INTEGRATION BY RBF OVER THE SPHERE

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Abstract. In this paper we consider numerical integration over the sphere by radial basis functions (RBF). After a brief introduction on RBF and spherical radial basis functions (SRBF), we show how to compute integrals of functions whose values are known at scattered data points. Numerical examples are given.

Key words. Scattered data, numerical integration on spheres, radial basis functions.

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1. Introduction. Radial Basis Functions (RBF) and Spherical Radial Basis Functions (SRBF) are well-established tools for approximating functions whose values are known on scattered data, respectively on $\mathbb{R}^n$ and the unit sphere $\mathbb{S}^{n-1} = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| = 1\} \subset \mathbb{R}^n$ (the so-called unit $n-1$-sphere). Given the values of a function f at N *centers* $X := \{\mathbf{x}_i\}_{i=1,\dots,N} \subseteq \Omega \subseteq \mathbb{R}^n$, one has to find scalars $\{\lambda_i\}_{i=1,\dots,N}$ such that

$$f(\mathbf{x}) \approx s_{f,X}(\mathbf{x}) := \sum_{i=1}^{N} \lambda_i \phi(\mathrm{dist}(\mathbf{x}, \mathbf{x}_i)) \tag{1.1}$$

where $\phi : \mathbb{R}_+ \to \mathbb{R}$ is the given RBF/SRBF and dist is the Euclidean or the geodesic distance [4], [32] respectively. The function $s_{f,X}$ can be obtained, for instance, by interpolation of f at the centers X, independently of their geometry. Such a property is particularly appealing since it is not necessarily possible in the case of polynomial interpolation, where the distribution of the centers may imply the singularity of the corresponding Vandermonde matrix. By the Curtis-Mairhuber theorem [32, Theorem 2.3] there is no $N-$dimensional space of continuous functions on the domain $\Omega \subseteq \mathbb{R}^n$ (having an interior point) that contains a unique interpolant for every set of centers X consisting of $N \geq 2$ data points. Consequently it is natural to choose a family of interpolants that depends on X, as in the case of RBF [7], [32].

A wide variety of RBF/SRBF have been proposed. Hardy introduced in 1971 the *Multiquadrics* (denoted by MQ) and *Inverse Multiquadrics* (IMQ) in conjunction with studies on topography, while Duchon in 1976-78 considered the *Thin Plate Splines* (TPS) and the *Polyharmonic Plate Splines* (see [32, §8.6]).

The interest in RBF has increased during the last two decades, especially after the work in 1982 by Franke on scattered data interpolation methods [10]. He showed that interpolation by multiquadrics and thin plate splines led to very good results, conjecturing also that the interpolation matrix that arises in the use of multiquadrics is invertible. This result was proved in 1986 by Micchelli [20] who linked this problem to the work of Schoenberg on positive definite functions [32, Theorem 7.13]. In [20] the analysis covers a wider class of RBF, including the Thin Plate Splines and the Gaussians (denoted by G). After these pioneering results, work blossomed in several directions: the discovery of RBF having compact support, fast methods for the

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computation of RBF interpolants with thousand points (in spite of the high condition number of the interpolation matrix), error estimates and the properties of the interpolation matrix and choice of a good scaling parameter.

Of course it is impossible to mention in detail all the research that has been done on this subject, and from this point of view we suggest to the interested reader the monographs [4], [5], [7], [32].

The problem of interpolation and approximation on scattered data of $\mathbb{S}^n$ by SRBF has also been considered. Starting from the concept of positive definiteness on the sphere (also introduced by Schoenberg [32, Corollary 17.9]), several authors have studied the requirements under which the interpolation matrix is nonsingular, independently of the distribution of the centers $X \subset \mathbb{S}^n$ (see [6] and references therein). For examples of effective SRBF, see [1], [8], [11], [13].

If the values of the function f are known on scattered data over a square, some results concerning numerical cubature by RBF have been obtained in [30]. On the other hand, there has been an increasing interest in integrating functions on the unit sphere by cubature rules on special sets of points ([23], [28], [29]) as well as on scattered data on $\mathbb{S}^n$, caps or more generally compact subsets of caps (see [18], [19] also for various applications over $\mathbb{S}^n$). The latter techniques allow for the construction of rules with positive weights and a certain polynomial degree of precision depending on the distribution of the points. In the framework of Galerkin methods, integrals of convolution type on $\mathbb{S}^n$ have been studied in [11], [16] and [22].

The purpose of this paper is to show how the more often used RBF and SRBF are able to approximate integrals of the form

$$If = \int_{\mathbb{S}^2} f(\mathbf{x}) d\mu(\mathbf{x}) \tag{1.2}$$

where $\mathbb{S}^2 \subset \mathbb{R}^3$ is the unit 2-sphere, $d\mu(\mathbf{x})$ denotes the surface measure on $\mathbb{S}^2$ and $f : \mathbb{S}^2 \to \mathbb{R}$ a function whose properties will be specified later. We will show that once an RBF/SRBF approximation is at hand, the numerical cubature can be achieved easily and that the rule is *optimal* in the sense of Golomb-Weinberger [32, Section 13]. In section 6 numerical experiments illustrate the performance of the method and the size of the weights of the cubature rule.

2. Numerical integration on the sphere by RBF and SRBF. Let $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\} \subseteq \mathbb{S}^2$ be a set of N centers and let $f : \mathbb{S}^2 \to \mathbb{R}$ be a continuous function defined on $\mathbb{S}^2$. Suppose that you know an approximation $s_{f,X}$ which is a linear combination of radial basis functions and low order polynomials

$$s_{f,X}(\mathbf{x}) := \sum_{i=1}^{N} \lambda_i \phi(\|\mathbf{x} - \mathbf{x}_i\|_2/\sigma) + \sum_{k=0}^{M-1} \sum_{l=0}^{2k} \gamma_{k,l} P_{k,l}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{S}^2. \tag{2.1}$$

Here $\sigma > 0$ is the scaling parameter [25], ϕ is a conditionally positive RBF of order M in $\mathbb{R}^3$ [32], and

$$P_{k,l}(\mathbf{x}) = \xi^{k_1} \eta^{k_2} \zeta^{k_3}, \tag{2.2}$$

where $\mathbf{x} = (\xi, \eta, \zeta)^T$ and $k = k_1 + k_2 + k_3$, k_1, k_2, $k_3 \geq 0$, so $P_{k,l}$ is a homogenous polynomial of degree k in three variables. For some RBF, $M = 0$ so there is no low order polynomial component. Typically the function $s_{f,X}$ is obtained by interpolating f on the centers but other strategies may be used (for instance least squares methods [5], [8]) .

