A TOPOLOGICAL OBSTRUCTION FOR THE REAL RADON TRANSFORM

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Abstract. A general framework to deal with problems of integral geometry is provided by the recently developed theory of integral transforms for sheaves and $\mathcal{D}$-modules. Our aim here is to illustrate such techniques by reconsidering the most classical example of integral transforms. Namely, the Radon hyperplane transform for C^∞-functions.

Introduction

The theory of integral transforms for sheaves and $\mathcal{D}$-modules provides a natural framework to deal with the problems of integral geometry. In particular, a general adjunction formula allows one to separate the analytical aspects of the problem from the topological ones. In [3, 1] such theory was applied to the study of the Radon hyperplane transform (see [2] for an exposition).

The Radon transform associates to a homogeneous C^∞ function on a real projective space P its integrals along the family of hyperplanes, thus yielding a homogeneous C^∞ function on the dual projective space P^*. It is a well-know fact that such transform is invertible in a suitable range of the degree of homogeneity. A classical approach (see e.g. [5]) is to deduce this result from the inversion formula for the Fourier transform. Another approach, closer to ours, is that of [8, Proposition 4.1.3], where the real Radon transform is considered as the "boundary value" of the complex one (see also [7] for a similar point of view).

Here we will use the methods of [3, 1] to discuss the case of arbitrary homogeneity. In particular, we will calculate exactly the finite dimensional obstruction to invertibility, and we will show how such obstruction is purely topological.

1. Notations

Let us recall some notations and results on the hyperplane Radon transform, referring the reader e.g. to [5].

Affine hyperplanes in $\mathbf{R}^n \ni y$ are described by equations

$$\langle y, \eta\rangle + \tau = 0,$$

and are thus parameterized by $(\eta, \tau) \in ((\mathbf{R}^n)^* \setminus \{0\}) \times \mathbf{R}$. The classical Radon transform associates to a rapidly decreasing C^∞ function f on $\mathbf{R}^n$ its integrals

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We present here the results of a paper that will appear elsewhere.

along affine hyperplanes, and reads

$$f(y) \mapsto g(\eta,\tau) = \int_{\langle y,\eta\rangle+\tau=0} f(y)\, d\mu_{\eta,\tau}, \tag{1}$$

where $d\mu_{\eta,\tau}$ is the measure on the hyperplane $\langle y,\eta\rangle+\tau=0$ induced by the volume element $dy_1\wedge\cdots\wedge dy_n$.

Let P be a real projective space of dimension n, and $[x]=[x_0,\ldots,x_n]$ a system of homogeneous coordinates on P. For $m\in\mathbf{Z}$ and $\varepsilon\in\mathbf{Z}/2\mathbf{Z}$, let us denote by $\mathcal{C}^\infty_{\mathsf{P}}(m|\varepsilon)$ the C^∞ line bundle on P whose sections φ satisfy

$$\varphi(\lambda x) = (\operatorname{sgn}\lambda)^\varepsilon \lambda^m \varphi(x) \qquad \text{for } \lambda\in\mathbf{R}^\times.$$

Let $\mathcal{V}_{\mathsf{P}} = \mathcal{C}^{\infty,(n)}_{\mathsf{P}}\otimes \mathrm{or}_{\mathsf{P}}$ be the sheaf of densities on P, where or_{P} denotes the orientation sheaf. Recall the isomorphism $\mathcal{V}_{\mathsf{P}}\simeq \mathcal{C}^\infty_{\mathsf{P}}(-n-1|-n-1)$. The Leray form

$$\omega(x) = \sum_{j=0}^n (-1)^j x_j dx_0\wedge\cdots\wedge\widehat{dx_j}\wedge\cdots\wedge dx_n$$

is the global section of $\mathcal{V}_{\mathsf{P}}(n+1|n+1)$ corresponding to $1\in\mathcal{C}^\infty_{\mathsf{P}}$. (Here, for a C^∞-module $\mathcal{F}$, we set $\mathcal{F}(m|\varepsilon)=\mathcal{F}\otimes_{\mathcal{C}^\infty_{\mathsf{P}}}\mathcal{C}^\infty_{\mathsf{P}}(m|\varepsilon)$.)

