mathcal{L}' \otimes_{\mathcal{O}_Y} g_*\mathcal{O}_S$ is nontrivial by the definition, and the morphism $g_*(g^*\mathcal{L}') \to g_*(\mathcal{D}_{Y\leftarrow S} \otimes_{\mathcal{D}_S} \mathcal{D}_{S\to X} \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{L})$ is a monomorphism by the left exactness of g_*. Thus the composition $\mathcal{L}' \to H^0(R(\mathcal{D}\mathcal{L}))$ is nontrivial. Hence it induces a canonical nontrivial morphism $\mathcal{D}\mathcal{L}' \to H^0(R(\mathcal{D}\mathcal{L}))$ of $\mathcal{D}_Y$-modules. □

1.3 Adjunction formulas

In this subsection we consider topological problems, and hence we work in the analytic category rather than the algebraic category.

For a complex manifold Z we denote by $\mathcal{O}_Z$ the sheaf of holomorphic functions on Z and by $\mathcal{D}_Z$ the sheaf of holomorphic differential operators. For an algebraic manifold Z over $\mathbf{C}$ we denote the corresponding complex manifold by Z_{an}, and for a morphism $f : Z \to Z'$ of algebraic manifolds we denote the corresponding holomorphic map by $f_{\mathrm{an}} : Z_{\mathrm{an}} \to Z'_{\mathrm{an}}$. For an algebraic manifold Z and an $\mathcal{O}_Z$-module $\mathcal{F}$ we set $\mathcal{F}_{\mathrm{an}} = \mathcal{O}_{Z_{\mathrm{an}}} \otimes_{\mathcal{O}_Z} \mathcal{F}$.

In the correspondence (1.1), let us consider also a functor in the derived category $\mathbf{D}^{\mathrm{b}}(\mathbf{C}.)$ of sheaves of $\mathbf{C}$-vector spaces, going in the opposite direction:

$$r : \mathbf{D}^{\mathrm{b}}(\mathbf{C}_{Y_{\mathrm{an}}}) \to \mathbf{D}^{\mathrm{b}}(\mathbf{C}_{X_{\mathrm{an}}}), \qquad r(F) = Rg_{\mathrm{an}*}f_{\mathrm{an}}^{-1}(F).$$

For example, let D be a Zariski locally closed subset of Y_{an} and take $F = \mathbf{C}_D$ (the constant sheaf with fiber $\mathbf{C}$ on D and zero on $Y_{\mathrm{an}} \setminus D$): then, for any $x \in X$ one has

$$r(\mathbf{C}_D)_x \simeq \mathrm{R}\Gamma_c(S_{D,x}; \mathbf{C}_{S_{D,x}}), \qquad S_{D,x} = \{y \in D : (x, y) \in S\}. \tag{1.4}$$

One has the following "adjunction formulas" (see [5]).

Proposition 1.7. *Let $\mathcal{L}$ be a locally free $\mathcal{O}_X$-module and $F \in \mathbf{D}^{\mathrm{b}}(\mathbf{C}_{Y_{\mathrm{an}}})$. Then, setting $l = \dim Y - \dim S$ and $m = \dim S + \dim Y - 2\dim X$, we have*

$$\mathrm{R}\Gamma(X_{\mathrm{an}}; r(F) \otimes \mathcal{L}^*_{\mathrm{an}}) \simeq \mathrm{RHom}_{\mathcal{D}_{Y_{\mathrm{an}}}}(R(\mathcal{D}\mathcal{L})_{\mathrm{an}}, F \otimes \mathcal{O}_{Y_{\mathrm{an}}})[l], \tag{1.5}$$

$$\mathrm{RHom}\,(r(F), \mathcal{L}^*_{\mathrm{an}}) \simeq \mathrm{RHom}_{\mathcal{D}_{Y_{\mathrm{an}}}}(R(\mathcal{D}\mathcal{L})_{\mathrm{an}}, R\mathcal{H}om\,(F, \mathcal{O}_{Y_{\mathrm{an}}}))[m]. \tag{1.6}$$

Once the calculation of $R(\mathcal{D}\mathcal{L})$ has been performed, these formulas will give different applications by computing $r(F)$ for different choices of the sheaf F (a problem of geometric nature).

1.4 Quasi-equivariant $\mathcal{D}$-modules

Let us recall the definition of (quasi-)equivariant $\mathcal{D}$-modules (we refer to Kashiwara [7]).

Let G be an algebraic group over $\mathbf{C}$, and let $\mathfrak{g}$ be its Lie algebra. We denote the enveloping algebra of $\mathfrak{g}$ by $\mathcal{U}(\mathfrak{g})$. Let Z be a G-manifold, i.e. an algebraic manifold endowed with an action of G. Let us denote by $\mu : G \times Z \to Z$ the action $\mu(g, z) = gz$ and by $p : G \times Z \to Z$ the projection $p(g, z) = z$. Moreover, define the morphisms $q_j : G \times G \times Z \to G \times Z$ $(j = 1, 2, 3)$ by $q_1(g_1, g_2, z) = (g_1, g_2 z)$, $q_2(g_1, g_2, z) = (g_1 g_2, z)$ and $q_3(g_1, g_2, z) = (g_2, z)$, and observe that $\mu \circ q_1 = \mu \circ q_2$, $p \circ q_2 = p \circ q_3$ and $\mu \circ q_3 = p \circ q_1$.

A G-equivariant $\mathcal{O}_Z$-module is an $\mathcal{O}_Z$-module $\mathcal{M}$ endowed with a $\mathcal{O}_{G\times Z}$-linear isomorphism $\beta : \mu^*\mathcal{M} \to p^*\mathcal{M}$ such that the following diagram commutes:

$$\begin{array}{ccc} {q_2}^*\mu^*\mathcal{M} & \xrightarrow{\quad {q_2}^*\beta \quad} & {q_2}^*p^*\mathcal{M} \\ \| & & \| \\ {q_1}^*\mu^*\mathcal{M} \xrightarrow{{q_1}^*\beta} & {q_1}^*p^*\mathcal{M} = {q_3}^*\mu^*\mathcal{M} & \xrightarrow{{q_3}^*\beta} {q_3}^*p^*\mathcal{M}. \end{array}$$

For a G-equivariant $\mathcal{O}_Z$-module $\mathcal{M}$ we have a canonical Lie algebra homomorphism $\rho_{\mathcal{M}} : \mathfrak{g} \to \mathrm{End}_{\mathbf{C}}(\mathcal{M})$.

Let $\mathcal{O}_G \boxtimes \mathcal{D}_Z$ denote the subalgebra $\mathcal{O}_{G\times Z} \otimes_{p^{-1}\mathcal{O}_Z} p^{-1}\mathcal{D}_Z$ of $\mathcal{D}_{G\times Z}$. A $\mathcal{D}_Z$-module $\mathcal{M}$ is called G-equivariant (resp. quasi-G-equivariant) if it is endowed with a G-equivariant $\mathcal{O}_Z$-module structure such that the isomorphism $\beta : \mu^*\mathcal{M} \to p^*\mathcal{M}$ is $\mathcal{D}_{G\times Z}$-linear (resp. $\mathcal{O}_G \boxtimes \mathcal{D}_Z$-linear). Note that for a morphism $f : Z \to Z'$ of algebraic manifolds and a $\mathcal{D}_{Z'}$-module $\mathcal{M}$ the $\mathcal{D}_Z$-module $H^0(\underline{f}^{-1}\mathcal{M})$ is naturally isomorphic to $f^*\mathcal{M}$ as an $\mathcal{O}_Z$-module.

For example for a G-equivariant $\mathcal{O}_Z$-module $\mathcal{F}$ the $\mathcal{D}_Z$-module $\mathcal{D}_Z \otimes_{\mathcal{O}_Z} \mathcal{F}$ is endowed with a natural quasi-G-equivariant $\mathcal{D}_Z$-module structure.

We denote by $\mathrm{Mod}_G(\mathcal{D}_Z)$ the category of quasi-G-equivariant $\mathcal{D}_Z$-modules, and by $\mathbf{D}^{\mathrm{b}}_G(\mathcal{D}_Z)$ the derived category of $\mathcal{D}_Z$-modules with bounded quasi-G-equivariant cohomology (see Kashiwara-Schmid [8]).

Let $\mathcal{M}$ be a quasi-G-equivariant $\mathcal{D}_Z$-module. The canonical Lie algebra homomorphism $\mathfrak{g} \to \mathcal{D}_Z$ induces a Lie algebra homomorphism $\kappa_{\mathcal{M}} : \mathfrak{g} \to \mathrm{End}_{\mathbf{C}}(\mathcal{M})$. Set $\gamma_{\mathcal{M}} = \rho_{\mathcal{M}} - \kappa_{\mathcal{M}}$.

Proposition 1.8 (Kashiwara [7]). *We have $\gamma_{\mathcal{M}}(a) \in \mathrm{End}_{\mathcal{D}_Z}(\mathcal{M})$ for any $a \in \mathfrak{g}$. Moreover, the linear map $\gamma_{\mathcal{M}} : \mathfrak{g} \to \mathrm{End}_{\mathcal{D}_Z}(\mathcal{M})$ is a Lie algebra homomorphism such that $\gamma_{\mathcal{M}} = 0$ if and only if $\mathcal{M}$ is G-equivariant.*

We also denote by

$$\gamma_{\mathcal{M}} : \mathcal{U}(\mathfrak{g}) \to \mathrm{End}_{\mathcal{D}_Z}(\mathcal{M}) \tag{1.7}$$

the corresponding algebra homomorphism.

Fix $x \in Z$ and set $H = \{g \in G : gx = x\}$. For a G-equivariant $\mathcal{O}_Z$-module $\mathcal{M}$, the fiber

$$\mathcal{M}(x) = \mathbf{C} \otimes_{\mathcal{O}_{Z,x}} \mathcal{M}_x$$

of $\mathcal{M}$ at x is endowed with a natural H-module structure. If $\mathcal{M}$ is a quasi-G-equivariant $\mathcal{D}_Z$-module, then $\mathcal{M}(x)$ is also endowed with a $\mathfrak{g}$-module structure induced from the $\mathcal{O}_Z$-linear action $\gamma_{\mathcal{M}}$. For $M = \mathcal{M}(x)$ we have the following.

(a) the action of the Lie algebra of H on M given by differentiating the H-module structure coincides with the restriction of the action of $\mathfrak{g}$,

(b) $hum = (\mathrm{Ad}(h)u)hm$ for any $h \in H$, $u \in \mathfrak{g}$, $m \in M$.

Here Ad denotes the adjoint action. A vector space M equipped with structures of an H-modules and a $\mathfrak{g}$-module is called a $(\mathfrak{g}, H)$-module if it satisfies the conditions (a) and (b) above.

The following result plays a crucial role in the rest of this paper.

Proposition 1.9. *Assume that $Z = G/H$, where H is a closed subgroup of G, and set $x = eH \in Z$.*

(i) *The category of G-equivariant $\mathcal{O}_Z$-modules is equivalent to the category of H-modules via the correspondence $\mathcal{M} \mapsto \mathcal{M}(x)$.*

(ii) *The category of quasi-G-equivariant $\mathcal{D}_Z$-modules is equivalent to the category of $(\mathfrak{g}, H)$-modules via the correspondence $\mathcal{M} \mapsto \mathcal{M}(x)$.*

The statement (i) is well-known (see [11]), and (ii) is due to Kashiwara [7].

2 Radon transforms for generalized flag manifolds

2.1 Quasi-equivariant $\mathcal{D}$-modules on generalized flag manifolds

Let G be a connected reductive algebraic group over $\mathbf{C}$, and $\mathfrak{g}$ the Lie algebra of G. The group G acts on $\mathfrak{g}$ by the adjoint action Ad. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, Δ the root system in $\mathfrak{h}^*$, $\{\alpha_i : i \in I_0\}$ a set of simple roots, Δ^+ the set of positive roots, Δ^- the set of negative roots, $\mathfrak{h}^*_{\mathbf{Z}} = \mathrm{Hom}\,(H, \mathbf{C}^\times) \subset \mathfrak{h}^*$ the weight lattice, and W the Weyl group. For $\alpha \in \Delta$ we denote by $\mathfrak{g}_\alpha$ the corresponding root space and by $\alpha^\vee \in \mathfrak{h}$ the corresponding coroot. For $i \in I_0$ we denote by $s_i \in W$ the reflection corresponding to i. For $w \in W$ we set $\ell(w) = \sharp(w\Delta^- \cap \Delta^+)$. Set $\rho = \frac{1}{2}\sum_{\alpha \in \Delta^+} \alpha$, and define a (shifted) affine action of W on $\mathfrak{h}^*$ by

$$w \circ \lambda = w(\lambda + \rho) - \rho. \tag{2.1}$$

For $I \subset I_0$, we set

$$\begin{aligned}
\Delta_I &= \Delta \cap \sum_{i \in I} \mathbf{Z}\alpha_i, \qquad \Delta_I^+ = \Delta_I \cap \Delta^+, \qquad W_I = \langle s_i : i \in I \ \rangle \subset W \\
\mathfrak{l}_I &= \mathfrak{h} \oplus \Big(\bigoplus\nolimits_{\alpha \in \Delta_I} \mathfrak{g}_\alpha\Big), \qquad \mathfrak{n}_I = \bigoplus\nolimits_{\alpha \in \Delta^+ \setminus \Delta_I} \mathfrak{g}_\alpha, \qquad \mathfrak{p}_I = \mathfrak{l}_I \oplus \mathfrak{n}_I, \\
(\mathfrak{h}^*_{\mathbf{Z}})_I &= \{\lambda \in \mathfrak{h}^*_{\mathbf{Z}} : \lambda(\alpha_i^\vee) \geq 0 \text{ for any } i \in I\}, \\
(\mathfrak{h}^*_{\mathbf{Z}})^0_I &= \{\lambda \in \mathfrak{h}^*_{\mathbf{Z}} : \lambda(\alpha_i^\vee) = 0 \text{ for any } i \in I\} \subset (\mathfrak{h}^*_{\mathbf{Z}})_I, \\
\rho_I &= (\sum_{\alpha \in \Delta^+ \setminus \Delta_I} \alpha)/2.
\end{aligned}$$

We denote by w_I the longest element of W_I. It is an element of W_I characterized by $w_I(\Delta_I^-) = \Delta_I^+$. Let L_I, N_I and P_I be the subgroups of G corresponding to $\mathfrak{l}_I$, $\mathfrak{n}_I$ and $\mathfrak{p}_I$.