We will restrict our attention to $\mathbb{S}^2$ to simplify the analysis but it is not difficult to extend the results to the $n-$dimensional case.

If for every $i = 1, \ldots, N$, the integral

$$\int_{\mathbb{S}^2} \phi(\|\mathbf{x} - \mathbf{x}_i\|_2/\sigma) d\mu(\mathbf{x}) \tag{2.3}$$

exists, then integrating (2.1) over $\mathbb{S}^2$ gives

$$\begin{aligned}\int_{\mathbb{S}^2} f(\mathbf{x}) d\mu(\mathbf{x}) \approx & \sum_{i=1}^{N} \lambda_i \int_{\mathbb{S}^2} \phi(\|\mathbf{x} - \mathbf{x}_i\|_2/\sigma) d\mu(\mathbf{x}) \\ & + \sum_{k=0}^{M-1} \sum_{l=0}^{2k} \gamma_{k,l} \int_{\mathbb{S}^2} P_{k,l}(\mathbf{x}) d\mu(\mathbf{x}).\end{aligned} \tag{2.4}$$

Every rotation $\mathcal{R} : \mathbb{S}^2 \to \mathbb{S}^2$ is an isometry, that is

$$\phi(\|\mathcal{R}\mathbf{x} - \mathcal{R}\mathbf{x}_i\|_2/\sigma) = \phi(\|\mathbf{x} - \mathbf{x}_i\|_2/\sigma). \tag{2.5}$$

Consider the rotation $\mathcal{R}$ that maps the center $\mathbf{x}_i$ to the North Pole $\overline{\mathbf{P}} = (0, 0, 1)$. As $\det(\mathrm{Jac}\, \mathcal{R}^{-1}) = 1$, the substitution $\mathbf{x}' = \mathcal{R}\mathbf{x}$ gives

$$\begin{aligned}& \int_{\mathbb{S}^2} \phi(\|\mathbf{x} - \mathbf{x}_i\|_2/\sigma) d\mu(\mathbf{x}) \\ = & \int_{\mathbb{S}^2} \phi(\|\mathcal{R}\mathbf{x} - \mathcal{R}\mathbf{x}_i\|_2/\sigma) d\mu(\mathbf{x}) \qquad (2.6) \\ = & \int_{\mathbb{S}^2} \phi(\|\mathbf{x}' - \overline{\mathbf{P}}\|_2/\sigma) \det(\mathrm{Jac}\, \mathcal{R}^{-1})\, d\mu(\mathbf{x}') \\ = & \int_{\mathbb{S}^2} \phi(\|\mathbf{x}' - \overline{\mathbf{P}}\|_2/\sigma)\, d\mu(\mathbf{x}'). \qquad (2.7)\end{aligned}$$

As this holds for every center $\{\mathbf{x}_i\}_{i=1,\ldots,N}$, (2.4) gives

$$\begin{aligned}\int_{\mathbb{S}^2} f(\mathbf{x}) d\mu(\mathbf{x}) \approx & \left(\int_{\mathbb{S}^2} \phi(\|\mathbf{x} - \overline{\mathbf{P}}\|_2/\sigma) d\mu(\mathbf{x}) \right) + \\ & \sum_{i=1}^{N} \lambda_i + \sum_{k=0}^{M-1} \sum_{l=0}^{2k} \gamma_{k,l} \int_{\mathbb{S}^2} P_{k,l}(\mathbf{x}) d\mu(\mathbf{x}).\end{aligned} \tag{2.8}$$

The first integral on the right side can be easily evaluated for most functions ϕ. A point on the unit sphere $\mathbb{S}^2$ having cartesian coordinates (ξ, η, ζ) can also be described by spherical coordinates (ψ, θ) with

$$\begin{aligned}\xi &= \cos\psi\, \sin\theta, \\ \eta &= \sin\psi\, \sin\theta, \\ \zeta &= \cos\theta.\end{aligned} \tag{2.9}$$

Here ψ is the azimuthal angle in the xy plane measured from the positive part of the x axis with $0 \leq \psi < 2\pi$ (the *longitude*) and θ the polar angle from the z axis with $0 \leq \theta \leq \pi$ (the *colatitude*).

RBF	$\phi(r)$	$\int \phi(\sqrt{s})\, ds$
MQ	$(1+r^2)^{\frac{1}{2}}$	$\frac{2}{3}(1+s)^{\frac{3}{2}}$
IMQ	$(1+r^2)^{-\frac{1}{2}}$	$2(1+s)^{\frac{1}{2}}$
G	$\exp(-r^2)$	$-\exp(-s)$
TPS	$r^2 \log r$	$\frac{1}{4}s^2 \log s - \frac{1}{8}s^2$
Buhm. C^2	$\tilde{r}^4 \log \tilde{r}^2 - \frac{7}{2}\tilde{r}^4 + \frac{16}{3}\tilde{r}^3 - 2\tilde{r}^2 + \frac{1}{6}$	$\frac{1}{3}\tilde{s}^3 \log \tilde{s} - \frac{23}{18}\tilde{s}^3 + \frac{32}{15}\tilde{s}^{\frac{5}{2}} - \tilde{s}^2 + \frac{1}{6}\tilde{s}$
Wend. C^0	$(1-r)_+^2$	$-\frac{4}{3}\tilde{s}^{3/2} + \frac{1}{2}\tilde{s}^2 + \tilde{s}$
Wend. C^2	$(1-r)_+^4(4r+1)$	$\frac{8}{7}\tilde{s}^{\frac{7}{2}} - 5\tilde{s}^3 + 8\tilde{s}^{\frac{5}{2}} - 5\tilde{s}^2 + \tilde{s}$
Wend. C^4	$(1-r)_+^6(35r^2+18r+3)$	$-\frac{128}{3}\tilde{s}^{\frac{9}{2}} + 7\tilde{s}^5 + 105\tilde{s}^4 -$ $-128\tilde{s}^{\frac{7}{2}} + 70\tilde{s}^3 - 14\tilde{s}^2 + 3\tilde{s}$

TABLE 2.1
Indefinite integrals of some RBF.