Let P^* be the dual projective space, and let $[\xi]$ be the system of homogeneous coordinates dual to $[x]$. The manifold P^* parameterizes projective hyperplanes in P by the correspondence $\xi \multimap \hat\xi = \{x\colon \langle x,\xi\rangle = 0\}$. The projective Radon transform is defined by

$$\varphi(x)\mapsto \psi(\xi) = \int_{\mathsf{P}} \varphi(x)\delta(\langle x,\xi\rangle)\omega(x), \tag{2}$$

where, in order for the integral to make sense, $\varphi\in\Gamma(\mathsf{P};\mathcal{C}^\infty_{\mathsf{P}}(-n|-n))$. It is clear that $\psi\in\Gamma(\mathsf{P}^*;\mathcal{C}^\infty_{\mathsf{P}^*}(-1|-1))$.

Remark 1. Let us identify $\mathbf{R}^n$ with the affine chart $x_0\neq 0$ of P, so that $y_j = x_j/x_0$ for $j=1,\ldots,n$. If $f(y)$ is rapidly decreasing, $\varphi(x)=x_0^{-n}f(y)$ is a well defined global section of $\mathcal{C}^\infty_{\mathsf{P}}(-n|-n)$, vanishing up to infinite order on the hyperplane at infinity $x_0=0$. In this sense, (2) is the natural projective extension of (1).

For $m<0$, consider the family $k_{(m|\varepsilon)}$ of distributions in $\mathbf{R}$ defined by

$$\text{for } m<0\colon \qquad k_{(m|1)}(t) = \left(\frac{d}{dt}\right)^{-m-1}\delta(t), \qquad k_{(m|0)}(t) = p.v.\,\frac{1}{t^{-m}}, \tag{3}$$

where *p.v.* stands for principal value. Note that $k_{(m|\varepsilon)}$ is $(m|\varepsilon)$-homogeneous. A natural generalization of (2) is then given by the following

Definition 1. The generalized projective Radon transform is defined by

$$\begin{aligned} \mathcal{R}_{(m^*|\varepsilon^*)}\colon \Gamma(\mathsf{P};\mathcal{C}^\infty_{\mathsf{P}}(m^*|\varepsilon^*)) &\to \Gamma(\mathsf{P}^*;\mathcal{C}^\infty_{\mathsf{P}^*}(m|\varepsilon)) \\ \varphi(x) &\mapsto \psi(\xi) = \int_{\mathsf{P}}\varphi(x)k_{(m|\varepsilon)}(\langle x,\xi\rangle)\omega(x), \end{aligned} \tag{4}$$

where

$$m^* = -n-1-m, \quad \varepsilon^*\equiv -n-1-\varepsilon \bmod 2.$$

Let $\mathcal{R}^*_{(m|\varepsilon)}$ be the transform obtained by interchanging the roles of P and P^*, i.e.

$$\tag{5} \begin{aligned} \mathcal{R}^*_{(m|\varepsilon)} \colon \Gamma(\mathsf{P}^*;\mathcal{C}^\infty_{\mathsf{P}^*}(m|\varepsilon)) &\to \Gamma(\mathsf{P};\mathcal{C}^\infty_{\mathsf{P}}(m^*|\varepsilon^*)) \\ \psi(\xi) &\mapsto \varphi(x) = \int_{\mathsf{P}^*} \psi(\xi) k_{(m^*|\varepsilon^*)}(\langle x,\xi\rangle)\omega(\xi). \end{aligned}$$

A classical result (see [5, Theorem 1', pg. 73]) asserts that for $m<0$ and $m^*<0$ (i.e. $-n-1<m<0$) the transform $\mathcal{R}_{(m^*|\varepsilon^*)}$ is invertible, and its inverse is $\mathcal{R}^*_{(m|\varepsilon)}$. (We will always neglect non-zero multiplicative constants.)

Our aim is to extend the above result to the case of arbitrary $m\in\mathbf{Z}$. In fact, interchanging the roles of P and P^* it is enough to consider the case $m<0$.