For $\lambda \in (\mathfrak{h}^*_{\mathbf{Z}})_I$ let $V_I(\lambda)$ be the irreducible L_I-module with highest weight λ. We regard $V_I(\lambda)$ as a P_I-module with the trivial action of N_I, and define the generalized Verma module with highest weight λ by

$$M_I(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p}_I)} V_I(\lambda). \tag{2.2}$$

Let $L(\lambda)$ be the unique irreducible quotient of $M_I(\lambda)$ (note that $L(\lambda)$ does not depend on the choice of I such that $\lambda \in (\mathfrak{h}^*_{\mathbf{Z}})_I$). Then any irreducible P_I-module is isomorphic to $V_I(\lambda)$ for some $\lambda \in (\mathfrak{h}^*_{\mathbf{Z}})_I$, and we have $\dim V_I(\lambda) = 1$ if and only if $\lambda \in (\mathfrak{h}^*_{\mathbf{Z}})^0_I$. Moreover, any irreducible $(\mathfrak{g}, P_I)$-module is isomorphic to $L(\lambda)$ for some $\lambda \in (\mathfrak{h}^*_{\mathbf{Z}})_I$.

Let

$$X_I = G/P_I$$

be the generalized flag manifold associated to I.

By the category equivalence given in Proposition 1.9 isomorphism classes of G-equivariant $\mathcal{O}_{X_I}$-modules (resp. quasi-G-equivariant $\mathcal{D}_{X_I}$-modules) are in one-to-one correspondence with isomorphism classes of P_I-modules (resp. $(\mathfrak{g}, P_I)$-modules). For $\lambda \in (\mathfrak{h}^*_{\mathbf{Z}})_I$ we denote by $\mathcal{O}_{X_I}(\lambda)$ the G-equivariant $\mathcal{O}_{X_I}$-module corresponding to the irreducible P_I-module $V_I(\lambda)$. We see easily the following.

Lemma 2.1. *Let $\lambda \in (\mathfrak{h}^*_{\mathbf{Z}})_I$. The quasi-$G$-equivariant $\mathcal{D}_{X_I}$-module corresponding to the $(\mathfrak{g}, P_I)$-module $M_I(\lambda)$ is isomorphic to*

$$\mathcal{D}\mathcal{O}_{X_I}(\lambda) = \mathcal{D}_{X_I} \otimes_{\mathcal{O}_{X_I}} \mathcal{O}_{X_I}(\lambda).$$

We need the following relative version of the Borel-Weil-Bott theorem later (see Bott [3]).

Proposition 2.2. *Let $I \subset J \subset I_0$ and let $\pi : X_I \to X_J$ be the canonical projection. For $\lambda \in (\mathfrak{h}^*_{\mathbf{Z}})_I$ we have the following.*

(i) *If there exists some $\alpha \in \Delta_J$ satisfying $(\lambda+\rho-2\rho_I)(\alpha^\vee) = 0$, then we have $R\pi_*(\mathcal{O}_{X_I}(\lambda)) = 0$.*

(ii) *Assume that $(\lambda + \rho - 2\rho_I)(\alpha^\vee) \neq 0$ for any $\alpha \in \Delta_J$. Take $w \in W_J$ satisfying $(w(\lambda + \rho - 2\rho_I))(\alpha^\vee) > 0$ for any $\alpha \in \Delta^+_J$. Then we have*

$$R\pi_*(\mathcal{O}_{X_I}(\lambda)) = \mathcal{O}_{X_J}(w(\lambda + \rho - 2\rho_I) - (\rho - 2\rho_J))[-(\ell(w_J w) - \ell(w_I))].$$

Let $I, J \subset I_0$ with $I \neq J$. The diagonal action of G on $X_I \times X_J$ has a finite number of orbits, and the only closed one $G(eP_I, eP_J)$ is identified with $X_{I\cap J} = G/(P_I \cap P_J)$. In the rest of this paper we shall consider the correspondence (1.1) for $X = X_I$, $Y = X_J$ and $S = X_{I\cap J}$:

$$X_I \xleftarrow{f} X_{I\cap J} \xrightarrow{g} X_J \tag{2.3}$$

and the Radon transform $R(\mathcal{D}\mathcal{O}_{X_I}(\lambda))$ for $\lambda \in (\mathfrak{h}^*_{\mathbf{Z}})_I$. Since f and g are morphisms of G-manifolds, the functor (1.2) induces a functor

$$R : \mathbf{D}^{\mathrm{b}}_G(\mathcal{D}_{X_I}) \to \mathbf{D}^{\mathrm{b}}_G(\mathcal{D}_{X_J}). \tag{2.4}$$

Note that we have

$$\Omega_g \simeq \mathcal{O}_{X_{I\cap J}}(\gamma_{I,J}) \qquad \text{for } \gamma_{I,J} = \textstyle\sum_{\alpha\in\Delta^+_J\setminus\Delta_I} \alpha. \tag{2.5}$$

2.2 Radon transforms of quasi-equivariant $\mathcal{D}$-modules

Let $\lambda \in (\mathfrak{h}^*_{\mathbf{Z}})_I$. We describe our method to analyze $R(\mathcal{DO}_{X_I}(\lambda)) = \underline{g}_* \underline{f}^{-1}(\mathcal{DO}_{X_I}(\lambda))$. By

$$(\underline{f}^{-1}(\mathcal{DO}_{X_I}(\lambda)))(e(P_I \cap P_J)) \simeq \mathcal{DO}_{X_I}(\lambda)(eP_I) \simeq M_I(\lambda)$$

the quasi-G-equivariant $\mathcal{D}_{X_{I\cap J}}$-module $\underline{f}^{-1}(\mathcal{DO}_{X_I}(\lambda))$ corresponds to the $(\mathfrak{g}, P_I \cap P_J)$-module $M_I(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p}_I)} V_I(\lambda)$ under the category equivalence given in Proposition 1.9.

Set

$$\Gamma = \{x \in W_I \,:\, x \text{ is the shortest element of } W_{I\cap J}x\}, \tag{2.6}$$
$$\Gamma_k = \{x \in \Gamma \,:\, \ell(x) = k\}. \tag{2.7}$$

It is well-known that an element $x \in W_I$ belongs to Γ if and only if $x^{-1}\Delta^+_{I\cap J} \subset \Delta^+_I$. This condition is also equivalent to

$$(x(\lambda+\rho))(\alpha^\vee) > 0 \text{ for any } \alpha \in \Delta^+_{I\cap J}. \tag{2.8}$$

In particular, we have $x \circ \lambda \in (\mathfrak{h}^*_{\mathbf{Z}})_{I\cap J}$ for $x \in \Gamma$.

By Lepowsky [9] and Rocha-Caridi [13] we have the following resolution of the finite dimensional $\mathfrak{l}_I$-module $V_I(\lambda)$:

$$0 \to N_n \to N_{n-1} \to \cdots \to N_1 \to N_0 \to V_I(\lambda) \to 0 \tag{2.9}$$

with $n = \dim \mathfrak{l}_I / \mathfrak{l}_I \cap \mathfrak{p}_J$ and

$$N_k = \bigoplus_{x\in\Gamma_k} \mathcal{U}(\mathfrak{l}_I) \otimes_{\mathcal{U}(\mathfrak{l}_I\cap\mathfrak{p}_J)} V_{I\cap J}(x \circ \lambda).$$

By the Poincaré-Birkhoff-Witt theorem we have the isomorphism

$$\mathcal{U}(\mathfrak{l}_I) \otimes_{\mathcal{U}(\mathfrak{l}_I\cap\mathfrak{p}_J)} V_{I\cap J}(x \circ \lambda) \simeq \mathcal{U}(\mathfrak{p}_I) \otimes_{\mathcal{U}(\mathfrak{p}_{I\cap J})} V_{I\cap J}(x \circ \lambda)$$

of $\mathcal{U}(\mathfrak{l}_I)$-modules, where $\mathfrak{n}_{I\cap J}$ acts trivially on $V_{I\cap J}(x \circ \lambda)$. Moreover, the action of $\mathfrak{n}_I$ on $\mathcal{U}(\mathfrak{p}_I) \otimes_{\mathcal{U}(\mathfrak{p}_{I\cap J})} V_{I\cap J}(x \circ \lambda)$ is trivial. Indeed, by $[\mathfrak{p}_I, \mathfrak{n}_I] \subset \mathfrak{n}_I$ we have $\mathfrak{n}_I\mathcal{U}(\mathfrak{p}_I) = \mathcal{U}(\mathfrak{p}_I)\mathfrak{n}_I$, and hence

$$\begin{aligned} \mathfrak{n}_I(\mathcal{U}(\mathfrak{p}_I) \otimes_{\mathcal{U}(\mathfrak{p}_{I\cap J})} V_{I\cap J}(x \circ \lambda)) &\subset \mathcal{U}(\mathfrak{p}_I)\mathfrak{n}_I \otimes V_{I\cap J}(x \circ \lambda) \\ &\subset \mathcal{U}(\mathfrak{p}_I) \otimes \mathfrak{n}_I V_{I\cap J}(x \circ \lambda) = 0 \end{aligned}$$

by $\mathfrak{n}_I \subset \mathfrak{n}_{I\cap J}$. Thus we obtain the following resolution of the finite dimensional $\mathfrak{p}_I$-module $V_I(\lambda)$ (with trivial action of $\mathfrak{n}_I$):

$$0 \to N'_n \to N'_{n-1} \to \cdots \to N'_1 \to N'_0 \to V_I(\lambda) \to 0 \tag{2.10}$$

with

$$N'_k = \bigoplus_{x\in\Gamma_k} \mathcal{U}(\mathfrak{p}_I) \otimes_{\mathcal{U}(\mathfrak{p}_{I\cap J})} V_{I\cap J}(x \circ \lambda).$$

By tensoring $\mathcal{U}(\mathfrak{g})$ to (2.10) over $\mathcal{U}(\mathfrak{p}_I)$ we obtain the following resolution of the $(\mathfrak{g}, P_{I\cap J})$-module $M_I(\lambda)$:

$$0 \to \tilde{N}_n \to \tilde{N}_{n-1} \to \cdots \to \tilde{N}_1 \to \tilde{N}_0 \to M_I(\lambda) \to 0 \tag{2.11}$$

with

$$\tilde{N}_k = \bigoplus_{x\in\Gamma_k} M_{I\cap J}(x\circ\lambda).$$

Since the quasi-equivariant $\mathcal{D}_{X_{I\cap J}}$-module corresponding to the $(\mathfrak{g}, P_{I\cap J})$-module $M_{I\cap J}(x\circ\lambda)$ is $\mathcal{D}\mathcal{O}_{X_{I\cap J}}(x\circ\lambda)$, we have obtained the following resolution of the quasi-G-equivariant $\mathcal{D}_{X_{I\cap J}}$-module $\underline{f}^{-1}(\mathcal{D}\mathcal{O}_{X_I}(\lambda))$:

$$0 \to \mathcal{N}_n \to \mathcal{N}_{n-1} \to \cdots \to \mathcal{N}_1 \to \mathcal{N}_0 \to \underline{f}^{-1}(\mathcal{D}\mathcal{O}_{X_I}(\lambda)) \to 0 \tag{2.12}$$

with

$$\mathcal{N}_k = \bigoplus_{x\in\Gamma_k} \mathcal{D}\mathcal{O}_{X_{I\cap J}}(x\circ\lambda). \tag{2.13}$$

Our next task is to investigate on $\underline{g}_*(\mathcal{D}\mathcal{O}_{X_{I\cap J}}(x\circ\lambda))$ for $x\in\Gamma$. We first remark that

$$\underline{g}_*(\mathcal{D}\mathcal{O}_{X_{I\cap J}}(x\circ\lambda)) = \mathcal{D}_{X_J}\otimes_{\mathcal{O}_{X_J}} Rg_*(\mathcal{O}_{X_{I\cap J}}(x\circ\lambda+\gamma_{I,J})). \tag{2.14}$$

Indeed, by (2.5) we have

$$\begin{aligned}
\underline{g}_*(\mathcal{D}\mathcal{O}_{X_{I\cap J}}(x\circ\lambda)) &= Rg_*(\mathcal{D}_{X_J\leftarrow X_{I\cap J}}\otimes^L_{\mathcal{D}_{X_{I\cap J}}} \mathcal{D}_{X_{I\cap J}}\otimes^L_{\mathcal{O}_{X_{I\cap J}}} \mathcal{O}_{X_{I\cap J}}(x\circ\lambda))\\
&= Rg_*(\mathcal{D}_{X_J\leftarrow X_{I\cap J}}\otimes^L_{\mathcal{O}_{X_{I\cap J}}} \mathcal{O}_{X_{I\cap J}}(x\circ\lambda))\\
&= Rg_*(g^{-1}\mathcal{D}_{X_J}\otimes_{g^{-1}\mathcal{O}_{X_J}} \Omega_g\otimes_{\mathcal{O}_{X_{I\cap J}}} \mathcal{O}_{X_{I\cap J}}(x\circ\lambda))\\
&= \mathcal{D}_{X_J}\otimes_{\mathcal{O}_{X_J}} Rg_*(\Omega_g\otimes_{\mathcal{O}_{X_{I\cap J}}} \mathcal{O}_{X_{I\cap J}}(x\circ\lambda))\\
&= \mathcal{D}_{X_J}\otimes_{\mathcal{O}_{X_J}} Rg_*(\mathcal{O}_{X_{I\cap J}}(x\circ\lambda+\gamma_{I,J})).
\end{aligned}$$

Lemma 2.3. *Let $\lambda\in(\mathfrak{h}^*_{\mathbf{Z}})_I$ and $x\in\Gamma$.*

(i) *If $(x(\lambda+\rho))(\alpha^\vee)=0$ for some $\alpha\in\Delta_J$, then we have $Rg_*(\mathcal{O}_{X_{I\cap J}}(x\circ\lambda+\gamma_{I,J}))=0$.*

(ii) *Assume that $(x(\lambda+\rho))(\alpha^\vee)\neq 0$ for any $\alpha\in\Delta_J$. Take $y\in W_J$ satisfying $(yx(\lambda+\rho))(\alpha^\vee)>0$ for any $\alpha\in\Delta_J^+$. Then we have*

$$Rg_*(\mathcal{O}_{X_{I\cap J}}(x\circ\lambda+\gamma_{I,J})) = \mathcal{O}_{X_J}((yx)\circ\lambda)[-(\ell(w_Jy)-\ell(w_{I\cap J}))].$$

Proof. Since $\Delta^+\setminus\Delta_J$ is stable under the action of W_J, we have $y\rho_J=\rho_J$ for any $y\in W_J$. In particular,

$$\rho_J = s_\alpha(\rho_J) = \rho_J - \rho_J(\alpha^\vee)\alpha$$

for any $\alpha\in\Delta_J$, and hence $\rho_J(\alpha^\vee)=0$ for any $\alpha\in\Delta_J$.