Using this change of coordinates, remembering that the determinant of the Jacobian matrix of the transformation is simply $\sin\theta$ and setting $s := 2(1-\cos\theta)/\sigma^2$, using

$$\|\mathbf{x} - \overline{\mathbf{P}}\|_2^2 = 2(1-\cos\theta), \tag{2.10}$$

we get

$$\begin{aligned} I_{\phi,\sigma} &= \int_{\mathbb{S}^2} \phi\left(\|\mathbf{x} - \overline{\mathbf{P}}\|_2/\sigma\right)\, d\mu(\mathbf{x}) \\ &= \int_0^{2\pi} \int_0^{\pi} \phi\left(\sqrt{\frac{2(1-\cos\theta)}{\sigma^2}}\right) \sin\theta\, d\theta\, d\psi \\ &= \pi\sigma^2 \int_0^{4/\sigma^2} \phi\left(\sqrt{s}\right) ds. \end{aligned} \tag{2.11}$$

For some RBF, an explicit expression for the indefinite integral $\int \phi(\sqrt{s})\, ds$ are provided in Table *I*. For RBF with (compact) support $[0,1]$ we used the notation $\tilde{r} = \min\{r,1\}$, $\tilde{s} = \min\{s,1\}$ and $s_+ := \max(s,0)$. In particular for the Wendland [31] and Buhmann [3] RBF the upper limit of integration in the last integral in (2.11) becomes $\min\{4/\sigma^2, 1\}$.

The approximation (2.8) of the integral also requires

$$I_M = \sum_{k=0}^{M-1} \sum_{l=0}^{2k} \gamma_{k,l} \int_{\mathbb{S}^2} P_{k,l}(\mathbf{x})\, d\mu(\mathbf{x}). \tag{2.12}$$

To this end, we observe that

$$\begin{aligned}
&\int_{\mathbb{S}^2} P_{k,l}(\mathbf{x}) d\mu(\mathbf{x}) \\
&= \int_{\mathbb{S}^2} \xi^{k_1} \eta^{k_2} \zeta^{k_3} d\mu(\mathbf{x}) \\
&= \int_0^{2\pi} \int_0^{\pi} (\cos\psi \sin\theta)^{k_1} (\sin\psi \sin\theta)^{k_2} (\cos\theta)^{k_3} \, d\theta \, d\psi \\
&= \int_0^{2\pi} \int_0^{\pi} \cos^{k_1}\psi \sin^{k_1+k_2}\theta \sin^{k_2}\psi \cos^{k_3}\theta \, d\theta \, d\psi \\
&= \left(\int_0^{2\pi} \cos^{k_1}\psi \sin^{k_2}\psi \, d\psi \right) \left(\int_0^{\pi} \cos^{k_3}\theta \sin^{k_1+k_2}\theta \, d\theta \right). \qquad (2.13)
\end{aligned}$$

The *Beta function* β satisfies

$$\int_0^{\pi/2} \sin^{m_1}\theta \cos^{m_2}\theta \, d\theta = \frac{1}{2} \beta \left(\frac{m_1+1}{2}, \frac{m_2+1}{2} \right), \qquad (2.14)$$

and $\beta(t_1, t_2) = \beta(t_2, t_1)$, so

$$\begin{aligned}
&\int_0^{\pi} \sin^{m_1}\theta \cos^{m_2}\theta \, d\theta \\
&= \int_0^{\pi/2} \sin^{m_1}\theta \cos^{m_2}\theta \, d\theta + \int_{\pi/2}^{\pi} \sin^{m_1}\theta \cos^{m_2}\theta \, d\theta \\
&= \int_0^{\pi/2} \sin^{m_1}\theta \cos^{m_2}\theta \, d\theta + (-1)^{m_1} \int_0^{\pi/2} \cos^{m_1}\theta \sin^{m_2}\theta \, d\theta \\
&= \frac{1}{2} \beta \left(\frac{m_1+1}{2}, \frac{m_2+1}{2} \right) + \frac{(-1)^{m_1}}{2} \beta \left(\frac{m_2+1}{2}, \frac{m_1+1}{2} \right) \\
&= \left(\frac{1+(-1)^{m_1}}{2} \right) \beta \left(\frac{m_1+1}{2}, \frac{m_2+1}{2} \right). \qquad (2.15)
\end{aligned}$$

Similarly if m_1 and m_2 are both even, then

$$\int_0^{2\pi} \sin^{m_1}\theta \cos^{m_2}\theta \, d\theta = 2 \beta \left(\frac{m_1+1}{2}, \frac{m_2+1}{2} \right), \qquad (2.16)$$

while $\int_0^{2\pi} \sin^{m_1}\theta \cos^{m_2}\theta \, d\theta = 0$ in all the other cases. Thus, from (2.13), (2.15) and (2.16), one can easily compute $\int_{\mathbb{S}^2} P_{k,l}(\mathbf{x}) d\mu(\mathbf{x})$.

The only RBF in Table 2.1 for which $M > 0$ are the Multiquadrics (MQ) and the Thin Plate Splines (TPS), which are conditionally positive of order 1 and 2 respectively. In these cases, when $\gamma_{k,l}$ are known, the computation of I_M is particularly easy. For the Multiquadrics it is simply required to integrate the constant 1 on $\mathbb{S}^2$, hence

$$\int_{\mathbb{S}^2} P_{0,0}(\mathbf{x}) d\mu(\mathbf{x}) = 4\pi. \qquad (2.17)$$

In the case of the Thin Plate Splines

$$\int_{\mathbb{S}^2} \xi \, d\mu(\mathbf{x}) = \int_{\mathbb{S}^2} \eta \, d\mu(\mathbf{x}) = \int_{\mathbb{S}^2} \zeta \, d\mu(\mathbf{x}) = 0, \qquad (2.18)$$

SRBF	*Analytic Expression* $\phi(\alpha)$
Sph. Multiq.	$\sqrt{1+h^2-2hc}$
Sph. Multiq. II	$(1-h^2)\,(1+h^2-2hc)^{3/2}$
Sph. Rec. Multiq.	$1/\sqrt{1+h^2-2hc}$
Abel-Poisson Spline	$(1-h^2)/(1+h^2-2hc)^{3/2}$
Log. Spline	$\frac{1}{h}\log(1+2h/(1-h+\sqrt{1+h^2-2hc}))$
Sph. Spline I	$1-a+b\log((a+b)/b)$
Sph. Spline II	$\frac{1}{2}(4-3ab-3c+((4-3c)c-1)\log b-$ $-b(3c-1)\log(a+b))$