2. Statement of the result

Let us complete the family (3) of distributions in $\mathbf{R}$ by setting

$$\tag{6} \text{for } m\geq 0\colon \qquad k_{(m|1)}(t)=t^m\operatorname{sgn}(t), \qquad k_{(m|0)}(t)=t^m\log|t|.$$

Since $k_{(m^*|1)}(t)$ is $(m^*|1)$-homogeneous, (5) defines the Radon transform $\mathcal{R}^*_{(m|1^*)}$ for any $m\in\mathbf{Z}$. On the other hand, for $m\geq 0$ and $\psi\in\Gamma(\mathsf{P}^*;\mathcal{C}^\infty_{\mathsf{P}^*}(m|0^*))$ one has

$$\tag{7} \begin{aligned} \psi(\lambda\xi)k_{(m^*|0)}(\langle x,\lambda\xi\rangle)\omega(\lambda\xi) = &(\operatorname{sgn}\lambda)^{n+1}\psi(\xi)k_{(m^*|0)}(\langle x,\xi\rangle)\omega(\xi) \\ &+ (\operatorname{sgn}\lambda)^{n+1}\log|\lambda|\psi(\xi)\langle x,\xi\rangle^{m^*}\omega(\xi). \end{aligned}$$

Denote by $\mathbf{C}[x]_{(m^*)}$ the space of homogeneous polynomials of degree m^*, and consider the transform

$$\tag{8} \begin{aligned} \mathcal{C}_m\colon \Gamma(\mathsf{P}^*;\mathcal{C}^\infty_{\mathsf{P}^*}(m|0^*)) &\to \mathbf{C}[x]_{(m^*)} \\ \psi(\xi) &\mapsto \int_{\mathsf{P}^*}\psi(\xi)\langle x,\xi\rangle^{m^*}\omega(\xi). \end{aligned}$$

It follows from (7) that $\mathcal{R}^*_{(m|0^*)}$ is well-defined on $\ker(\mathcal{C}_m)$ for any $m\in\mathbf{Z}$. Note also that $\mathcal{R}_{(m^*|0)}(P)=0$ for any $P\in\mathbf{C}[x]_{(m^*)}\subset\Gamma(\mathsf{P};\mathcal{C}^\infty_{\mathsf{P}}(m^*|0))$ (cf [5, pp. 87–88]). We can then state our result.

Theorem 1. *Assume that $m<0$. Then the transforms*

$$\tag{9} \Gamma(\mathsf{P};\mathcal{C}^\infty_{\mathsf{P}}(m^*|1)) \underset{\mathcal{R}^*_{(m|1^*)}}{\overset{\mathcal{R}_{(m^*|1)}}{\rightleftarrows}} \Gamma(\mathsf{P}^*;\mathcal{C}^\infty_{\mathsf{P}^*}(m|1^*))$$

$$\tag{10} \frac{\Gamma(\mathsf{P};\mathcal{C}^\infty_{\mathsf{P}}(m^*|0))}{\mathbf{C}[x]_{(m^*)}} \underset{\mathcal{R}^*_{(m|0^*)}}{\overset{\mathcal{R}_{(m^*|0)}}{\rightleftarrows}} \ker(\mathcal{C}_m)$$

are (up to non-zero multiplicative constants) mutually inverse.

3. Review on integral transforms

Following [3]*, let us briefly recall how the theory of integral transforms for sheaves and $\mathcal{D}$-modules applies to the Radon transform (see* [2] *for an exposition).*

Let X, Y be complex analytic manifolds, and denote by q_1 and q_2 the first and second projection from $X\times Y$ to X and Y, respectively. The $\mathcal{D}$-module analogue of an integral transform like (4) is given by the functor

$$\cdot \overset{D}{\circ} \mathcal{K}\colon \mathcal{M}\mapsto \mathcal{M}\overset{D}{\circ}\mathcal{K} = Dq_{2!}(Dq_1^*\mathcal{M}\overset{D}{\otimes}\mathcal{K}),$$

where $\mathcal{M}$ is a $\mathcal{D}_X$-module, $\mathcal{K}$ is a $\mathcal{D}_{X\times Y}$-module, and $Dq_{2!}$, Dq_1^*, and $\overset{D}{\otimes}$ denote the (derived) functors of proper direct image, inverse image, and tensor product for $\mathcal{D}$-modules. Similarly, for sheaves G on Y and K on $X\times Y$ (better, objects of the bounded derived categories of sheaves of $\mathbf{C}$-vector spaces), one considers