By the definition we have

$$x\circ\lambda+\gamma_{I,J}+\rho-2\rho_{I\cap J} = x(\lambda+\rho)+\gamma_{I,J}-2\rho_{I\cap J} = x(\lambda+\rho)-2\rho_J,$$

and

$$y(x(\lambda+\rho)-2\rho_J)-(\rho-2\rho_J)=yx(\lambda+\rho)-2\rho_J-(\rho-2\rho_J)=(yx)\circ\lambda$$

for any $y\in W_J$. Hence the assertion follows from Proposition 2.2. □

Set

$$\begin{aligned}\Gamma(\lambda) &= \{x\in\Gamma\,:\,(x(\lambda+\rho))(\alpha^\vee)\neq 0 \text{ for any } \alpha\in\Delta_J\}, &(2.15)\\ \Gamma_k(\lambda) &= \{x\in\Gamma(\lambda)\,:\,\ell(x)=k\}. &(2.16)\end{aligned}$$

and for $x\in\Gamma(\lambda)$ denote by y_x the element of W_J satisfying $(y_x x(\lambda+\rho))(\alpha^\vee)>0$ for any $\alpha\in\Delta_J^+$. Set

$$m(x)=\ell(w_J y_x)-\ell(w_{I\cap J})\qquad\text{for } x\in\Gamma(\lambda). \tag{2.17}$$

Lemma 2.4. *For $\lambda\in(\mathfrak{h}^*_{\mathbf{Z}})_I$ and $x\in\Gamma(\lambda)$ we have*

$$\begin{aligned}\ell(x) &= \sharp\{\alpha\in\Delta_I^+\setminus\Delta_J\,:\,(x(\lambda+\rho))(\alpha^\vee)<0\}, &(2.18)\\ m(x) &= \sharp\{\alpha\in\Delta_J^+\setminus\Delta_I\,:\,(x(\lambda+\rho))(\alpha^\vee)>0\}. &(2.19)\end{aligned}$$

Proof. We have

$$\begin{aligned}\ell(x) &= \sharp(x^{-1}\Delta_I^-\cap\Delta_I^+)\\ &= \sharp\{\alpha\in\Delta_I^+\,:\,(x(\lambda+\rho))(\alpha^\vee)<0\}\\ &= \sharp\{\alpha\in\Delta_I^+\setminus\Delta_J\,:\,(x(\lambda+\rho))(\alpha^\vee)<0\},\end{aligned}$$

and

$$\begin{aligned}m(x) &= \ell(w_J)-\ell(y_x)-\ell(w_{I\cap J})\\ &= \sharp(\Delta_J^+\setminus\Delta_I)-\sharp(y_x^{-1}\Delta_J^-\cap\Delta_J^+)\\ &= \sharp(\Delta_J^+\setminus\Delta_I)-\sharp\{\alpha\in\Delta_J^+\,:\,(x(\lambda+\rho))(\alpha^\vee)<0\}\\ &= \sharp(\Delta_J^+\setminus\Delta_I)-\sharp\{\alpha\in\Delta_J^+\setminus\Delta_I\,:\,(x(\lambda+\rho))(\alpha^\vee)<0\}\\ &= \sharp\{\alpha\in\Delta_J^+\setminus\Delta_I\,:\,(x(\lambda+\rho))(\alpha^\vee)>0\}\end{aligned}$$

by (2.8). □

Proposition 2.5. *For any $\lambda\in(\mathfrak{h}^*_{\mathbf{Z}})_I$ there exists a family $\{\mathcal{M}(k)^\bullet\}_{k\geq 0}$ of objects of $\mathbf{D}^{\mathrm{b}}_G(\mathcal{D}_{X_J})$ satisfying the following conditions.*

(i) $\mathcal{M}(0)^\bullet\simeq R(\mathcal{DO}_{X_I}(\lambda))$.

(ii) $\mathcal{M}(k)^\bullet=0$ *for* $k>\dim\mathfrak{l}_I/\mathfrak{l}_I\cap\mathfrak{p}_J$.

(iii) *We have a distinguished triangle*

$$\mathcal{C}(k)^\bullet\to\mathcal{M}(k)^\bullet\to\mathcal{M}(k+1)^\bullet\xrightarrow{+1}$$

where

$$\mathcal{C}(k)^\bullet=\bigoplus_{x\in\Gamma_k(\lambda)}\mathcal{DO}_{X_J}((y_x x)\circ\lambda)[\ell(x)-m(x)].$$

Proof. For $0 \leq k \leq \dim \mathfrak{l}_I/\mathfrak{l}_I \cap \mathfrak{p}_J$ define an object $\mathcal{N}(k)^\bullet$ of $\mathbf{D}^{\mathrm{b}}_G(\mathcal{D}_{X_{I\cap J}})$ by

$$\mathcal{N}(k)^\bullet = [\cdots \to 0 \to \mathcal{N}_n \to \mathcal{N}_{n-1} \to \cdots \to \mathcal{N}_k \to 0 \cdots],$$

where $\mathcal{N}_j$ has degree $-j$ (see (2.12) and (2.13) for the notation). For $k > \dim \mathfrak{l}_I/\mathfrak{l}_I \cap \mathfrak{p}_J$ we set $\mathcal{N}(k)^\bullet = 0$. By $\mathcal{N}(0)^\bullet \simeq \underline{f}^{-1}(\mathcal{D}\mathcal{O}_{X_I}(\lambda))$ we have $\underline{g}_*\mathcal{N}(0)^\bullet \simeq R(\mathcal{D}\mathcal{O}_{X_I}(\lambda))$. Set $\mathcal{M}(k)^\bullet = \underline{g}_*\mathcal{N}(k)^\bullet$. Then the statements (i) and (ii) are obvious. Let us show (iii). Applying $\underline{g}_*$ to the distinguished triangle

$$\mathcal{N}_k[k] \to \mathcal{N}(k)^\bullet \to \mathcal{N}(k+1)^\bullet \xrightarrow{+1}$$

we obtain a distinguished triangle

$$\underline{g}_*\mathcal{N}_k[k] \to \mathcal{M}(k)^\bullet \to \mathcal{M}(k+1)^\bullet \xrightarrow{+1}.$$

By (2.13), (2.14) and Lemma 2.3 we have

$$\underline{g}_*\mathcal{N}_k = \bigoplus_{x \in \Gamma_k(\lambda)} \mathcal{D}\mathcal{O}_{X_J}((y_x x) \circ \lambda)[-m(x)].$$

The statement (iii) is proved. □

Theorem 2.6. *Let $\lambda \in (\mathfrak{h}^*_{\mathbf{Z}})_I$.*

(i) *We have*

$$\sum_p (-1)^p [H^p(R(\mathcal{D}\mathcal{O}_{X_I}(\lambda)))] = \sum_{x \in \Gamma(\lambda)} (-1)^{\ell(x)-m(x)} [\mathcal{D}\mathcal{O}_{X_J}((y_x x) \circ \lambda)]$$

in the Grothendieck group of the category of quasi-G-equivariant $\mathcal{D}_{X_J}$-modules.

(ii) *If $\Gamma(\lambda) = \emptyset$, then $R(\mathcal{D}\mathcal{O}_{X_I}(\lambda)) = 0$.*

(iii) *If $\Gamma(\lambda)$ consists of a single element x, then*

$$R(\mathcal{D}\mathcal{O}_{X_I}(\lambda)) = \mathcal{D}\mathcal{O}_{X_J}((y_x x) \circ \lambda)[\ell(x) - m(x)].$$

(iv) *If $\ell(x) \geq m(x)$ for any $x \in \Gamma(\lambda)$, then we have $H^p(R(\mathcal{D}\mathcal{O}_{X_I}(\lambda))) = 0$ unless $p = 0$.*

(v) *If $(\lambda + \rho)(\alpha^\vee) < 0$ for any $\alpha \in \Delta^+_J \setminus \Delta_I$, then there exists a canonical morphism*

$$\Phi : \mathcal{D}\mathcal{O}_{X_J}((w_J w_{I\cap J}) \circ \lambda) \to H^0(R(\mathcal{D}\mathcal{O}_{X_I}(\lambda))).$$

Moreover, Φ is an epimorphism if $\ell(x) > m(x)$ for any $x \in \Gamma(\lambda) \setminus \{e\}$.

Proof. The statements (i), (ii), (iii) are obvious from Proposition 2.5. The statement (iv) follows from Proposition 2.5 and Lemma 1.4. Assume that λ satisfies the assumption in (v). Then we have $e \in \Gamma(\lambda)$ and $y_e = w_J w_{I\cap J}$. Hence (v) follows from Proposition 2.5. □

Lemma 2.7. (i) *The map $W_J \times \Gamma \to W_J W_I$ ($(y, x) \mapsto yx$) is bijective.*

(ii) *For $\lambda \in (\mathfrak{h}^*_{\mathbf{Z}})_I$ we have*

$$\{y_x x \,:\, x \in \Gamma(\lambda)\} = \{w \in W_J W_I \,:\, (w(\lambda+\rho))(\alpha^\vee) > 0 \text{ for any } \alpha \in \Delta_J^+\}$$

and we have

$$\ell(x) - m(x) = \ell(y_x) + \ell(x) - \sharp(\Delta_J^+ \setminus \Delta_I) = \ell(y_x x) - \sharp(\Delta_J^+ \setminus \Delta_I).$$

Proof. (i) is a consequence of the definition of Γ, and the first statement in (ii) follows from (i) and the definition of y_x. By

$$\ell(x) - m(x) = \ell(x) - (\ell(w_J) - \ell(y_x) - \ell(w_{I\cap J})) = \ell(x) + \ell(y_x) - \sharp(\Delta_J^+ \setminus \Delta_I)$$

we have only to show $\ell(y_x x) = \ell(x) + \ell(y_x)$ for $x \in \Gamma(\lambda)$. We have

$$x\Delta^+ \cap \Delta^- = x\Delta_I^+ \cap \Delta_I^- \subset \Delta_I^- \setminus \Delta_{I\cap J} \subset \Delta^- \setminus \Delta_J$$

by $x \in W_I$ and $x^{-1}\Delta_{I\cap J}^+ \subset \Delta_I^+$. Since $w \in W_J$, we obtain $y_x(x\Delta^+ \cap \Delta^-) \subset \Delta^-$. Hence

$$\begin{aligned}
\ell(y_x x) &= \sharp(y_x x\Delta^- \cap \Delta^+) \\
&= \sharp(y_x(x\Delta^- \cap \Delta^+) \cap \Delta^+) + \sharp(y_x(x\Delta^- \cap \Delta^-) \cap \Delta^+) \\
&= \sharp(y_x(x\Delta^- \cap \Delta^+) \cap \Delta^+) + \sharp(y_x\Delta^- \cap \Delta^+) - \sharp(y_x(x\Delta^+ \cap \Delta^-) \cap \Delta^+) \\
&= \ell(x) + \ell(y_x).
\end{aligned}$$

□

For $\lambda \in (\mathfrak{h}^*_{\mathbf{Z}})_I$ we set

$$\Xi(\lambda) = \{w \in W_J W_I \,:\, (w(\lambda+\rho))(\alpha^\vee) > 0 \text{ for any } \alpha \in \Delta_J^+\}. \tag{2.20}$$

Using Lemma 2.7 above we can reformulate Theorem 2.6 as follows.

Theorem 2.8. *Let $\lambda \in (\mathfrak{h}^*_{\mathbf{Z}})_I$.*

(i) *We have*

$$\sum_p (-1)^p [H^p(R(\mathcal{DO}_{X_I}(\lambda)))] = (-1)^{\sharp(\Delta_J^+\setminus\Delta_I)} \sum_{w\in\Xi(\lambda)} (-1)^{\ell(w)} [\mathcal{DO}_{X_J}(w\circ\lambda)]$$

in the Grothendieck group of the category of quasi-G-equivariant $\mathcal{D}_{X_J}$-modules.