TABLE 2.2
SRBF, with $a=\sqrt{2-2c}$, $b=1-c$, $c=\cos(\alpha)$, $\alpha\in[0,\pi]$ *and* $h\in(0,1)$.

which implies that

$$\int_{\mathbb{S}^2} P_{0,0}(\mathbf{x})d\mu(\mathbf{x}) = 4\pi,$$

$$\int_{\mathbb{S}^2} P_{1,l}(\mathbf{x})d\mu(\mathbf{x}) = 0,\ l=1,2,3. \tag{2.19}$$

For the integration by SRBF, suppose that

$$f(\mathbf{x}) \approx \sum_{i=1}^{N} \lambda_i \phi(d(\mathbf{x},\mathbf{x}_i)) + \sum_{k=0}^{M-1}\sum_{l=0}^{2k} \gamma_{k,l} Y_{k,l}(\mathbf{x}). \tag{2.20}$$

where $Y_{k,l}$, $l=0,\ldots,2k$ are spherical harmonics of degree k and $d(\mathbf{x},\mathbf{y})=\arccos(\mathbf{x}^T\mathbf{y})$ is the geodesic distance (see [23], [29]).

As $d(\mathcal{R}\mathbf{x},\mathcal{R}\mathbf{y}) = d(\mathbf{x},\mathbf{y})$ for any rotation $\mathcal{R}:\mathbb{S}^2\to\mathbb{S}^2$ then for any center $\mathbf{x}_i\in X$

$$I_{\phi,h} = \int_{\mathbb{S}^2} \phi(d(\mathbf{x},\mathbf{x}_i))d\mu(\mathbf{x}) = \int_{\mathbb{S}^2} \phi(\, d(\mathbf{x},\overline{\mathbf{P}})\,)\,d\mu(\mathbf{x}). \tag{2.21}$$

Using spherical coordinates θ, ψ from (2.9) with $d(\mathbf{x},\mathbf{y})=\arccos(\mathbf{x}^T\mathbf{y})$ (so $d(\mathbf{x},\bar{\mathbf{P}}) = \theta$) we have

$$I_{\phi,h} = \int_0^{2\pi}\int_0^{\pi} \phi\,(\theta)\sin\theta\,d\theta\,d\psi = 2\pi\int_0^{\pi}\phi\,(\theta)\sin\theta\,d\theta. \tag{2.22}$$

Some of the more common SRBF are listed in Table 2.2, in which the argument $\alpha\in[0,\pi]$ is the geodesic distance and $h\in(0,1)$ is a spherical localization parameter. For more information about the Spherical Multiquadrics and Spherical Inverse Multiquadric see [8], for Spherical Multiquadrics II [1], while for Spherical Spline I, Spherical Spline II, Poisson Spline and Logarithmic Spline consider [1], [11] and [13]. Note that in [8] the argument of the SRBF is the geodesic distance $d(\mathbf{x},\mathbf{y})$ while in [13] it is a function of $c=\mathbf{x}^T\mathbf{y}=\cos d(\mathbf{x},\mathbf{y})$.

As in the case of the RBF, we show in Table III that for most of the SRBF the indefinite integral

$$\int \phi\,(\theta)\sin\theta\,d\theta \tag{2.23}$$

SRBF	*Indefinite integrals* $\int \phi(\theta)\,\sin\theta\,d\theta$
Sph. Multiq.	$(1+h^2-2hc)^{3/2}/(3h)$
Sph. Multiq. II	$(1-h^2)(1+h^2-2hc)^{5/2}/(5h)$
Sph. Rec. Multiq.	$\sqrt{1+h^2-2hc}/h$
Abel-Poisson Spline	$-(1-h^2)/(h\sqrt{1+h^2-2hc})$
Log. Spline	Long Expression
Sph. Spline I	see (2.24)
Sph. Spline II	see (2.25)

TABLE 2.3
Indefinite integrals of SRBF, with $a=\sqrt{2-2c}$, $b=1-c$, $c=\cos(\theta)$, $\theta\in[0,\pi]$ *and* $h\in(0,1)$.

is known explicitly. For the Log Spline and Spherical Spline I and II a short espression is not available. According to Maple and putting $c=\cos(\theta)$, $a=\sqrt{2-2c}$, $b=1-c$, we get in the case of Spherical Spline I

$$\ln\left(1+\frac{2}{a}\right)I_{\phi,1}=\frac{(c^2-2c-3)}{2}\ln\left(\frac{a+2}{a}\right)+\frac{(ac-c+a-1)}{2}-\ln(2-2c)+2\ln 2 \tag{2.24}$$

while for Spherical Spline II

$$\begin{aligned}I_{\phi,1}=&-c+\frac{2a}{3}+\frac{5ac}{6}+\frac{(-ac^2+c^2+c\ln(b)+c^3\ln(b))}{2}-c^2\ln(b)-\frac{3}{4}\\&-\frac{(c-2c^2+c^3)\ln(a+1-c)}{2}-2\ln(a+2)\,.\end{aligned} \tag{2.25}$$

To finish our analysis we simply observe that the integral of the spherical harmonics $Y_{k,l}$ over $\mathbb{S}^2$, is always equal to 0, except for the case $k=l=0$ when, depending on the normalization of $Y_{k,l}$, it is 4π.

Summarizing, numerical integration based on approximation/interpolation by a linear combination of RBF/SRBF is particularly inexpensive when the coefficients $\{\lambda_i\}$, $\{\gamma_{k,l}\}$ in (2.1) or (2.20) are known. In practice, one has to compute only one definite integral where the indefinite integral is known explicitly.

3. Cubature and optimal recovery. The results in this section are an application to integration of a more general theory on linear functionals developed by Golomb-Weinberger [32, Section 13], in the framework of *optimal recovery in Hilbert spaces*. It is required that the space is endowed with a reproducing kernel, a tool that is well-known to researchers in approximation theory from 1950 [32, Section 10].