$$K\circ\cdot\colon G\mapsto K\circ G=Rq_{1!}(K\otimes q_2^{-1}G).$$

Assume that G is $\mathbf{R}$-constructible, $\mathcal{K}$ is regular holonomic, and $K=\mathcal{S}\mathrm{ol}(\mathcal{K},\mathcal{O}_{X\times Y})$ is its associated perverse sheaf. Under a natural transversality hypothesis—satisfied in the case of the Radon transform—one has the following adjunction formula between global solution complexes

$$\mathrm{Sol}(\mathcal{N},\mathcal{C}^\infty(K\circ G_{[\dim X]}))\simeq\mathrm{Sol}(\mathcal{M}\overset{D}{\circ}\mathcal{K},\mathcal{C}^\infty(G)),\tag{11}$$

where $G\mapsto G_{[1]}$ is the shift functor in the derived category, and $\mathcal{C}^\infty(G)=G\overset{\mathrm{w}}{\otimes}\mathcal{O}_Y$ is the formal cohomology functor of [9]. (Recall that $\mathcal{C}^\infty(G)=\mathcal{C}^\infty_N$ if $G=\mathbf{C}_N$ is the constant sheaf along a totally real submanifold $N\subset Y$, of which Y is a complexification.) Finally, recall that in the transversality hypothesis, one has the following isomorphisms, asserting that the morphisms $\mathcal{N}\to\mathcal{M}\overset{D}{\circ}\mathcal{K}$ and $\mathcal{M}\overset{D}{\circ}\mathcal{K}\to\mathcal{N}$ are described by an integral kernel.

$$\alpha\colon\ \mathrm{Sol}(\mathcal{M}^\vee\overset{D}{\boxtimes}\mathcal{N},\mathcal{K})\xrightarrow{\sim}\mathrm{Hom}_{\mathcal{D}_Y}(\mathcal{N},\mathcal{M}\overset{D}{\circ}\mathcal{K}),\tag{12}$$

$$\beta\colon\ \mathrm{Sol}(\mathcal{M}\overset{D}{\boxtimes}\mathcal{N}^\vee,\mathcal{K}^\vee)\xrightarrow{\sim}\mathrm{Hom}_{\mathcal{D}_Y}(\mathcal{M}\overset{D}{\circ}\mathcal{K},\mathcal{N}),\tag{13}$$

where $\mathcal{M}^\vee=R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{D}_X)\otimes_{\mathcal{O}_X}\Omega_X^*$ is the dual of $\mathcal{M}$ as a left $\mathcal{D}_Y$-module.

Let $\mathbb{P}\ni[z]$ be a complex projective space of dimension n, $\mathbb{P}^*\ni[\zeta]$ the dual space, $\mathbb{S}\subset\mathbb{P}\times\mathbb{P}^*$ the incidence relation $\langle z,\zeta\rangle=0$, and $\mathbb{S}^c=(\mathbb{P}\times\mathbb{P}^*)\setminus\mathbb{S}$. One denotes by $\mathcal{B}_{\mathbb{S}^c}$ the regular holonomic $\mathcal{D}_{\mathbb{P}\times\mathbb{P}^*}$-module of meromorphic functions on $\mathbb{P}\times\mathbb{P}^*$, with poles along $\mathbb{S}$. For $m\in\mathbf{Z}$, let $\mathcal{O}_{\mathbb{P}(m)}$ be the holomorphic line bundle whose sections Φ satisfy

$$\Phi(\lambda z)=\lambda^m\Phi(z)\qquad\text{for }\lambda\in\mathbf{C}^\times.$$

If $\mathcal{F}$ is an $\mathcal{O}_{\mathbb{P}}$-module, set $\mathcal{F}_{(m)}=\mathcal{F}\otimes_{\mathcal{O}_{\mathbb{P}}}\mathcal{O}_{\mathbb{P}(m)}$.