(ii) *If $\Xi(\lambda) = \emptyset$, then $R(\mathcal{DO}_{X_I}(\lambda)) = 0$.*

(iii) *If $\Xi(\lambda)$ consists of a single element w, then*

$$R(\mathcal{DO}_{X_I}(\lambda)) = \mathcal{DO}_{X_J}(w\circ\lambda)[\ell(w) - \sharp(\Delta_J^+ \setminus \Delta_I)].$$

(iv) *If $\ell(w) \geq \sharp(\Delta_J^+ \setminus \Delta_I)$ for any $w \in \Xi(\lambda)$, then $H^p(R(\mathcal{DO}_{X_I}(\lambda))) = 0$ unless $p = 0$.*

(v) *If $(\lambda+\rho)(\alpha^\vee)<0$ for any $\alpha\in\Delta_J^+\setminus\Delta_I$, then there exists a canonical morphism*

$$\Phi:\mathcal{DO}_{X_J}((w_Jw_{I\cap J})\circ\lambda)\to H^0(R(\mathcal{DO}_{X_I}(\lambda))).$$

Moreover, Φ is an epimorphism if $\ell(w)>\sharp(\Delta_J^+\setminus\Delta_I)$ for any $w\in\Xi(\lambda)\setminus\{w_Jw_{I\cap J}\}$.

Remark 2.9. The following result which is a little weaker than Theorem 2.8(ii) can be obtained by observing that an integral transform for $\mathcal{D}$-modules with equivariant kernel preserves the infinitesimal character of a quasi-equivariant $\mathcal{D}$-module (see e.g. [8]):

$$\text{If } (W\circ\lambda)\cap(\mathfrak{h}^*_{\mathbf{Z}})_J=\emptyset,\text{ then } R(\mathcal{DO}_{X_I}(\lambda))=0. \tag{2.21}$$

An advantage of the argument using the infinitesimal character is that it also works for a broader class of integral transforms in equivariant contexts.

Let us briefly recall this argument (suggested to us by M. Kashiwara). Let Z be a G-manifold, denote by $\mathfrak{z}(\mathfrak{g})$ the center of $\mathcal{U}(\mathfrak{g})$ and set $\mathfrak{n}^+=\mathfrak{n}_\emptyset=\bigoplus_{\alpha\in\Delta^+}\mathfrak{g}_\alpha$, $\mathfrak{n}^-=\bigoplus_{\alpha\in\Delta^-}\mathfrak{g}_\alpha$. One says that a quasi-$G$-equivariant $\mathcal{D}_Z$-module $\mathcal{M}$ has infinitesimal character χ (for some $\chi\in\operatorname{Hom}(\mathfrak{z}(\mathfrak{g}),\mathbf{C})$) if $\gamma_{\mathcal{M}}(a)$ is the multiplication by $\chi(a)$ for any $a\in\mathfrak{z}(\mathfrak{g})$. Define a linear map $\sigma:\mathfrak{z}(\mathfrak{g})\to\mathcal{U}(\mathfrak{h})\simeq S(\mathfrak{h})$ as the composition of the embedding $\mathfrak{z}(\mathfrak{g})\to\mathcal{U}(\mathfrak{g})$ and the projection $\mathcal{U}(\mathfrak{g})\to\mathcal{U}(\mathfrak{h})$ with respect to the direct sum decomposition $\mathcal{U}(\mathfrak{g})=\mathcal{U}(\mathfrak{h})\oplus(\mathfrak{n}^-\mathcal{U}(\mathfrak{g})+\mathcal{U}(\mathfrak{g})\mathfrak{n}^+)$. Then σ is an injective homomorphism of $\mathbf{C}$-algebras. For $\lambda\in\mathfrak{h}^*$ define an algebra homomorphism $\chi_\lambda:\mathfrak{z}(\mathfrak{g})\to\mathbf{C}$ by $\chi_\lambda(a)=\langle\sigma(a),\lambda\rangle$. By a result of Harish-Chandra, any algebra homomorphism from $\mathfrak{z}(\mathfrak{g})$ to $\mathbf{C}$ coincides with χ_λ for some $\lambda\in\mathfrak{h}^*$, and for $\lambda,\mu\in\mathfrak{h}^*$ one has $\chi_\lambda=\chi_\mu$ if and only if $\mu\in W\circ\lambda$. By the category equivalence of Proposition 1.9, the infinitesimal characters of quasi-G-equivariant $\mathcal{D}_{X_I}$-modules are of the form χ_λ for $\lambda\in(\mathfrak{h}^*_{\mathbf{Z}})_I$. Therefore, recalling Harish-Chandra's result, if $(W\circ\lambda)\cap(\mathfrak{h}^*_{\mathbf{Z}})_J=\emptyset$, then $R(\mathcal{DO}_{X_I}(\lambda))=0$.

2.3 Extremal cases

We characterize the extremal cases (see Definition 1.5) in the correspondence (2.3). We shall only deal with the invertible $\mathcal{O}$-modules. Given $\lambda\in(\mathfrak{h}^*_{\mathbf{Z}})^0_I$ and $\mu\in(\mathfrak{h}^*_{\mathbf{Z}})^0_J$, we write for short (λ,μ) instead of $(\mathcal{O}_{X_I}(\lambda),\mathcal{O}_{X_J}(\mu))$.

Proposition 2.10. *The pair (λ,μ) is an extremal case if and only if $\mu=\lambda+\gamma_{I,J}$. This condition is also equivalent to the following system*

$$\begin{cases}\lambda(\alpha_i^\vee)=\mu(\alpha_i^\vee)=0 & (i\in I\cap J),\\ \lambda(\alpha_i^\vee)=0,\ \mu(\alpha_i^\vee)=\gamma_{I,J}(\alpha_i^\vee) & (i\in I\setminus J),\\ \lambda(\alpha_i^\vee)=-\gamma_{I,J}(\alpha_i^\vee),\ \mu(\alpha_i^\vee)=0 & (i\in J\setminus I),\\ \mu(\alpha_i^\vee)-\lambda(\alpha_i^\vee)=\gamma_{I,J}(\alpha_i^\vee) & (i\in I_0\setminus(I\cup J)).\end{cases} \tag{2.22}$$

Proof. The first statement is obvious by (2.5). Since $\Delta^+\setminus\Delta_I$ and Δ_J are stable under the action of W_I and W_J respectively, we have $w(\gamma_{I,J})=\gamma_{I,J}$ for any $w\in W_{I\cap J}=W_I\cap W_J$. In particular, we have

$$\gamma_{I,J}=s_i(\gamma_{I,J})=\gamma_{I,J}-\gamma_{I,J}(\alpha_i^\vee)\alpha_i$$

for any $i\in I\cap J$. Hence we obtain

$$\gamma_{I,J}(\alpha_i^\vee)=0\quad\text{ for any } i\in I\cap J.$$

Therefore, the relation $\mu=\lambda+\gamma_{I,J}$ is equivalent to the system (2.22). □

By (2.22) we have the following

Corollary 2.11. *If $\mathfrak{g}$ is semisimple and $I \cup J = I_0$, there exists a unique extremal case for (2.3).*

Proposition 2.12. *Let (λ, μ) be an extremal case.*

(i) *We have*

$$(\lambda+\rho)(\alpha^\vee)\begin{cases} <0 & \text{for any } \alpha \in \Delta_J^+ \setminus \Delta_I, \\ >0 & \text{for any } \alpha \in \Delta_I^+, \end{cases}$$

and $(w_J w_{I\cap J}) \circ \lambda = \mu$. In particular, $e \in \Gamma(\lambda)$ and $\ell(e) = m(e) = 0$.

(ii) *Let*

$$I = I_0 \setminus \{p_1, \dots, p_l, t_1, \dots, t_n\}, \qquad J = I_0 \setminus \{q_1, \dots, q_m, t_1, \dots, t_n\}$$

(where $l \geq 1$, $m \geq 1$, $n \geq 0$ and all p_i's, q_h's and t_j's are different from each other)*, and let $\lambda = \sum_{i=1}^l r_i \varpi_{p_i} + \sum_{j=1}^n r'_j \varpi_{t_j}$. Then $\nu = \sum_{i=1}^l k_i \varpi_{p_i} + \sum_{j=1}^n k'_j \varpi_{t_j}$ satisfies the property*

$$(\nu+\rho)(\alpha^\vee) < 0 \qquad \text{for any } \alpha \in \Delta_J^+ \setminus \Delta_I \tag{2.23}$$

(cf. Theorem 2.6(v)) *if and only if $k_i \leq r_i$ for any $i = 1, \dots, l$.*

Proof. (i) Since μ and $\gamma_{I,J}$ are fixed by the action of W_J and $W_{I\cap J}$ respectively, We have

$$(w_J w_{I\cap J}) \circ \lambda = w_J w_{I\cap J}(\mu - \gamma_{I,J} + \rho) - \rho = \mu - w_J(\gamma_{I,J} - w_{I\cap J}\rho + w_J\rho).$$

By

$$w_{I\cap J}\rho - w_J\rho = (\rho - w_J\rho) - (\rho - w_{I\cap J}\rho) = \sum_{\alpha\in\Delta_J^+} \alpha - \sum_{\alpha\in\Delta_{I\cap J}^+} \alpha = \gamma_{I,J}$$

we obtain $(w_J w_{I\cap J}) \circ \lambda = \mu$. Hence by $w_J w_{I\cap J}(\Delta_J^+ \setminus \Delta_I) \subset \Delta_J^-$ and $\mu \in (\mathfrak{h}_{\mathbf{Z}}^*)_J^0$, we have

$$(\lambda+\rho)(\alpha^\vee) = (w_{I\cap J} w_J(\mu+\rho))(\alpha^\vee) = (\mu+\rho)(w_J w_{I\cap J}\alpha^\vee) < 0$$

for any $\alpha \in \Delta_J^+ \setminus \Delta_I$. Moreover, we have $(\lambda+\rho)(\alpha^\vee) > 0$ for any Δ_I^+ by (2.22).
(ii) We may assume that $n = 0$. Let $U = \{(k_1, \dots, k_l) \in \mathbf{Z}^l : \nu = \sum_{i=1}^l k_i \varpi_{p_i}$satisfies (2.23)$\} \subset (\mathbf{Z}_{<0})^l$. Since $(r_1, \dots, r_l) \in U$ by (i), then $U \neq \emptyset$, and hence $U = \prod_i^l \mathbf{Z}_{\leq k_i^0}$ for some $r_i \leq k_i^0 < 0$ $(i = 1, \dots, l)$. Take any $1 \leq \bar{i} \leq l$, and let $\beta = w_{I\cap J}(\alpha_{p_{\bar{i}}})$: then $\beta \in \Delta_J^+ \setminus \Delta_I$, and from $\lambda + \rho = w_{I\cap J} w_J(\mu+\rho)$ we get $(\lambda + \varpi_{p_{\bar{i}}} + \rho)(\beta^\vee) = \rho(w_J \alpha_{p_{\bar{i}}}^\vee) + \varpi_{p_{\bar{i}}}(\beta^\vee) = \varpi_{p_{\bar{i}}}(\beta^\vee) - 1 \geq 0$: hence $r_{\bar{i}} = k_{\bar{i}}^0$. □

By Proposition 1.6, if the pair (λ, μ) is an extremal case we get a nontrivial $\mathcal{D}_{X_J}$-linear morphism

$$\Phi : \mathcal{D}\mathcal{O}_{X_J}(\mu) \to H^0(R(\mathcal{D}\mathcal{O}_{X_I}(\lambda))). \tag{2.24}$$

Theorem 2.13. *Let (λ, μ) be an extremal case.*

(i) *We have $H^p(R(\mathcal{D}\mathcal{O}_{X_I}(\lambda))) = 0$ for any $p \neq 0$ if and only if $\ell(x) \geq m(x)$ for any $x \in \Gamma(\lambda)$.*

(ii) *Assume that* $H^p(R(\mathcal{D}\mathcal{O}_{X_I}(\lambda))) = 0$ *for any* $p \neq 0$. *Then* Φ *is an epimorphism if and only if* $\ell(x) > m(x)$ *for any* $x \in \Gamma(\lambda) \setminus \{e\}$.

(iii) *Assume that* $H^p(R(\mathcal{D}\mathcal{O}_{X_I}(\lambda))) = 0$ *for any* $p \neq 0$. *Then* Φ *is an isomorphism if and only if* $\Gamma(\lambda) = \{e\}$.

We need the following result in order to prove Theorem 2.13.

Lemma 2.14. *Let* (λ, μ) *be an extremal case, and let* $x_1, x_2 \in \Gamma(\lambda)$. *Set* $y_k = y_{x_k}$ *for* $k = 1, 2$. *If* $L((y_1x_1)\circ\lambda)$ *appears as a subquotient of* $M_J((y_2x_2)\circ\lambda)$, *then we have* $\ell(x_2)-\ell(y_2) \leq \ell(x_1)-\ell(y_1)$.

Proof. For $\xi \in \mathfrak{h}^*_{\mathbf{Z}}$ we set

$$\Delta^+_0(\xi) = \{\alpha \in \Delta^+ \,:\, (\xi + \rho)(\alpha^\vee) = 0\}, \quad W_0(\xi) = \{w \in W \,:\, w \circ \xi = \xi\}.$$

Take $\nu \in W \circ \lambda$ such that $(\nu + \rho)(\alpha^\vee) \geq 0$ for any $\alpha \in \Delta^+$, and let $w \in W$ such that $\lambda = w \circ \nu$. We can assume that $\ell(w) \leq \ell(x)$ for any $x \in W$ satisfying $\lambda = x \circ \nu$. Then w is the (unique) element of $wW_0(\nu)$ with minimal length.

Let us first show:

$$y_kx_kw \text{ is the element of } y_kx_kwW_0(\nu) \text{ with minimal length.} \tag{2.25}$$

It is sufficient to show $y_kx_kw\Delta^+_0(\nu) \subset \Delta^+$. Since w is the element of $wW_0(\nu)$ with minimal length, we have $w\Delta^+_0(\nu) \subset \Delta^+$, and therefore $w\Delta^+_0(\nu) = \Delta^+_0(\lambda)$. By Proposition 2.12 we have $\Delta^+_0(\lambda) \subset \Delta^+ \setminus \Delta_I$. Hence by $W_I(\Delta^+ \setminus \Delta_I) = \Delta^+ \setminus \Delta_I$ we have $x_k\Delta^+_0(\lambda) \subset \Delta^+$. Thus $x_k\Delta^+_0(\lambda) = \Delta^+_0(x_k \circ \lambda)$. By $x_k \in \Gamma(\lambda)$ we have $\Delta^+_0(x_k \circ \lambda) \subset \Delta^+ \setminus \Delta_J$, and hence $y_k\Delta^+_0(x_k \circ \lambda) \subset \Delta^+$. The statement (2.25) is proved.