Suppose that $\mathcal{H}$ is a real Hilbert space of functions $f:\Omega\to\mathbb{R}$, where Ω is a region containing at least one point. Furthermore let $(\cdot,\cdot)_{\mathcal{H}}$ and $\|f\|_{\mathcal{H}}=(f,f)_{\mathcal{H}}^{1/2}$ be respectively the inner product and the corresponding norm in $\mathcal{H}$. A function $K_{\mathcal{H}}:\Omega\times\Omega\to\mathbb{R}$ is called a *reproducing kernel* for $\mathcal{H}$ if

1. $K_{\mathcal{H}}(\cdot,\mathbf{x})\in\mathcal{H}$ for all $\mathbf{x}\in\Omega$;
2. $f(\mathbf{x})=(f,K_{\mathcal{H}}(\cdot,\mathbf{x}))_{\mathcal{H}}$ for all $f\in\mathcal{H}$ and all $\mathbf{x}\in\Omega$.

Such a Hilbert space $\mathcal{H}$ is often referred to by the acronym RKHS. Among the properties of reproducing kernels we mention the following

1. $K_{\mathcal{H}}(\mathbf{x},\mathbf{y})=(K_{\mathcal{H}}(\cdot,\mathbf{y}),K_{\mathcal{H}}(\cdot,\mathbf{y}))_{\mathcal{H}}$ for all $\mathbf{x}\in\Omega$;
2. $K_{\mathcal{H}}(\mathbf{x},\mathbf{y})=K_{\mathcal{H}}(\mathbf{y},\mathbf{x})$ for $\mathbf{x}$, $\mathbf{y}\in\Omega$;

3. If $f,\ f_n \in \mathcal{H}$, $n \in \mathbb{N}$ are given such that f_n converges to f in the Hilbert space norm then f_n also converges pointwise to f.

It is not difficult to show that every RKHS has exactly one reproducing kernel. Let us consider now the linear functional

$$I : f \to \int_\Omega f(\mathbf{x})\, d\mu(\mathbf{x}), \qquad f \in \mathcal{H} \tag{3.1}$$

where $\mu(\mathbf{x})$ is a measure on Ω. It is easy to prove that if

$$\int_\Omega \sqrt{K_\mathcal{H}(\mathbf{x},\mathbf{x})}\, d\mu(\mathbf{x}) < +\infty, \tag{3.2}$$

then the functional I is also bounded because

$$\begin{aligned}
\left| \int_\Omega f(\mathbf{x})\, d\mu(\mathbf{x}) \right| &= \left| \int_\Omega (f, K_\mathcal{H}(\cdot,\mathbf{x}))_\mathcal{H}\, d\mu(\mathbf{x}) \right| \\
&\le \|f\|_\mathcal{H} \int_\Omega \|K_\mathcal{H}(\cdot,\mathbf{x})\|_\mathcal{H}\, d\mu(\mathbf{x}) \\
&\le \|f\|_\mathcal{H} \int_\Omega \sqrt{(K_\mathcal{H}(\cdot,\mathbf{x}), K_\mathcal{H}(\cdot,\mathbf{x}))_\mathcal{H}}\, d\mu(\mathbf{x}) \\
&= \|f\|_\mathcal{H} \int_\Omega \sqrt{K_\mathcal{H}(\mathbf{x},\mathbf{x})}\, d\mu(\mathbf{x}).
\end{aligned} \tag{3.3}$$

By the Riesz theorem [32], there is a unique $v_I \in \mathcal{H}$ such that

$$If = (f, v_I)_\mathcal{H}, \qquad \text{for all } f \in \mathcal{H}. \tag{3.4}$$

As $K_\mathcal{H}(\cdot,\mathbf{x}) \in \mathcal{H}$,

$$IK_\mathcal{H}(\cdot,\mathbf{x}) = \int_\Omega K_\mathcal{H}(\mathbf{x}',\mathbf{x}) d\mu(\mathbf{x}') = (K_\mathcal{H}(\cdot,\mathbf{x}), v_I)_\mathcal{H} = v_I(\mathbf{x}), \tag{3.5}$$

gives

$$v_I = \int_\Omega K_\mathcal{H}(\cdot,\mathbf{x}) d\mu(\mathbf{x}). \tag{3.6}$$

Given a fixed set of *nodes* $X = \{\mathbf{x}_i\}_{i=1,\dots,N} \subset \Omega$, if the N by N matrix A defined by $A_{i,j} = K_\mathcal{H}(\mathbf{x}_i,\mathbf{x}_j)$, $i,j = 1,\dots,N$ is non singular, then there are N *weights* $\{w_{i,N}\}_{i=1,\dots,N}$ (uniquely determined) such that

$$\int_\Omega K_\mathcal{H}(\mathbf{x},\mathbf{x}_j)\, d\mu(\mathbf{x}) = \sum_{i=1}^N w_{i,N} K_\mathcal{H}(\mathbf{x}_i,\mathbf{x}_j), \qquad j = 1,\dots,N. \tag{3.7}$$

Note that by (3.7), the function

$$s_{I,X}^{(\mathrm{int})}(\mathbf{x}) := \sum_{i=1}^N w_{i,N} K_\mathcal{H}(\mathbf{x}_i,\mathbf{x}) \tag{3.8}$$

is the interpolant of the representer v_I on the set of nodes X.

At this point, if $V := \text{span}\{K_{\mathcal{H}}(\mathbf{x}_i, \cdot)\}$ then one can define the cubature rule

$$Q_V^{(\text{int})} f := \sum_{i=1}^{N} w_{i,N} f(\mathbf{x}_i) = (f, s_{I,X}^{(\text{int})})_{\mathcal{H}} \tag{3.9}$$

which can be interpreted as approximating $(f, v_I)_{\mathcal{H}}$ by $(f, s_{I,X}^{(\text{int})})_{\mathcal{H}}$.