The first homotopy group of the real projective space $\mathsf{P}\subset\mathbb{P}$ is $\mathbf{Z}/2\mathbf{Z}$, and hence there are essentially two locally constant sheaves of rank one over P. For $\varepsilon\in\mathbf{Z}/2\mathbf{Z}$ we denote them by $\mathbf{C}_{\mathsf{P}}^{(\varepsilon)}$, asking $\mathbf{C}_{\mathsf{P}}^{(0)}$ to be the constant sheaf. One then has

$$\begin{aligned}\mathcal{C}^\infty_{\mathsf{P}(m^*|\varepsilon^*)}&\simeq\mathcal{C}^\infty(\mathbf{C}_{\mathsf{P}}^{(\varepsilon^*)})_{(m^*)}\\&\simeq\mathcal{S}\mathrm{ol}(\mathcal{D}_{\mathbb{P}(-m^*)},\mathcal{C}^\infty(\mathbf{C}_{\mathsf{P}}^{(\varepsilon^*)})).\end{aligned}$$

With the notations of the beginning of this section, let us consider

$$X=\mathbb{P},\quad Y=\mathbb{P}^*,\quad \mathcal{K}=\mathcal{B}_{\mathbb{S}^c},\quad K=\mathbf{C}_{\mathbb{S}^c},\quad \mathcal{M}=\mathcal{D}_{\mathbb{P}(-m^*)},\quad G=\mathbf{C}_{\mathsf{P}^*}^{(\varepsilon)}.$$

We denote by $K'=R\mathcal{H}om(K,\mathbf{C}_{\mathbb{P}\times\mathbb{P}^*})$ the dual of K.

Theorem 2. *([3, 4])*

(i) *The two functors $\cdot\overset{D}{\circ}\mathcal{B}_{\mathbb{S}}$ and $\mathcal{B}^\vee_{\mathbb{S}^c}\overset{D}{\circ}\cdot$, as well as the two functors $\mathbf{C}_{\mathbb{S}^c[n]}\circ\cdot$ and $\cdot\circ\mathbf{C}'_{\mathbb{S}^c[n]}$ are quasi-inverse to each other.*

(ii) *For $m < 0$, the sections*

$$s_m(z,\zeta) = \frac{\omega(z)}{\langle z,\zeta\rangle^{-m}}, \tag{14}$$

$$t_{m^*}(z,\zeta) = \begin{cases} \delta^{(-m^*-1)}(\langle z,\zeta\rangle)\omega(\zeta) & \textit{for } m^* < 0, \\ \langle z,\zeta\rangle^{m^*} Y(\langle z,\zeta\rangle)\omega(\zeta) & \textit{for } m^* \geq 0, \end{cases} \tag{15}$$

induce by (12) *isomorphisms*

$$\alpha(s_m)\colon \mathcal{D}_{\mathbb{P}^*}(-m) \to \mathcal{D}_{\mathbb{P}}(-m^*) \overset{D}{\circ} \mathcal{B}_{\mathbb{S}^c},$$
$$\alpha(t_{m^*})\colon \mathcal{D}_{\mathbb{P}}(-m^*) \to \mathcal{B}^{\vee}_{\mathbb{S}^c} \overset{D}{\circ} \mathcal{D}_{\mathbb{P}^*}(-m).$$

(iii) *By* (11), *$\alpha(s_m)$ and $\alpha(t_{m^*})$ induce mutually inverse isomorphisms*

$$\mathrm{R}\Gamma(\mathbb{P}; \mathcal{C}^\infty(\mathbf{C}_{\mathbb{S}^c} \circ \mathbf{C}^{(\varepsilon)}_{\mathsf{P}^*}[n])(m^*)) \underset{\widetilde{\alpha}(t_{m^*})}{\overset{\widetilde{\alpha}(s_m)}{\rightleftarrows}} \Gamma(\mathsf{P}^*; \mathcal{C}^\infty_{\mathsf{P}^*}(\varepsilon^*|m)). \tag{16}$$

(To explain the notations in (15), recall that on $\mathbf{C} \ni \tau$ one has $\mathcal{B}_{\mathbf{C}^\times} = \mathcal{D}_{\mathbf{C}}/(\mathcal{D}_{\mathbf{C}} \cdot \partial_\tau \tau)$, $\mathcal{B}^{\vee}_{\mathbf{C}^\times} = \mathcal{D}_{\mathbf{C}}/(\mathcal{D}_{\mathbf{C}} \cdot \tau\partial_\tau)$. One denotes by $Y(\tau)$ the canonical generator of $\mathcal{B}^{\vee}_{\mathbf{C}^\times}$, and sets $\delta^{(k)}(\tau) = \partial^k_\tau Y(\tau)$.)