We next show

$$\ell(y_kx_kw) = \ell(w) + \ell(x_k) - \ell(y_k). \tag{2.26}$$

For any $\alpha \in \Delta^+_I$ we have

$$(\nu + \rho)(w^{-1}\alpha^\vee) = (\lambda + \rho)(\alpha^\vee) > 0,$$

and hence $w^{-1}\Delta^+_I \subset \Delta^+$ by the choice of ν. Thus we have

$$w^{-1}(x_k^{-1}\Delta^+ \cap \Delta^-) = w^{-1}(x_k^{-1}\Delta^+_I \cap \Delta^-_I) \subset w^{-1}\Delta^-_I \subset \Delta^-.$$

Hence $\ell(x_kw) = \ell(w) + \ell(x_k)$. Here, we have used the well-known fact that for $u, v \in W$ we have $\ell(uv) = \ell(u) + \ell(v)$ if and only if $u(v\Delta^+ \cap \Delta^-) \subset \Delta^-$. Similarly, we have

$$(\nu + \rho)(w^{-1}x_k^{-1}y_k^{-1}\alpha^\vee) = (y_kx_k(\lambda + \rho))(\alpha^\vee) > 0$$

for any $\alpha \in \Delta^+_J$ by the definition of y_k and hence $w^{-1}x_k^{-1}y_k^{-1}\Delta^+_J \subset \Delta^+$. Thus we have

$$w^{-1}x_k^{-1}y_k^{-1}(y_k\Delta^+ \cap \Delta^-) = w^{-1}x_k^{-1}y_k^{-1}(y_k\Delta^+_J \cap \Delta^-_J) \subset w^{-1}x_k^{-1}y_k^{-1}\Delta^-_J \subset \Delta^-.$$

Hence $\ell(x_kw) = \ell(y_kx_kw) + \ell(y_k)$. The statement (2.26) is proved.

Note that $L((y_1x_1)\circ\lambda) = L((y_1x_1w)\circ\nu)$ and that $M_J((y_2x_2)\circ\lambda)$ is a quotient of the ordinary Verma module $M((y_2x_2w) \circ \nu) = M_\emptyset((y_2x_2w) \circ \nu)$. Hence by our assumption and by (2.25) we obtain $y_1x_1w \geq y_2x_2w$ with respect to the standard partial order on W by a result of Bernstein-Gelfand-Gelfand [2] concerning the composition factors of Verma modules. In particular, we have $\ell(y_1x_1w) \geq \ell(y_2x_2w)$. Hence we obtain the desired result by (2.26). $\square$

Proof of Theorem 2.13. We shall use the notation in Proposition 2.5.

We first show the following.

$$\text{If } H^r(\mathcal{M}(k)^\bullet) = 0 \text{ for any } k \geq \ell, \text{ then } H^r(\mathcal{C}(k)^\bullet) = 0 \text{ for any } k \geq \ell. \tag{2.27}$$

Assume that there exists some $k \geq \ell$ such that $H^r(\mathcal{C}(k)^\bullet) \neq 0$. Let k_0 be the largest such k. Then we have exact sequences

$$H^{r-1}(\mathcal{M}(k_0+1)^\bullet) \to H^r(\mathcal{C}(k_0)^\bullet) \to 0, \tag{2.28}$$

$$H^{r-1}(\mathcal{C}(k)^\bullet) \to H^{r-1}(\mathcal{M}(k)^\bullet) \to H^{r-1}(\mathcal{M}(k+1)^\bullet) \to 0 \quad (k > k_0). \tag{2.29}$$

By $H^r(\mathcal{C}(k_0)^\bullet) \neq 0$ there exists some $x_1 \in \Gamma(\lambda)$ such that $\ell(x_1) - m(x_1) = -r$, $\ell(x_1) = k_0$ and $\mathcal{D}\mathcal{O}_{X_J}((y_{x_1}x_1) \circ \lambda)$ is a direct summand of $H^r(\mathcal{C}(k_0)^\bullet)$. On the other hand, any irreducible subquotient of $H^r(\mathcal{C}(k_0)^\bullet)$ is isomorphic to an irreducible subquotient of $H^{r-1}(\mathcal{C}(k)^\bullet)$ for some $k \geq k_0 + 1$ by (2.28) and (2.29). Moreover, $H^{r-1}(\mathcal{C}(k)^\bullet)$ is isomorphic to the direct sum of $\mathcal{D}\mathcal{O}_{X_J}((y_{x_2}x_2) \circ \lambda)$ for $x_2 \in \Gamma(\lambda)$ such that $\ell(x_2) - m(x_2) = -(r-1)$, $\ell(x_2) = k$. By the category equivalence given in Proposition 1.9 we see that there exists some $x_2 \in \Gamma(\lambda)$ such that $\ell(x_2) - m(x_2) = -(r-1)$, $\ell(x_2) \geq k_0 + 1$, and that $L((y_{x_1}x_1) \circ \lambda)$ is isomorphic to an irreducible subquotient of $M_J((y_{x_2}x_2) \circ \lambda)$. Then by Lemma 2.14 we have

$$\ell(x_2) - \ell(y_{x_2}) \leq \ell(x_1) - \ell(y_{x_1}). \tag{2.30}$$

On the other hand we have

$$\ell(x_2) + \ell(y_{x_2}) = \ell(x_1) + \ell(y_{x_1}) + 1. \tag{2.31}$$

by Lemma 2.7. Hence we have $2\ell(x_2) \leq 2\ell(x_1) + 1$. Since $\ell(x_1)$ and $\ell(x_2)$ are integers, we obtain $\ell(x_2) \leq \ell(x_1)$. This is a contradiction. The statement (2.27) is proved.

Let us show (i). By Theorem 2.6(iv) we have $H^p(R(\mathcal{D}\mathcal{O}_{X_I}(\lambda))) = 0$ for any $p \neq 0$ if $\ell(x) \geq m(x)$ for any $x \in \Gamma(\lambda)$. Assume $H^p(R(\mathcal{D}\mathcal{O}_{X_I}(\lambda))) = 0$ for any $p > 0$ and $\ell(x) < m(x)$ for some $x \in \Gamma(\lambda)$. Then we have $H^p(\mathcal{M}(0)^\bullet) = 0$ for any $p > 0$ and $H^p(\mathcal{C}(k)^\bullet) \neq 0$ for some $p > 0$ and some $k \geq 0$. Let r be the largest positive integer such that $H^r(\mathcal{C}(k)^\bullet) \neq 0$ for some $k \geq 0$. Then we have an exact sequence

$$H^r(\mathcal{M}(k)^\bullet) \to H^r(\mathcal{M}(k+1)^\bullet) \to 0 \qquad (k \geq 0).$$

Since $H^r(\mathcal{M}(0)^\bullet) = 0$, we see by induction on k that $H^r(\mathcal{M}(k)^\bullet) = 0$ for any $k \geq 0$. Hence by (2.27) we have $H^r(\mathcal{C}(k)^\bullet) = 0$ for any $k \geq 0$. This is a contradiction. The statement (i) is proved.

Let us show (ii). By (i) and the assumption we have $\ell(x) \geq m(x)$ for any $x \in \Gamma(\lambda)$; in other words $H^p(\mathcal{C}(k)^\bullet) = 0$ for any $p > 0$ and any $k \geq 0$. By Theorem 2.6(v) Φ is an epimorphism if $\ell(x) > m(x)$ for any $x \in \Gamma(\lambda) \setminus \{e\}$. Assume that Φ is an epimorphism. Since $\Phi : H^0(\mathcal{C}(0)^\bullet) \to H^0(\mathcal{M}(0)^\bullet)$ is an epimorphism, we have $H^0(\mathcal{M}(k)^\bullet) = 0$ for any $k > 0$ by the exact sequences

$$H^0(\mathcal{C}(0)^\bullet) \to H^0(\mathcal{M}(0)^\bullet) \to H^0(\mathcal{M}(1)^\bullet) \to 0,$$
$$H^0(\mathcal{M}(k)^\bullet) \to H^0(\mathcal{M}(k+1)^\bullet) \to 0$$

Hence by (2.27) we have $H^0(\mathcal{C}(k)^\bullet) = 0$ for any $k > 0$. It implies that $\ell(x) > m(x)$ for any $x \in \Gamma(\lambda) \setminus \{e\}$. The statement (ii) is proved.

Let us finally show (iii). By (i) and the assumption we have $H^p(\mathcal{C}(k)^\bullet) = 0$ for any $p > 0$ and any $k \geq 0$. By Theorem 2.6(v) Φ is an isomorphism if $\Gamma(\lambda) = \{e\}$. Hence it is sufficient to show that $H^{-p}(\mathcal{C}(k)^\bullet) = 0$ for any $k > 0$ and any $p \geq 0$ if Φ is an isomorphism. Let us show it by induction on p. If $p = 0$, then we have $H^0(\mathcal{C}(k)^\bullet) = 0$ for any $k > 0$ by the proof of (ii). Assume that the statement is proved up to p. Consider the exact sequence

$$H^{-(p+1)}(\mathcal{M}(0)^\bullet) \to H^{-(p+1)}(\mathcal{M}(1)^\bullet) \to H^{-p}(\mathcal{C}(0)^\bullet) \to H^{-p}(\mathcal{M}(0)^\bullet).$$

We have $H^{-p}(\mathcal{C}(0)^\bullet) = 0$ for $p > 0$, and $\Phi : H^{-p}(\mathcal{C}(0)^\bullet) \to H^{-p}(\mathcal{M}(0)^\bullet)$ is an isomorphism for $p = 0$. Moreover, we have $H^{-(p+1)}(\mathcal{M}(0)^\bullet) = 0$ by Lemma 1.4. Hence we have $H^{-(p+1)}(\mathcal{M}(1)^\bullet) = 0$. Thus we obtain $H^{-(p+1)}(\mathcal{M}(k)^\bullet) = 0$ for any $k > 0$ by the exact sequence

$$H^{-(p+1)}(\mathcal{M}(k)^\bullet) \to H^{-(p+1)}(\mathcal{M}(k+1)^\bullet) \to H^{-p}(\mathcal{C}(k)^\bullet)$$

and the hypothesis of induction. Hence we have $H^{-(p+1)}(\mathcal{C}(k)^\bullet) = 0$ for any $k > 0$ by (2.27). The statement (iii) is proved. □

Remark 2.15. Let (λ, μ) be an extremal case. For $x \in \Gamma$ and $\alpha \in \Delta_J^+ \setminus \Delta_I$ we have

$$(x(\lambda + \rho))(\alpha^\vee) = (\lambda + x\rho))(\alpha^\vee) = (\mu - \gamma_{I,J} + x\rho)(\alpha^\vee) = (x\rho - \gamma_{I,J})(\alpha^\vee),$$

and hence we have $H^p(R(\mathcal{D}\mathcal{O}_{X_I}(\lambda))) = 0$ for any $p > 0$ if and only if

$$\left\{ \begin{array}{l} \text{for } x \in \Gamma \text{ satisfying } (x\rho - \gamma_{I,J})(\alpha^\vee) \neq 0 \text{ for any } \alpha \in \Delta_J^+ \setminus \Delta_I \text{ we have} \\ \sharp S(x) \leq \ell(x), \text{ where } S(x) = \{\alpha \in \Delta_J^+ \setminus \Delta_I \,:\, (x\rho - \gamma_{I,J})(\alpha^\vee) > 0\}\ . \end{array} \right. \tag{2.32}$$

We do not know an example of (G, I, J) such that (2.32) is not satisfied. Anyway, we can prove that condition (2.32) is satisfied in the following cases (where we say that $A, B \subset I_0$ are "contiguous" if there exists $(i, j) \in A \times B$ such that $\alpha_i(\alpha_j^\vee) \neq 0$):

(1) G has rank ≤ 6;

(2) G is of classical type, and I and J are maximal proper subsets of I_0;

(3) $I \setminus J$ is not contiguous to J, or $J \setminus I$ is not contiguous to I;

(4) I and J are contiguous disjoint irreducible subsystems of I_0.

(For (3) one easily sees that $S(x) = \varnothing$ for any $x \in \Gamma$, while (1), (2) and (4) are obtained with a case-by-case analysis; details are omitted.)

In the next section we shall give conditions in order that Φ is an epimorphism and that Φ is an isomorphism in the case where I and J are maximal proper subsets of I_0. In particular, Φ is not necessarily an epimorphism nor a monomorphism. It seems to be an interesting problem to determine the kernel and the cokernel of Φ.

Remark 2.16. In §3 of Tanisaki [15] we investigated the morphism Φ when $\mathfrak{n}_J$ is commutative using a more geometric method. In particular, we proved that KerΦ corresponds to the unique maximal proper submodule of $M_J(\mu)$ under the category equivalence given in Proposition 1.9 of the present paper (see the proof of Theorem 3.4 of [15]). We also gave sufficient conditions in order that the higher cohomology groups of $R(\mathcal{D}\mathcal{O}_{X_I}(\lambda))$ vanish and that Φ is an epimorphism in terms of geometry of the moment map. These geometric conditions were checked in the case $\mathfrak{g} = \mathfrak{sl}_n$; however, they do not hold in general. This point is overcome using the representation theoretic method employed in the present paper.