Observe that $Q_V^{(\text{int})} f := \sum_{i=1}^{N} w_{i,N} f(\mathbf{x}_i)$ is meaningless when $f \in L^2(\Omega)$, but $L^2(\Omega)$ is not a RKHS so what has been described above cannot be applied. In general, $\mathcal{H}$ is a RKHS if and only if the point evaluation functionals are continuous, i.e. $\delta_x \in \mathcal{H}^*$ for all $x \in \Omega$, where $\mathcal{H}^*$ is the dual space of $\mathcal{H}$. This property suggests why (3.9) makes sense when $\mathcal{H}$ is a RKHS.
The rule $Q_V^{(\text{int})}$ is exact on V, as for $f = \sum_{i=1}^{N} \gamma_i K_{\mathcal{H}}(\mathbf{x}_i, \cdot)$

$$\begin{aligned}
\int_\Omega f(\mathbf{x})\, d\mu(\mathbf{x}) &= \int_\Omega \sum_{i=1}^{N} \gamma_i K_{\mathcal{H}}(\mathbf{x}_i, \mathbf{x})\, d\mu(\mathbf{x}) \\
&= \sum_{i=1}^{N} \gamma_i \int_\Omega K_{\mathcal{H}}(\mathbf{x}_i, \mathbf{x})\, d\mu(\mathbf{x}) \\
&= \sum_{i=1}^{N} \gamma_i \sum_{j=1}^{N} w_{j,N} K_{\mathcal{H}}(\mathbf{x}_i, \mathbf{x}_j) \\
&= \sum_{j=1}^{N} w_{j,N} \left(\sum_{j=1}^{N} \gamma_i K_{\mathcal{H}}(\mathbf{x}_i, \mathbf{x}_j) \right) \\
&= \sum_{i=1}^{N} w_{i,N} f(\mathbf{x}_i).
\end{aligned} \tag{3.10}$$

Notice that in (3.10) the weights $\{w_{i,N}\}$ are independent of the integrand f as they were obtained by interpolating the representer v_I on the set of nodes X.

Another approach consists in approximating If via the orthogonal projection P_V onto $V = \text{span}\{K_{\mathcal{H}}(\mathbf{x}_i, \cdot)\}$ with respect to the inner product $(\cdot, \cdot)_{\mathcal{H}}$ of the RKHS $\mathcal{H}$. If v_I is the representer previously defined, then from

$$(P_V f, g)_{\mathcal{H}} = (f, P_V g)_{\mathcal{H}} \text{ for all } f, g \in \mathcal{H} \tag{3.11}$$

we can define another quadrature rule $Q_V^{(\text{proj})}$ by

$$Q_V^{(\text{proj})} f := \int_\Omega (P_V f)(\mathbf{x})\, d\mu(\mathbf{x})_{\mathcal{H}} = (P_V f, v_I)_{\mathcal{H}} = (P_V v_I, f)_{\mathcal{H}}. \tag{3.12}$$

Maybe a little surprisingly, the quadrature rule $Q_V^{(\text{int})}$ coincides with $Q_V^{(\text{proj})}$ when applied to functions $f \in \mathcal{H}$. To this purpose, we first notice that by the properties of P_V

$$Q_V^{(\text{proj})} f = (f, P_V v_I)_{\mathcal{H}} = (P_V f, v_I)_{\mathcal{H}}, \qquad f \in \mathcal{H}, \tag{3.13}$$

$$Q_V^{(\text{int})} f = (f, s_I^{(\text{int})})_{\mathcal{H}} = (P_V f, s_I^{(\text{int})})_{\mathcal{H}}, \quad f \in \mathcal{H}. \tag{3.14}$$

Thus $Q_V^{(\mathrm{proj})} f = Q_V^{(\mathrm{int})} f$ on $\mathcal{H}$, if and only if they coincide on V, i.e.

$$(K_{\mathcal{H}}(x_i,\cdot), v_I) = (K_{\mathcal{H}}(x_i,\cdot), s_I^{(\mathrm{int})}), \quad i = 1,\dots,N. \tag{3.15}$$

But (3.15) holds since, v_I is the representer of I on $\mathcal{H}$ and $Q^{(\mathrm{int})}$ is exact on V giving

$$(K_{\mathcal{H}}(x_i,\cdot), v_I) = \int_\Omega K_{\mathcal{H}}(x_i,\cdot)\, d\mu(\mathbf{x}) = (K_{\mathcal{H}}(x_i,\cdot), s_I^{(\mathrm{int})}),\ i = 1,\dots,N. \tag{3.16}$$

4. An abstract setting. Denoting by $\mathcal{H}^*$ the dual space of $\mathcal{H}$, a cubature rule

$$Q_{\mathbf{w}} f := \sum_{i=1}^{N} w_{i,N} f(\mathbf{x}_{i,N}) \tag{4.1}$$

with weights $\mathbf{w} = \{w_{i,N}\}$ is said *optimal* in the RKHS $\mathcal{H}$ if it minimizes the worst case error

$$E_{\mathcal{H}}(Q_{\mathbf{w}}) := \left\| I - \sum_{i=1}^{N} w_{i,N} \delta_{x_{j,N}} \right\|_{\mathcal{H}^*} \tag{4.2}$$

over all the rules with nodes $\{\mathbf{x}_{i,N}\}$. From (3.6) $v_I = \int_\Omega K_{\mathcal{H}}(\cdot,\mathbf{x}) d\mu(\mathbf{x})$ is the Riesz representer, so we get as in [12]

$$\begin{aligned} E_{\mathcal{H}}(Q_{\mathbf{w}}) &= \sup_{f\in\mathcal{H}, \|f\|_{\mathcal{H}}=1} \left| \int_\Omega f(\mathbf{x}) d\mu(\mathbf{x}) - \sum_{i=1}^{N} w_{i,N} f(\mathbf{x}_{i,N}) \right| \\ &= \sup_{f\in\mathcal{H}, \|f\|_{\mathcal{H}}=1} \left| \left(f, \int_\Omega K_{\mathcal{H}}(\cdot,\mathbf{x}) d\mu(\mathbf{x}) - \sum_{i=1}^{N} w_{i,N} K_{\mathcal{H}}(\cdot,\mathbf{x}_{i,N}) \right)_{\mathcal{H}} \right| \\ &= \left\| \int_\Omega K_{\mathcal{H}}(\cdot,\mathbf{x}) d\mu(\mathbf{x}) - \sum_{i=1}^{N} w_{i,N} K_{\mathcal{H}}(\cdot,\mathbf{x}_{i,N}) \right\|_{\mathcal{H}} \\ &= \left\| v_I - \sum_{i=1}^{N} w_{i,N} K_{\mathcal{H}}(\cdot,\mathbf{x}_{i,N}) \right\|_{\mathcal{H}}. \end{aligned} \tag{4.3}$$

This implies that the minimum of $Q_V^{(\mathrm{opt})}$ is attained by

$$Q_V^{(\mathrm{opt})} = Q_V^{(\mathrm{proj})} = Q_V^{(\mathrm{int})}. \tag{4.4}$$

It is interesting at this point to see some examples of RKHS in which optimal recovery can be applied.