4. Sketch of proof

As it was done in [3, 1] *for the case $-n-1 < m < 0$, we will show here how Theorem 1 can be obtained as a corollary of Theorem 2.*

A purely topological computation (see [3, Proposition 5.16]) gives

$$\mathbf{C}_{\mathbb{S}^c} \circ \mathbf{C}^{(1^*)}_{\mathsf{P}^*}[n] \simeq \mathbf{C}^{(1)}_{\mathsf{P}}, \tag{17}$$

$$\mathbf{C}_{\mathbb{P}\setminus\mathsf{P}} \to \mathbf{C}_{\mathbb{S}^c} \circ \mathbf{C}^{(0^*)}_{\mathsf{P}^*}[n] \to \mathbf{C}_{\mathbb{P}} \xrightarrow[+1]{} . \tag{18}$$

The isomorphism (9) is obtained by plugging (17) in (16).

In order to get (10), we need to describe the complex

$$F = \mathrm{R}\Gamma(\mathbb{P}; \mathcal{C}^\infty(\mathbf{C}_{\mathbb{S}^c} \circ \mathbf{C}^{(0^*)}_{\mathsf{P}^*}[n])(m^*)).$$

By (18), we get a distinguished triangle

$$0 \to \mathrm{R}\Gamma(\mathbb{P}; \mathcal{C}^\infty(\mathbf{C}_{\mathbb{P}\setminus\mathsf{P}})(m^*)) \to F \to \mathbf{C}[x]_{(m^*)} \to 0,$$

where we used Serre's isomorphism

$$\mathrm{R}\Gamma(\mathbb{P}; \mathcal{C}^\infty(\mathbf{C}_{\mathbb{P}})(m^*)) \simeq \mathrm{R}\Gamma(\mathbb{P}; \mathcal{O}_{\mathbb{P}}(m^*)) \simeq \mathbf{C}[x]_{(m^*)}.$$

Moreover, the short exact sequence $0 \to \mathbf{C}_{\mathbb{P}\setminus\mathsf{P}} \to \mathbf{C}_{\mathbb{P}} \to \mathbf{C}_{\mathsf{P}} \to 0$ gives

$$0 \to \mathbf{C}[x]_{(m^*)} \to \Gamma(\mathsf{P}; \mathcal{C}^\infty_{\mathsf{P}}(m^*|0)) \to \mathrm{R}\Gamma(\mathbb{P}; \mathcal{C}^\infty(\mathbf{C}_{\mathbb{P}\setminus\mathsf{P}})(m^*)) \to 0.$$

Combining the two short exact sequences above, we get the following commutative diagram, whose first row is exact

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{C}[x]_{(m^*)} \arrow[r] & \Gamma(\mathsf{P}; \mathcal{C}^\infty_{\mathsf{P}}(m^*|0)) \arrow[r] \arrow[dr, "\mathcal{R}_{(m^*|0)}"'] & F \arrow[r] \arrow[d, "\wr"', "\widetilde{\alpha}(s_m)"] & \mathbf{C}[x]_{(m^*)} \arrow[r] & 0. \\ & & & \Gamma(\mathsf{P}^*; \mathcal{C}^\infty_{\mathsf{P}^*}(m|0^*)) \arrow[ur, "\widetilde{\mathcal{C}}_m"'] & & \end{tikzcd}$$

To conclude, let us show that the morphism $\widetilde{\mathcal{C}}_m$ above coincides with the morphism $\mathcal{C}_m$ of (8).