3 The maximal parabolic case

In this section we apply our results to the case where P_I and P_J are maximal parabolic subgroups, and obtain results for the Radon transform $R(\mathcal{D}\mathcal{O}_{X_I}(\lambda))$ with respect to the geometric correspondence

$$X_I \xleftarrow{f} X_{I\cap J} \xrightarrow{g} X_J$$

for $\lambda \in (\mathfrak{h}^*_{\mathbf{Z}})^0_I$. In this case we have

$$I = I_0 \setminus \{p\} \text{ and } J = I_0 \setminus \{q\} \text{ for some } p \neq q, \tag{3.1}$$

and $(\mathfrak{h}^*_{\mathbf{Z}})^0_I = \{r\varpi_p : r \in \mathbf{Z}\}$, where ϖ_k denotes the fundamental weight corresponding to $k \in I_0$.

We keep the standard notations of Bourbaki [4]. In particular, if G is of rank n, then $I_0 = \{1, 2, \dots, n\}$.

3.1 The case $(\mathbf{A_n})$

In this subsection we consider the case where $G = SL(V)$ for an $n+1$-dimensional complex vector space V. By the symmetry of the Dynkin diagram we may (and shall) assume that $p > q$. We have the identifications:

$$\begin{aligned} X_I &= \{p\text{-dimensional subspace of } V\}, \\ X_J &= \{q\text{-dimensional subspace of } V\}, \\ X_{I\cap J} &= \{(U_1, U_2) \in X_I \times X_J : U_1 \supset U_2\}, \end{aligned}$$

and f, g are natural projections. The invertible $\mathcal{O}_{X_I}$-module $\mathcal{O}_{X_I}(\varpi_p)$ corresponds to the tautological line bundle whose fiber at $U \in X_I$ is $\bigwedge^p U$ (a subbundle of the product bundle $X_I \times \bigwedge^p V$), and we have $\mathcal{O}_{X_I}(r\varpi_p) = \mathcal{O}_{X_I}(\varpi_p)^{\otimes r}$. Hence in the standard notation of algebraic geometry we have $\mathcal{O}_{X_I}(r\varpi_p) = \mathcal{O}_{X_I}(-r)$.

For $k \in I_0 = \{1, \dots, n\}$ set

$$k_* = n + 1 - k, \quad k_+ = \max\{k, k_*\}, \quad k_- = \min\{k, k_*\}.$$

We first give consequences of Theorem 2.6. A weight $\lambda = \sum_{i=1}^{n+1} \lambda_i \varepsilon_i$ ($\lambda_i \in \mathbf{Z}$, $\sum_{i=1}^{n+1} \lambda_i = 0$) belongs to $(\mathfrak{h}^*_{\mathbf{Z}})_J$ if and only if $\lambda_1 \geq \dots \geq \lambda_q$ and $\lambda_{q+1} \geq \dots \geq \lambda_{n+1}$. The Weyl group W is identified with the symmetric group S_{n+1}, and it acts on the weights by permutations of the components, i. e. $\sigma\lambda = \sum_{i=1}^{n+1} \lambda_i \varepsilon_{\sigma(i)}$ for any $\sigma \in W$. Then we have $W_I = S_p \times S_{p_*}$ and $W_J = S_q \times S_{q_*}$. We have

$$\begin{aligned} \varpi_p &= \frac{1}{n+1}\left[(n+1-p)(\varepsilon_1 + \cdots \varepsilon_p) - p(\varepsilon_{p+1} + \cdots + \varepsilon_{n+1})\right] \\ &= \varepsilon_1 + \cdots + \varepsilon_p + \mathrm{const}(\varepsilon_1 + \cdots + \varepsilon_{n+1}) \\ \rho &= \frac{1}{2}\left[n\varepsilon_1 + (n-2)\varepsilon_2 + \cdots + (-n)\varepsilon_{n+1}\right] \\ &= -\varepsilon_2 - \cdots - n\varepsilon_{n+1} + \mathrm{const}(\varepsilon_1 + \cdots + \varepsilon_{n+1}), \end{aligned}$$

and therefore we get

$$\begin{aligned} r\varpi_p + \rho &= r\varepsilon_1 + (-1+r)\varepsilon_2 + \cdots + (-(p-1)+r)\varepsilon_p - p\varepsilon_{p+1} - \cdots \\ &\qquad \cdots - n\varepsilon_{n+1} + \mathrm{const}(\varepsilon_1 + \cdots + \varepsilon_{n+1}). \end{aligned}$$

By the assumption $q < p$ the set $\Gamma(r\varpi_p)$ consists of $(\sigma, \tau) \in S_p \times S_{p_*}$ satisfying

$$\begin{cases} \tau = e, \\ \sigma^{-1}(1) < \cdots < \sigma^{-1}(q), \\ \sigma^{-1}(q+1) < \cdots < \sigma^{-1}(p), \\ \{\sigma^{-1}(q+1), \ldots, \sigma^{-1}(p)\} \cap \{p+r+1, \ldots, n+r+1\} = \emptyset, \end{cases}$$

and we have

$$\begin{aligned} \ell((\sigma, e)) &= \sharp\{(a,b) \,:\, 1 \le a \le q,\, q+1 \le b \le p,\, \sigma^{-1}(a) > \sigma^{-1}(b)\}, \\ m((\sigma, e)) &= \sharp\{(b,c) \,:\, q+1 \le b \le p,\, p+1 \le c \le n+1,\, \sigma^{-1}(b) < r + c\}. \end{aligned}$$

Hence by Theorem 2.6 we obtain the following results.

Proposition 3.1. (i) *We have $R(\mathcal{D}\mathcal{O}_{X_I}(-a\varpi_p)) = 0$ if $p_- > q$ and $q < a < q_*$.*

(ii) *We have $R(\mathcal{D}\mathcal{O}_{X_I}(-a\varpi_p)) = \mathcal{D}\mathcal{O}_{X_J}(-b\varpi_q)[-c]$ for $(a,b,c) =$*

$$\begin{cases} (q_*, p_*, 0) & (p_- > q), \\ (q, p, (p-q)(p_* - q)) & (p_- > q), \\ (r, r_*, 0),\ \ q \le r \le q_* & (p_- = q,\ \ i.e.\ p = q_*). \end{cases}$$

(iii) *We have $H^k(R(\mathcal{D}\mathcal{O}_{X_I}(-a\varpi_p))) = 0$ for any $k \neq 0$ in the following cases:*

$$\begin{cases} a \ge 1 & (p_- < q), \\ a \ge q & (p_- = q,\ \ i.e.\ p = q_*), \\ a > q & (p_- > q). \end{cases}$$

Let us consider the extremal case. By

$$\gamma_{I,J} = p_* \sum_{i=q+1}^{p} \varepsilon_i - (p-q) \sum_{i=p+1}^{n+1} \varepsilon_i.$$

and (2.22) the extremal case is given by $(-q_*\varpi_p, -p_*\varpi_q)$. By Theorem 2.13 we obtain the following.

Proposition 3.2. *We have $H^k((R(\mathcal{D}\mathcal{O}_{X_I}(-q_*\varpi_p))) = 0$ for any $k \neq 0$, and there exists a canonical nontrivial epimorphism*

$$\Phi : \mathcal{D}\mathcal{O}_{X_J}(-p_*\varpi_q) \to H^0(R(\mathcal{D}\mathcal{O}_{X_I}(-q_*\varpi_p))).$$

Moreover, Φ is an isomorphism if and only if $p_- \geq q$.

Remark 3.3. In the situation of Proposition 3.2 it is proved in [15] that for $p^* \leq q$ the kernel of Φ is the maximal proper G-stable submodule of $\mathcal{D}\mathcal{O}_{X_J}(-p_*\varpi_q)$.

In the rest of this subsection we assume that $q < p_-$ and give application to topological problems. By Propositions 3.1 and 1.7 we have the following.

Proposition 3.4. *For any $F \in \mathbf{D}^{\mathrm{b}}(\mathbf{C}_{X_{J,\mathrm{an}}})$ and $q+1 \leq a \leq q_* - 1$ we have*

$$\begin{aligned} \mathrm{R}\Gamma(X_{I,\mathrm{an}}; r(F) \otimes \mathcal{O}_{X_I}(a\varpi_p)_{\mathrm{an}}) &= 0, \\ \mathrm{RHom}\,(r(F), \mathcal{O}_{X_I}(a\varpi_p)_{\mathrm{an}}) &= 0, \end{aligned}$$

and for $(a,b,c,d) = (q_*, p_*, (p-q)p_*, pp_* - qq_* - q(p-q))$ or $(a,b,c,d) = (q, p, q(p-q), -q(p-q))$ we have

$$\begin{aligned} \mathrm{R}\Gamma(X_{I,\mathrm{an}}; r(F) \otimes \mathcal{O}_{X_I}(a\varpi_p)_{\mathrm{an}}) &\simeq \mathrm{R}\Gamma(X_{J,\mathrm{an}}; F \otimes \mathcal{O}_{X_J}(b\varpi_q)_{\mathrm{an}})[-c], \\ \mathrm{RHom}\,(r(F), \mathcal{O}_{X_I}(a\varpi_p)_{\mathrm{an}}) &\simeq \mathrm{RHom}\,(F, \mathcal{O}_{X_J}(b\varpi_q)_{\mathrm{an}})[-d]. \end{aligned}$$

Let us treat some particular cases. In the following we set $N = qq_*$.

(1) Let $y_\circ \in X_J$, and set $F = \mathbf{C}_{\{y_\circ\}}$. Since $g^{-1}(y_\circ) \to X_{I,y_\circ}$ is a closed embedding, one has

$$r(F) \simeq \mathbf{C}_{X_{I,y_\circ,\mathrm{an}}}, \tag{3.2}$$

where $X_{I,y_\circ} = fg^{-1}(\{y_\circ\}) = \{x \in X_I : y_\circ \subset x\}$ (identified with the Grassmannian of $(p-q)$-subspaces of $V/y_\circ$). By Proposition 3.4 and (3.2) we obtain the following.

Proposition 3.5. *For any $q+1 \leq a \leq q_* - 1$ we have*

$$\mathrm{R}\Gamma(X_{I,y_\circ,\mathrm{an}}; \mathcal{O}_{X_I}(a\varpi_p)_{\mathrm{an}}) \simeq 0, \qquad \mathrm{R}\Gamma_{X_{I,y_\circ,\mathrm{an}}}(X_I; \mathcal{O}_{X_I}(a\varpi_p)_{\mathrm{an}}) \simeq 0,$$

and for $(a,c,d) = (q_*, (p-q)p_*, pp_* - qq_* + p_*q)$ or $(a,c,d) = (q, q(p-q), p_*q)$ we have

$$H^c(X_{I,y_\circ,\mathrm{an}}; \mathcal{O}_{X_I}(a\varpi_p)_{\mathrm{an}}) \simeq \mathbf{C}\{z\}, \quad H^d_{X_{I,y_\circ,\mathrm{an}}}(X_I; \mathcal{O}_{X_I}(a\varpi_p)_{\mathrm{an}}) \simeq \mathcal{B}^\infty_{0|\mathbf{C}^N}$$

where $\mathbf{C}\{z\}$ (resp. $\mathcal{B}^\infty_{0|\mathbf{C}^N}$) is the ring of convergent power series in $z = (z_1, \dots, z_N) \in \mathbf{C}^N$ (resp. the ring of hyperfunctions in $\mathbf{C}^N$ along $\{0\}$ of infinite order), and all other cohomology groups vanish.

Namely, there are natural identifications $\mathrm{R}\Gamma(X_{J,\mathrm{an}}; \mathbf{C}_{y_\circ} \otimes \mathcal{O}_{X_J}(b\varpi_q)_{\mathrm{an}}) \simeq \mathrm{R}\Gamma(\{0\}; \mathcal{O}_{\mathbf{C}^N_{\mathrm{an}}}) = \mathbf{C}\{z\}$ and $\mathrm{RHom}\,(\mathbf{C}_{y_\circ}; \mathcal{O}_{X_J}(b\varpi_q)_{\mathrm{an}}) \simeq \mathrm{R}\Gamma_{\{0\}}(\mathbf{C}^N_{\mathrm{an}}; \mathcal{O}_{\mathbf{C}^N_{\mathrm{an}}}) = \mathcal{B}^\infty_{0|\mathbf{C}^N}[-N]$.

(2) Let $z_\circ$ be a q_*-subspace of V, $E_{z_\circ} = \{y \in X_J : y \cap z_\circ = 0\} \simeq \mathbf{C}^N$ and set $F = \mathbf{C}_{E_{z_\circ,\mathrm{an}}}$. One has

$$r(F) \simeq \mathbf{C}_{\widehat{E}_{z_\circ,\mathrm{an}}}[-2q(p-q)], \tag{3.3}$$

where $\widehat{E_{z_\circ}} = fg^{-1}(E_{z_\circ}) = \{x \in X_I : \dim(x \cap z_\circ) = p-q\}$ (i.e. the p-dimensional subspaces of V in generic position w.r.t. $z_\circ$). Namely, the map $\tilde{f} = (f|_{g^{-1}(E_{z_\circ})})_{\mathrm{an}} : (g^{-1}(E_{z_\circ}))_{\mathrm{an}} \to \widehat{E}_{z_\circ,\mathrm{an}}$ is a complex vector bundle of rank $q(p-q)$ (the fiber over $x \in \widehat{E}_{z_\circ}$ is $S_{E_{z_\circ},x} = \{y \in E_{z_\circ} : y \subset x\} \simeq \mathbf{C}^{q(p-q)}$); hence there is a morphism of functors $R\tilde{f}_*\tilde{f}^{-1}[2q(p-q)] \to \mathrm{id}_{\mathbf{D}^{\mathrm{b}}(\mathbf{C}_{\widehat{E_{z_\circ},\mathrm{an}}})}$ defining a natural morphism $r(F) = Rf_{\mathrm{an}!}\mathbf{C}_{(g^{-1}(E_{z_\circ}))_{\mathrm{an}}} \to \mathbf{C}_{\widehat{E}_{z_\circ,\mathrm{an}}}[-2q(p-q)]$, which is an isomorphism since, by (1.4), one has $r(F)_x \simeq \mathbf{C}[-2q(p-q)]$ (for $x \in \widehat{E}_{z_\circ,\mathrm{an}}$) and $= 0$ (otherwise).