It is well known that if $\varphi : \Omega \times \Omega \to \mathbb{R}$ is a symmetric (completely) positive definite kernel, then its associated native space $K_{\mathcal{H}} = \mathcal{N}_\varphi(\Omega)$ is a Hilbert space with reproducing kernel φ (see, e.g. [32, p. 138]). By (3.7), we have that for a fixed set of nodes X (that are also centers of the radial basis function), the weights $\{w_{i,N}\}$ such that

$$\int_\Omega \varphi(\mathbf{x},\mathbf{x}_j)\, d\mu(\mathbf{x}) = \sum_{i=1}^{N} w_{i,N}\, \varphi(\mathbf{x}_i,\mathbf{x}_j), \quad j = 1,\dots,N. \tag{4.5}$$

This provides the cubature rule

$$Q_V^{(\mathrm{int})} f := \sum_{i=1}^{N} w_{i,N} f(\mathbf{x}_i) \tag{4.6}$$

that is optimal with respect to the inner product of the RKHS $\mathcal{N}_\varphi(\Omega)$. Notice that by (4.5) $Q_V^{(\mathrm{int})}$ is the rule of our numerical experiments since

$$\sum_{i=1}^{N} w_{i,N} f(\mathbf{x}_i) = \sum_{i=1}^{N} w_{i,N} \sum_{j=1}^{N} \lambda_j \varphi(\mathbf{x}_i, \mathbf{x}_j)$$
$$= \sum_{j=1}^{N} \lambda_j \sum_{i=1}^{N} w_{i,N} \varphi(\mathbf{x}_i, \mathbf{x}_j) = \sum_{j=1}^{N} \lambda_j \int_\Omega \varphi(\mathbf{x}, \mathbf{x}_j)\, d\mu(\mathbf{x}).$$

The case of conditionally positive definite kernels is more complicated, but again it is possible to show that their native spaces $\mathcal{N}_\varphi(\Omega)$ are RKHS with respect to a known inner product dependent on φ (see, e.g. [32], p.146).

For a detailed error analysis see [30] where upper bounds of the cubature error have been achieved in terms of mesh-norm and separation distance.

5. Numerical implementation. The results this section are probably known to experts in RBF, but in our opinion they are still useful for a general audience. In our numerical implementation, the coefficients $\{\lambda_i\}$, $\{\gamma_{k,l}\}$ have been computed by interpolating the values $\{f_i\}$ at the centers $\{x_i\}_{i=1,\dots,N} \subset \Omega := \mathbb{R}^n$. In the case of conditionally positive RBF of degree M one has to solve the linear system

$$\sum_{i=1}^{N} \lambda_i \phi(\|\mathbf{x}_j - \mathbf{x}_i\|_2/\sigma) + \sum_{k=0}^{M-1} \sum_{l=0}^{2k} \gamma_{k,l} P_{k,l}(\mathbf{x}_j) = f(\mathbf{x}_j),\ j = 1, \dots, N \tag{5.1}$$

where $\mathbf{x}_j \in \mathbb{S}^2$, $j = 1, \dots, N$, plus the additional conditions

$$\sum_{i=1}^{N} \lambda_i P_{k,l}(\mathbf{x}_i) = 0, \quad k = 0, \dots, M-1, \quad l = 0, \dots, 2k. \tag{5.2}$$

The linear system defined by (5.1), (5.2), can be rewritten as

$$\begin{bmatrix} A & Q \\ Q^T & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\gamma} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{0} \end{bmatrix} \tag{5.3}$$

where $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_N)^T$, $\boldsymbol{\gamma} = (\gamma_{0,0}, \gamma_{1,0}, \gamma_{1,1}, \gamma_{1,2}, \dots, \gamma_{M-1,2(M-1)})^T$

$$A_{i,j} = \phi(\|\mathbf{x}_i - \mathbf{x}_j\|_2/\sigma), \qquad i, j = 1, \dots, N \tag{5.4}$$
$$Q_{i,j} = P_{k,l}(\mathbf{x}_i),\ i = 1, \dots, N,\ k = 0, \dots, M,\ l = 0, \dots, 2k,\ j = k^2 + l + 1 \tag{5.5}$$

and solved by an appropriate numerical method [5].

The case of the SRBF is similar, with

$$A_{i,j} = \phi(d(\mathbf{x}_i, \mathbf{x}_j)),\ i, j = 1, \dots, N \tag{5.6}$$
$$Q_{i,j} = Y_{k,l}(\mathbf{x}_i),\ i = 1, \dots, N,\ j = k^2 + l + 1. \tag{5.7}$$

where $d(\mathbf{x}, \mathbf{y})$ is the geodesic distance and $Y_{k,l}$, $k = 0, \ldots, M-1$, $l = 0, \ldots, 2k$ the spherical harmonics.

By the *uncertainty relation* [26] the attainable error and the condition number $\kappa(V)$ of the matrix

$$V = \begin{bmatrix} A & Q \\ Q^T & 0 \end{bmatrix} \tag{5.8}$$

cannot both be kept small.

As result, a large number of centres N in (2.1) can lead to a better approximation but also to a very ill-conditioned interpolation matrix. To mitigate this problem several techniques have been proposed [2], [5], [32, p. 253]. One of the more common approaches is due to Floater and Iske [9], which is used here for $\mathbb{S}^2$ although more general domains $\Omega \subseteq \mathbb{R}^n$ can be treated. The condition number of V depends on the *separation distance* (including the factor 2 means this is the *packing radius* on the sphere)

$$\rho_X = \frac{1}{2} \min_{\mathbf{x}_i \in X} \min_{\substack{\mathbf{x}_j \in X \\ \mathbf{x}_j \neq \mathbf{x}_i}} \|\mathbf{x}_i - \mathbf{x}_j\|_2, \tag{5.9}$$

while the available error estimates (see [5], [26], [32]) depend on the *mesh norm* (in some papers also called *fill-in distance* or the *covering radius* on the sphere)

$$h_{X,\Omega} = \sup_{\mathbf{x} \in \Omega} \min_{i=1,\ldots,N} \|\mathbf{x} - \mathbf{x}_i\|_2. \tag{5.10}$$

The ideal situation is to have a large separation distance ρ_X and a small mesh norm $h_{X,\Omega}$. A common criterion is the mesh ratio

$$q_{X,\Omega} := \frac{h_{X,\Omega}}{\rho_X} \geq 1, \tag{5.11}$$

which one wants to keep as small as possible.