The isomorphism (13) is functorial in $\mathcal{K}$. Applying it to the natural morphism $q\colon \mathcal{B}^\vee_{\mathbb{S}^c} \to \mathcal{O}_{\mathbb{P}\times\mathbb{P}^*}$, dual to the embedding $\mathcal{O}_{\mathbb{P}\times\mathbb{P}^*} \to \mathcal{B}_{\mathbb{S}^c}$, we get a commutative diagram

$$\begin{tikzcd}
\mathrm{Sol}(\mathcal{D}_{\mathbb{P}(-m^*)} \overset{D}{\boxtimes} \mathcal{D}_{\mathbb{P}^*(-m^*)}, \mathcal{B}^\vee_{\mathbb{S}^c}) \arrow[r, "\sim", "\beta"'] \arrow[d, "q"] & \mathrm{Hom}_{\mathcal{D}_{\mathbb{P}^*}}(\mathcal{D}_{\mathbb{P}(-m^*)} \overset{D}{\circ} \mathcal{B}_{\mathbb{S}^c}, \mathcal{D}_{\mathbb{P}^*(-m)}) \arrow[d] \\
\mathrm{Sol}(\mathcal{D}_{\mathbb{P}(-m^*)} \overset{D}{\boxtimes} \mathcal{D}_{\mathbb{P}^*(-m^*)}, \mathcal{O}_{\mathbb{P}\times\mathbb{P}^*}) \arrow[r, "\sim", "\beta"'] & \mathrm{Hom}_{\mathcal{D}_{\mathbb{P}^*}}(\mathcal{D}_{\mathbb{P}(-m^*)} \overset{D}{\circ} \mathcal{O}_{\mathbb{P}\times\mathbb{P}^*}, \mathcal{D}_{\mathbb{P}^*(-m)})
\end{tikzcd}$$

Note that the morphism $\beta(t_{m^*})$ is the inverse of the morphism $\alpha(s_m)$. Moreover, $q(t_{m^*}) = \langle z, \zeta\rangle^{m^*}\omega(\zeta)$ is the complex analogue of the integral kernel defining $\mathcal{C}_m$. The equality $\widetilde{\mathcal{C}}_m = \mathcal{C}_m$ then follows by considering the commutative diagram

$$\begin{tikzcd}
\Gamma(\mathsf{P}^*; \mathcal{C}^\infty_{\mathsf{P}^*(m|0^*)}) \arrow[r, phantom, "\simeq"] \arrow[ddd, "\widetilde{\mathcal{C}}_m"'] & \mathrm{Sol}(\mathcal{D}_{\mathbb{P}^*(-m)}, \mathcal{C}^\infty(\mathbf{C}^{(0^*)}_{\mathsf{P}^*})) \arrow[d, "\widetilde{\beta}(t_{m^*})"'] \arrow[dr, "\widetilde{\beta}(q(t_{m^*}))"] & \\
 & \mathrm{Sol}(\mathcal{D}_{\mathbb{P}(-m^*)} \overset{D}{\circ} \mathcal{B}_{\mathbb{S}^c}, \mathcal{C}^\infty(\mathbf{C}^{(0^*)}_{\mathsf{P}^*})) \arrow[r] \arrow[d, dash, "\wr"] & \mathrm{Sol}(\mathcal{D}_{\mathbb{P}(-m^*)} \overset{D}{\circ} \mathcal{O}_{\mathbb{P}\times\mathbb{P}^*}, \mathcal{C}^\infty(\mathbf{C}^{(0^*)}_{\mathsf{P}^*})) \arrow[d, dash, "\wr"] \\
 & \mathrm{Sol}(\mathcal{D}_{\mathbb{P}(-m^*)}, \mathcal{C}^\infty(\mathbf{C}_{\mathbb{S}^c} \circ \mathbf{C}^{(0^*)}_{\mathsf{P}^*}[n])) \arrow[r] \arrow[d] & \mathrm{Sol}(\mathcal{D}_{\mathbb{P}(-m^*)}, \mathcal{C}^\infty(\mathbf{C}_{\mathbb{P}\times\mathbb{P}^*} \circ \mathbf{C}^{(0^*)}_{\mathsf{P}^*}[n])) \arrow[dl] \\
\mathbf{C}[x]_{(m^*)} \arrow[r, phantom, "\simeq"] & \mathrm{Sol}(\mathcal{D}_{\mathbb{P}(-m^*)}, \mathcal{C}^\infty(\mathbf{C}_{\mathbb{P}})) &
\end{tikzcd}$$

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