By Proposition 3.4 and (3.3) we obtain the following.

Proposition 3.6. *For any $q+1 \le a \le q_* - 1$ we have*

$$\mathrm{R}\Gamma_{\mathrm{c}}(\widehat{E}_{z_\circ,\mathrm{an}}; \mathcal{O}_{X_I}(a\varpi_p)_{\mathrm{an}}) \simeq 0, \qquad \mathrm{R}\Gamma(\widehat{E}_{z_\circ,\mathrm{an}}; \mathcal{O}_{X_I}(a\varpi_p)_{\mathrm{an}}) \simeq 0,$$

and for $(a,c,d) = (q_*, p(p_*-q)+q^2, p_*(p-q))$ or $(a,c,d) = (q, p_*q, q(p-q))$ we have

$$\begin{aligned} H^c_{\mathrm{c}}(\widehat{E}_{z_\circ,\mathrm{an}}; \mathcal{O}_{X_I}(a\varpi_p)_{\mathrm{an}}) &\simeq H^N_{\mathrm{c}}(E_{z_\circ,\mathrm{an}}; \mathcal{O}_{E_{z_\circ,\mathrm{an}}}), \\ H^d(\widehat{E}_{z_\circ,\mathrm{an}}; \mathcal{O}_{X_I}(a\varpi_p)_{\mathrm{an}}) &\simeq \Gamma(E_{z_\circ,\mathrm{an}}; \mathcal{O}_{E_{z_\circ,\mathrm{an}}}) \end{aligned}$$

where $H^N_{\mathrm{c}}(E_{z_\circ,\mathrm{an}}; \mathcal{O}_{E_{z_\circ,\mathrm{an}}}) \simeq \Gamma(E_{z_\circ,\mathrm{an}}; \Omega_{E_{z_\circ,\mathrm{an}}})'$ (resp. $\Gamma(E_{z_\circ,\mathrm{an}}; \mathcal{O}_{E_{z_\circ,\mathrm{an}}})$) are Martineau's analytic functionals (resp. the entire functions) in $E_{z_\circ,\mathrm{an}} \simeq \mathbf{C}^N$, and all other cohomology groups vanish.

Namely, one identifies $\mathrm{R}\Gamma(X_{J,\mathrm{an}}; \mathbf{C}_{E_{z_\circ,\mathrm{an}}} \otimes \mathcal{O}_{X_J}(b\varpi_q)_{\mathrm{an}}) \simeq H^N_{\mathrm{c}}(E_{z_\circ,\mathrm{an}}; \mathcal{O}_{E_{z_\circ,\mathrm{an}}})[-N]$ and $\mathrm{RHom}\,(\mathbf{C}_{E_{z_\circ,\mathrm{an}}}; \mathcal{O}_{X_J}(b\varpi_q)_{\mathrm{an}}) \simeq \Gamma(E_{z_\circ,\mathrm{an}}; \mathcal{O}_{E_{z_\circ,\mathrm{an}}})$.

3.2 The case $(\mathbf{B_n})$

In this subsection we consider the case where G is (the universal covering group of) $SO(V)$ for an $2n+1$-dimensional complex vector space V equipped with a non-degenerate symmetric bilinear form $(\ ,\ ) : V \times V \to \mathbf{C}$. Then we have the identifications:

$$\begin{aligned} X_I &= \{p\text{-dimensional subspace } U \text{ of } V \text{ such that } (U,U) = 0\}, \\ X_J &= \{q\text{-dimensional subspace } U \text{ of } V \text{ such that } (U,U) = 0\}, \\ X_{I\cap J} &= \begin{cases} \{(U_1,U_2) \in X_I \times X_J : U_1 \subset U_2\} & (p < q) \\ \{(U_1,U_2) \in X_I \times X_J : U_1 \supset U_2\} & (p > q), \end{cases} \end{aligned}$$

and f, g are natural projections. The invertible $\mathcal{O}_{X_I}$-module $\mathcal{O}_{X_I}(\varpi_p)$ corresponds to the tautological line bundle whose fiber at $U \in X_I$ is $\bigwedge^p U$.

By Theorem 2.6 we have the following.

Proposition 3.7. (i) *We have $R(\mathcal{D}\mathcal{O}_{X_I}(-a\varpi_p)) = 0$ in the following cases:*

$$\begin{cases} 2n-p-q < a < q & \text{if } p < q \le n, \\ q < a < 2n-p-q & \text{if } q < p < n, \\ 2q < a < 2(n-q) & \text{if } p = n. \end{cases}$$

(ii) *We have* $R(\mathcal{D}\mathcal{O}_{X_I}(-a\varpi_p)) = \mathcal{D}\mathcal{O}_{X_J}(-b\varpi_q)[-c]$ *for* $(a,b,c) =$

$$\begin{cases} (q,p,0),\ (2n-p-q, 2n-p-q, c_1) & (p<q<n,\ 2n-2p-q \le 0), \\ (n,2p,0) & (q=n,\ n-2p\le 0), \\ (2n-p-q, 2n-p-q, 0) & (q<p<n,\ 2n-2p-q\ge 0), \\ (q,p,c_2) & (q<p<n,\ 2n-2p-q\ge 0), \end{cases}$$

where

$$c_1 = \frac{(q-p)(3p+3q-4n-1)}{2}, \qquad c_2 = \frac{(p-q)(4n+1-3p-3q)}{2}.$$

By Theorem 2.13 we have the following.

Proposition 3.8. *Let*

$$(r,s) = \begin{cases} (q,p) & \text{if } 1\le p<q\le n-1, \\ (2n-p-q, 2n-p-q) & \text{if } 1\le q<p\le n-1, \\ (2(n-q), n-q) & \text{if } p=n,\ 1\le q\le n-1, \\ (n,2p) & \text{if } 1\le p\le n-1,\ q=n. \end{cases}$$

Then we have $H^k(R(\mathcal{D}\mathcal{O}_{X_I}(-r\varpi_p))) = 0$ *for any* $k\neq 0$, *and there exists a canonical nontrivial morphism*

$$\Phi : \mathcal{D}\mathcal{O}_{X_J}(-s\varpi_q) \to H^0(R(\mathcal{D}\mathcal{O}_{X_I}(-r\varpi_p))).$$

Moreover, Φ *is an epimorphism if and only if we have either*

(a) $p<q\le n$,

(b) $q<p<n$ *and* $2n-2p-q\ge 0$,

and an isomorphism if and only if we have either

(a) $p<q\le n$ *and* $2n-2p-q\le 0$,

(b) $q<p<n$ *and* $2n-2p-q\ge 0$.

3.3 The case $(\mathbf{C_n})$

In this subsection we consider the case where $G = Sp(V)$ for an $2n$-dimensional complex vector space V equipped with a non-degenerate anti-symmetric bilinear form $(\ ,\ ) : V\times V\to \mathbf{C}$. Then we have the identifications:

$$\begin{aligned} X_I &= \{p\text{-dimensional subspace } U \text{ of } V \text{ such that } (U,U)=0\}, \\ X_J &= \{q\text{-dimensional subspace } U \text{ of } V \text{ such that } (U,U)=0\}, \\ X_{I\cap J} &= \begin{cases} \{(U_1,U_2)\in X_I\times X_J : U_1\subset U_2\} & (p<q) \\ \{(U_1,U_2)\in X_I\times X_J : U_1\supset U_2\} & (p>q), \end{cases} \end{aligned}$$

and f, g are natural projections. The invertible $\mathcal{O}_{X_I}$-module $\mathcal{O}_{X_I}(\varpi_p)$ corresponds to the tautological line bundle whose fiber at $U\in X_I$ is $\bigwedge^p U$.

By Theorem 2.6 we have the following.

Proposition 3.9. (i) *We have $R(\mathcal{D}\mathcal{O}_{X_I}(-a\varpi_p)) = 0$ in the following cases:*

$$\begin{cases} 2n-p-q+1 < a < q & \text{if } \ p<q, \\ q < a < 2n-p-q+1 & \text{if } \ q<p. \end{cases}$$

(ii) *We have $R(\mathcal{D}\mathcal{O}_{X_I}(-a\varpi_p)) = \mathcal{D}\mathcal{O}_{X_J}(-b\varpi_q)[-c]$ for $(a,b,c) =$*

$$\begin{cases} (q,p,0) & (p<q\le n,\ 2n-2p-q+1\le 0), \\ (2n-p-q+1, 2n-p-q+1, c_1) & (p<q\le n,\ 2n-2p-q+1\le 0), \\ (2n-p-q+1, 2n-p-q+1, 0) & (q<p\le n,\ 2n-2p-q+1\ge 0), \\ (q,p,c_2) & (q<p\le n,\ 2n-2p-q+1\ge 0), \end{cases}$$

where

$$c_1 = \frac{(q-p)(3p+3q-4n-1)}{2}, \qquad c_2 = \frac{(p-q)(4n+1-3p-3q)}{2}.$$

By Theorem 2.13 we have the following.

Proposition 3.10. *Let*

$$(r,s) = \begin{cases} (q,p) & \text{if } 1\le p<q\le n, \\ (2n-p-q+1, 2n-p-q+1) & \text{if } 1\le q<p\le n. \end{cases}$$

Then we have $H^k(R(\mathcal{D}\mathcal{O}_{X_I}(-r\varpi_p))) = 0$ for any $k \neq 0$, and there exists a canonical nontrivial morphism

$$\Phi : \mathcal{D}\mathcal{O}_{X_J}(-s\varpi_q) \to H^0(R(\mathcal{D}\mathcal{O}_{X_I}(-r\varpi_p))).$$

Moreover, Φ is an epimorphism if and only if we have either

(a) $p<q<n$ *and* $n-p-q\ge 0$,

(b) $p<q\le n$ *and* $2n-2p-q+1\le 0$,

(c) $q<p\le n$,

and an isomorphism if and only if we have either

(a) $p<q\le n$ *and* $2n-2p-q+1\le 0$,

(b) $q<p\le n$ *and* $2n-2p-q+1\ge 0$.

Remark 3.11. In the situation of Proposition 3.10 it is proved in [15] that $\mathrm{Ker}\Phi$ is the maximal proper G-stable submodule of $\mathcal{D}\mathcal{O}_{X_J}(-s\varpi_q)$ if $q = n$ and $2p \le n-1$.

3.4 The case $(\mathbf{D_n})$

In this subsection we consider the case where G is (the universal covering group of) $SO(V)$ for an $2n$-dimensional complex vector space V equipped with a non-degenerate symmetric bilinear form $(\ ,\ ) : V \times V \to \mathbf{C}$.

For $1 \le k \le n$ set

$$X(k) = \{k\text{-dimensional subspace } U \text{ of } V \text{ such that } (U, U) = 0\}.$$

Then $X(k)$ is connected for $1 \le k \le n-1$, and $X(n)$ has two connected components, say $X_1(n)$ and $X_2(n)$. Then we have the identification:

$$\begin{aligned} &X(k) = X_{I_0\setminus\{k\}} \qquad (1 \le k \le n-2),\\ &X(n-1) = X_{I_0\setminus\{n-1,n\}},\\ &X_1(n) = X_{I_0\setminus\{n\}},\\ &X_2(n) = X_{I_0\setminus\{n-1\}}. \end{aligned}$$

If $\{p, q\} \ne \{n-1, n\}$, then

$$X_{I\cap J} = \begin{cases} \{(U_1, U_2) \in X_I \times X_J : U_1 \subset U_2\} & (p < q) \\ \{(U_1, U_2) \in X_I \times X_J : U_1 \supset U_2\} & (p > q), \end{cases}$$

and if $p = n-1$ and $q = n$, then f (resp. g) assigns $U \in X_{I\cap J} = X(n-1)$ to the unique $U' \in X_I = X_2(n)$ (resp. $U' \in X_J = X_1(n)$) such that $U \subset U'$. The invertible $\mathcal{O}_{X_I}$-module $\mathcal{O}_{X_I}(\varpi_p)$ corresponds to the tautological line bundle whose fiber at $U \in X_I$ is $\bigwedge^k U$ where $k = p$ for $1 \le k \le n-2$ and $k = n$ for $p \in \{n-1, n\}$.

By Theorem 2.6 we have the following.

Proposition 3.12. (i) *We have $R(\mathcal{D}\mathcal{O}_{X_I}(-a\varpi_p)) = 0$ in the following cases:*

$$\begin{cases} 2n-p-q-1 < a < q & \text{if } p < q \le n-2, \\ q < a < 2n-p-q-1 & \text{if } q < p \le n-2 \\ 2q < a < 2(n-q-1) & \text{if } p \in \{n-1, n\},\ 1 \le q \le n-2, \\ n-p-1 < a < n & \text{if } 1 \le p \le n-2,\ q \in \{n-1, n\}, \\ a = n-1 & \text{if } \{p, q\} = \{n-1, n\} \text{ and } n \text{ is even.} \end{cases}$$

(ii) *We have $R(\mathcal{D}\mathcal{O}_{X_I}(-a\varpi_p)) = \mathcal{D}\mathcal{O}_{X_J}(-b\varpi_q)[-c]$ for $(a, b, c) =$*

$$\begin{cases} (q, p, 0) & (p < q \le n-2,\ 2n-2p-q-1 \le 0), \\ (2n-p-q-1, 2n-p-q-1, c_1) & (p < q \le n-2,\ 2n-2p-q-1 \le 0), \\ (n, 2p, 0) & (p \le n-2,\ q \in \{n-1, n\},\ n-2p-1 \le 0), \\ (n-p-1, 2(n-p-1), c_2) & (p \le n-2,\ q \in \{n-1, n\},\ n-2p-1 \le 0), \\ (2n-p-q-1, 2n-p-q-1, 0) & (q < p \le n-2,\ 2n-2p-q-1 \ge 0), \\ (q, p, c_3) & (q < p \le n-2,\ 2n-2p-q-1 \ge 0), \\ (n, n-2, 0) & (\{p, q\} = \{n-1, n\},\ n : \textit{odd}), \\ (n-1, n-1, 0) & (\{p, q\} = \{n-1, n\},\ n : \textit{odd}), \\ (n-2, n, 0) & (\{p, q\} = \{n-1, n\},\ n : \textit{odd}), \end{cases}$$

where

$$c_1 = \frac{(q-p)(3p+3q-4n+1)}{2}, \qquad c_2 = \frac{(n-p)(3p-n+1)}{2},$$
$$c_3 = \frac{(p-q)(4n-3p-3q-1)}{2}.$$

By Theorem 2.13 we have the following.

Proposition 3.13. *Let* $(r,s) =$

$$\begin{cases} (q,p) & \text{if } 1 \le p < q \le n-2, \\ (2n-p-q-1, 2n-p-q-1) & \text{if } 1 \le q < p \le n-2, \\ (2(n-q-1), n-q-1) & \text{if } p \in \{n-1,n\},\ 1 \le q \le n-2, \\ (n, 2p) & \text{if } 1 \le p \le n-2,\ q \in \{n-1,n\}, \\ (n, n-2) & \text{if } \{p,q\} = \{n-1,n\}. \end{cases}$$

Then we have $H^k(R(\mathcal{D}\mathcal{O}_{X_I}(-r\varpi_p))) = 0$ *for any* $k \neq 0$*, and there exists a canonical nontrivial epimorphism*

$$\Phi : \mathcal{D}\mathcal{O}_{X_J}(-s\varpi_q) \to H^0(R(\mathcal{D}\mathcal{O}_{X_I}(-r\varpi_p))).$$

Moreover, Φ *is an isomorphism if and only if we have either*

(a) $p < q < n-1$ *and* $2n-2p-q-1 \le 0$,

(b) $q < p < n-1$ *and* $2n-2p-q-1 \ge 0$,

(c) $p < n-1$, $q \in \{n-1,n\}$ *and* $n-2p-1 \le 0$,

(d) $\{p,q\} = \{n-1,n\}$ *and* n *is odd.*

Remark 3.14. In the situation of Proposition 3.13 it is proved in [15] that $\mathrm{Ker}\Phi$ is the maximal proper G-stable submodule of $\mathcal{D}\mathcal{O}_{X_J}(-s\varpi_q)$ if $q \in \{n-1,n\}, 2p \le n-2$ and if $q = 1, p \in \{n-1,n\}$.

3.5 The exceptional cases ($\mathbf{G_2}$), ($\mathbf{F_4}$), ($\mathbf{E_6}$)

We write here the tables for the maximal parabolic cases in the exceptional algebras G_2, F_4, E_6. We obtained them by a case-by-case analysis.

As above, here we define $I = I_0 \setminus \{p\}$ and $J = I_0 \setminus \{q\}$. In the first line we write the $a \in \mathbf{Z}$ such that $R(\mathcal{D}\mathcal{O}_{X_{I_p}}(a\varpi_p)) = 0$ and the $a \in \mathbf{Z}$ such that $H^j(R(\mathcal{D}\mathcal{O}_{X_{I_p}}(a\varpi_p))) = 0$ for $j \neq 0$. In the second line, we write the $[a,b,c] \in \mathbf{Z}^3$ such that $R(\mathcal{D}\mathcal{O}_{X_{I_p}}(a\varpi_p)) = \mathcal{D}\mathcal{O}_{X_{I_q}}(\mu)[-c]$ with $\mu = b\varpi_q$ or, sometimes, $[a,(b_1,\dots,b_r),c] \in \mathbf{Z} \times \mathbf{Z}^r \times \mathbf{Z}$ and $\mu = \sum_{i=1}^r b_i\varpi_i$ (here r is the rank of the Lie algebra). In the third line we write the $(b_1,\dots,b_r) \in \mathbf{Z}^r$ such that there exists $\Phi : \mathcal{D}\mathcal{O}_{X_{I_q}}(\mu) \to H^0(R(\mathcal{D}\mathcal{O}_{X_{I_p}}(a\varpi_p)))$, with $\mu = \sum_{i=1}^r b_i\varpi_i$, as well as some informations about Φ. Finally, in the fourth line we write $[a,b] \in \mathbf{Z}^2$ such that $(\lambda,\mu) = (a\varpi_p, b\varpi_q)$ is the extremal case, and some informations about Φ.

G2	$q = p+1$ (mod 2)
$p=1$	— ; $a \le -1$ $[-1,(2,-2),0]$, $[-4,(2,-3),0]$ $(-a-2,a+1)$, $a \le -2$; epi $a \le -4$ $[-2,-1]$; no cohom. >0; no epi
$p=2$	— ; $a \le -1$ $[-1,-2,0]$, $[-2,-3,0]$ $(3a+3,-a-2)$, $a \le -2$; epi $[-2,-3]$; no cohom. >0; iso

F4	$q = p+1$ (mod 4)	$q = p+2$ (mod 4)	$q = p+3$ (mod 4)
$p=1$	— ; $a \le -1$ — $(-a-2,a+1,0,0)$, $a \le -2$; epi $[-2,-1]$; no cohom. >0; epi; no iso	— ; $a \le -1$ — $(0,-a-3,2a+4,0)$, $a \le -3$; epi $[-3,-2]$; no cohom. >0; epi; no iso	$a=-4$; $a \le -4$ — $(-a-5,0,0,2a+5)$, $a \le -5$; epi $[-5,-5]$; no cohom. >0; epi; no iso
$p=2$	— ; $a \le -2$ $[-2,-3,0]$, $[-3,-4,0]$ $(-a-3,0,2a+2,0)$, $a \le -3$; epi $[-3,-4]$; no cohom. >0; iso	$a=-3,-2$; $a \le -2$ $[-1,-3,5]$, $[-4,-8,0]$ $(0,-a-4,0,4a+8)$, $a \le -4$; epi $[-4,-8]$; no cohom. >0; iso	$a=-3,-2$; $a \le -2$ $[-1,-2,5]$, $[-4,-6,0]$ $(3a+6,-a-4,0,0)$, $a \le -4$; epi $[-4,-6]$; no cohom. >0; iso
$p=3$	$a=-5,-4,-3,-2$; $a \le -2$ $[-1,-2,5]$, $[-6,-9,0]$ $(0,,0,-a-6,3a+9)$, $a \le -6$; epi $[-6,-9]$; no cohom. >0; iso	$a=-4,-3$; $a \le -3$ $[-1,(-3,0,1,0),5]$, $[-2,-3,5]$, $[-5,-5,0]$ $(2a+5,0,-a-5,0)$, $a \le -5$; epi $[-5,-5]$; no cohom. >0; iso	— ; $a \le -3$ — $(0,a+1,0,-a-3)$, $a \le -3$; epi $a \le -5$ $[-3,-2]$; no cohom. >0; no epi
$p=4$	— ; $a \le -6$ — $(a+3,0,0,-a-6)$, $a \le -6$; epi $[-6,-3]$; no cohom. >0; epi; no iso	— ; $a \le -1$ — $(0,a+2,-a-3,0)$, $a \le -3$; epi $a \le -4$ $[-3,-1]$; no cohom. >0; no epi	— ; $a \le -1$ — $(0,0,a+1,-a-2)$, $a \le -2$; epi $[-2,-1]$; no cohom. >0; epi; no iso

E6	$q = p+1$ (mod 6)	$q = p+2$ (mod 6)	$q = p+3$ (mod 6)
$p=1$	— ; $a \le -4$ — $(0,a+3,0,0,0,-a-6)$, $a \le -6$; epi $[-6,-3]$; no cohom. >0; epi; no iso	— ; $a \le -1$ — $(-a-2,0,a+1,0,0,0)$, $a \le -2$; epi $[-2,-1]$; no cohom. >0; epi; no iso	— ; $a \le -1$ — $(0,0,-a-3,a+2,0,0)$, $a \le -3$; epi $[-3,-1]$; no cohom. >0; epi; no iso
$p=2$	— ; $a \le -4$ — $(0,0,a+2,0,0,-a-5)$, $a \le -5$; epi $[-5,-3]$; no cohom. >0; epi; no iso	— ; $a \le -1$ — $(0,-a-2,0,a+1,0,0)$, $a \le -2$; epi $[-2,-1]$; no cohom. >0; epi; no iso	— ; $a \le -4$ — $(-a-5,0,0,0,a+2,0)$, $a \le -5$; epi $[-5,-3]$; no cohom. >0; epi; no iso
$p=3$	— ; $a \le -2$ — $(-a-3,0,0,a+1,0,0)$, $a \le -3$; epi $[-3,-2]$; no cohom. >0; epi; no iso	— ; $a \le -4$ $[-4,-5,0]$, $[-5,-4,0]$ $(0,0,0,-a-5,2a+6,0)$, $a \le -5$; epi $[-5,-4]$; no cohom. >0; iso	$a=-6,-5,-4,-3$; $a \le -3$ $[-2,-5,9]$, $[-7,-7,0]$ $(0,0,-a-7,0,0,2a+7)$, $a \le -7$; epi $[-7,-7]$; no cohom. >0; iso
$p=4$	$a=-4,-3$; $a \le -3$ $[-2,-3,4]$; $[-5,-6,0]$ $(0,0,-a-5,0,2a+4,0)$, $a \le -5$; epi $[-5,-6]$; no cohom. >0; iso	$a=-5,-4,-3,-2$; $a \le -2$ $[-1,-3,13]$, $[-6,-9,0]$ $(0,0,0,-a-6,0,3a+9)$, $a \le -6$; epi $[-6,-9]$; no cohom. >0; iso	$a=-5,-4,-3,-2$; $a \le -2$ $[-1,-3,13]$, $[-6,-9,0]$ $(3a+9,0,0,-a-6,0,0)$, $a \le -6$; epi $[-6,-9]$; no cohom. >0; iso
$p=5$	$a=-7,-6,-5,-4,-3,-2$; $a \le -2$ $[-1,-2,9]$, $[-8,-10,0]$ $(0,-a-8,0,0,0,2a+6)$, $a \le -8$; epi $[-8,-10]$; no cohom. >0; iso	$a=-6,-5,-4,-3$; $a \le -3$ $[-2,-5,9]$, $[-7,-7,0]$ $(2a+7,0,0,0,-a-7,0)$, $a \le -7$; epi $[-7,-7]$; no cohom. >0; iso	$a=-5,-4$; $a \le -4$ $[-3,-5,4]$, $[-6,-6,0]$ $(0,2a+6,-a-6,0,0,0)$, $a \le -6$; epi $[-6,-6]$; no cohom. >0; iso
$p=6$	— ; $a \le -4$ $[-a,a+12,0]$, $-8 \le a \le -4$ $(a+4,0,0,0,0,-a-8)$, $a \le -8$; epi $[-8,-4]$; no cohom. >0; iso	— ; $a \le -4$ — $(-a-6,a+3,0,0,0,0)$, $a \le -6$; epi $[-6,-3]$; no cohom. >0; epi; no iso	— ; $a \le -4$ — $(0,-a-5,a+3,0,0,0)$, $a \le -5$; epi $[-5,-2]$; no cohom. >0; epi; no iso

E6	$q = p+4$ (mod 6)	$q = p+5$ (mod 6)
$p=1$	— ; $a \le -4$ — $(0,-a-5,0,0,a+3,0)$, $a \le -5$; epi $[-5,-2]$; no cohom. >0; epi; no iso	— ; $a \le -4$ $[-a,a+12,0]$, $-8 \le a \le -4$ $(-a-8,0,0,0,0,a+4)$, $a \le -8$; epi $[-8,-4]$; no cohom. >0; iso
$p=2$	$a=-7,-6,-5,-4$; $a \le -4$ $[-3,-6,5]$, $[-8,-6,0]$ $(0,0,0,0,-a-8,2a+10)$, $a \le -8$; epi $[-8,-6]$; no cohom. >0; iso	$a=-7,-6,-5,-4$; $a \le -4$ $[-3,-6,5]$, $[-8,-6,0]$ $(2a+10,0,-a-8,0,0,0)$, $a \le -8$; epi $[-8,-6]$; no cohom. >0; iso
$p=3$	$a=-7,-6,-5,-4,-3,-2$; $a \le -2$ $[-1,-2,9]$, $[-8,-10,0]$ $(2a+6,-a-8,0,0,0,0)$, $a \le -8$; epi $[-8,-10]$; no cohom. >0; iso	$a=-5,-4$; $a \le -4$ $[-3,-5,4]$, $[-6,-6,0]$ $(0,2a+12,0,0,-a-6,0)$, $a \le -6$; epi $[-6,-6]$; no cohom. >0; iso
$p=4$	$a=-5,-4,-3,-2$; $a \le -2$ $[-1,-2,8]$, $[-6,-9,0]$ $(3a+9,0,0,-a-6,0,0)$, $a \le -6$; epi $[-6,-9]$; no cohom. >0; iso	$a=-4,-3$; $a \le -3$ $[-2,-3,4]$, $[-5,-6,0]$ $(0,0,2a+4,0,-a-5,0)$, $a \le -5$; epi $[-5,-6]$; no cohom. >0; iso
$p=5$	— ; $a \le -4$ $[-4,-5,0]$, $[-5,-4,0]$ $(0,0,2a+6,-a-5,0,0)$, $a \le -5$; epi $[-5,-4]$; no cohom. >0; iso	— ; $a \le -2$ — $(0,0,0,a+1,0,-a-3)$, $a \le -3$; epi $[-3,-2]$; no cohom. >0; epi; no iso
$p=6$	— ; $a \le -1$ — $(0,0,0,a+2,-a-3,0)$, $a \le -3$; epi $[-3,-1]$; no cohom. >0; epi; no iso	— ; $a \le -1$ — $(0,0,0,0,a+1,-a-2)$, $a \le -2$; epi $[-2,-1]$; no cohom. >0; epi; no iso

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