The given set of centers $X = \{\mathbf{x}_i\}$ is often decomposed into a hierarchy of nested subsets

$$X_1 \subset X_2 \subset \ldots X_{L-1} \subset X_L = X \tag{5.12}$$

in which, roughly speaking, the ratio $q_{X_\ell,\Omega}$ is "small" and at each level the interpolation problem has a "small" condition number. To reach this target, in [9], [14] different *thinning* algorithms have been proposed and analysed in connection to a variant of the so called *k-center problem*.

Once the subsets $\{X_\ell\}_{\ell=1,\ldots,L}$ are at hand, we can interpolate the data $f|_{X_\ell} = \{f_{i,\ell}\}$ relative to the centers $X_\ell = \{\mathbf{x}_{i,\ell}\}$ by the *Multilevel Scheme* described in [9], [14] and analysed in [21]. To be more precise, starting with $\ell = 1$, at the ℓ-th level one matches the error function

$$f - (s_1 + \ldots + s_{\ell-1}) \tag{5.13}$$

by a RBF (or SRBF) s_ℓ, as described respectively in (2.1) and (2.20).

Denoting by $g|_{X_\ell}$ the restriction of the function $g : \Omega \to \mathbb{R}$ to the set $X_\ell \subset \Omega$ we can rewrite the sequence of interpolation problems (5.13) as

$$\begin{aligned} & s_1|_{X_1} = f|_{X_1} \\ & s_2|_{X_2} = (f - s_1)|_{X_2} \\ & \dots \\ & s_{L-1}|_{X_{L-1}} = (f - \sum_{k=1}^{L-2} s_k)|_{X_{L-1}} \\ & s_L|_{X_L} = (f - \sum_{k=1}^{L-1} s_k)|_{X_L}. \end{aligned} \tag{5.14}$$

As $X = X_L$ it follows that that

$$(s_1 + \ldots + s_L)|_X = f|_X \tag{5.15}$$

i.e. $s_{f,X} = s_1 + \ldots + s_L$. Note that in general at level ℓ of the scheme (5.14) we can choose the RBF scaling parameter σ_ℓ (or the spherical localization paramater h) as well as which RBF/SRBF to use. For a wide class of domains Ω, Iske [15] suggests using at the first level globally supported Thin Plates Splines, and in the next levels compactly supported RBF with scaling parameters σ_ℓ monotonically decreasing. A procedure for computing a good σ_ℓ is also given.

The multilevel interpolation method (5.14) can be easily used for our numerical integration purposes. This is a consequence of the fact that from (5.15) we can assume that

$$f \approx s_{X_1} + \ldots + s_{X_L} \tag{5.16}$$

hence in the domain $\Omega \subset \mathbb{R}^n$

$$\int_\Omega f(\mathbf{x})\, d\mathbf{x} \approx \int_\Omega s_1(\mathbf{x})\, d\mathbf{x} + \ldots + \int_\Omega s_L(\mathbf{x})\, d\mathbf{x}. \tag{5.17}$$

When $\Omega = \mathbb{S}^2$, one can of course use the technique described in the previous section for computing the L terms on the right hand side of (5.17).

In our Matlab program, we have implemented these *thinning* algorithms and *multilevel scheme* to provide a good approximation of the function f in the centers X. The set X is a portion of the data from the orbit of a MAGSAT/NASA satellite and decomposed into subsets X_ℓ as established respectively by the *separation distance* based thinning algorithm [9]. As alternative one can use the *progressive* algorithm proposed and analysed in [14].

A sketch of our code is hence the following:

1. Set the value of the integral I equal to 0;
2. Compute by thinning a hierarchical decomposition $\{X_\ell\}_{\ell=1,\ldots,L}$ of the initial set of centers X;
3. At level ℓ, once it has been decided the RBF/SRBF to use (and its parameters), compute the function s_ℓ by solving the linear system required by (5.14);
4. Compute $I_\ell = \int_{\mathbb{S}^2} s_\ell$ and put $I = I + I_\ell$;
5. Iterate steps 3. and 4. of this algorithm until a stopping criterion is satisfied or $\ell = L$.

First example: the Franke function f_1

RBF	σ	e_2	e_∞	e_I	$\kappa(V)$	$\|V^{-1}\|_2$
MQ	0.200	6-E04	1-E02	1-E06	2+E07	2+E07
G	0.250	7-E04	9-E03	7-E05	8+E07	2+E07
IMQ	0.250	8-E04	1-E02	2-E05	5+E05	2+E07
Wend. C^2	1.350	7-E04	1-E02	8-E06	6+E04	2+E07
TPS	1.000	7-E04	1-E02	9-E04	3+E05	2+E07

SRBF	h	e_2	e_∞	e_I	$\kappa(V)$	$\|V^{-1}\|_2$
Abel-Poisson spline	0.675	8-E04	9-E03	5-E05	1+E06	1+E03
Log. spline	0.825	7-E04	1-E02	5-E06	2+E06	2+E03
Sph. Rec. Multiq.	0.775	6-E04	9-E03	2-E05	1+E06	5+E02

TABLE 6.1
Numerical results for the Franke function f_1.

6. Numerical examples. We now show the behaviour of these cubature rules when applied to two well-known test functions restricted to the unit sphere $\mathbb{S}^2$. Before passing to the examples, we briefly describe some technical details.

The initial set of centers X_T consists of 10000 points obtained by thinning a selected portion of MAGSAT data X_M [17]. In practice, we use a subset X containing only the first 1000 points of X_T that already provides a good compromise between separation distance ρ_X and mesh norm $h_{X,\mathbb{S}^2}$. In our specific case $h_{X,\mathbb{S}^2} \approx 0.11$, $\rho_X \approx 0.037$ and $q_{X,\mathbb{S}^2} = h_{X,\mathbb{S}^2}/\rho_X \approx 3.22$. We perform our cubature tests on two Franke functions [10] adapted by Renka to the three dimensional case [24], namely

$$\begin{aligned} f_1(x,y,z) = {} & 0.75\, \exp\left(-(9x-2)^2/4 - (9y-2)^2/4 - (9z-2)^2/4\right) \\ & + 0.75\, \exp\left(-(9x+1)^2/49 - (9y+1)/10 - (9z+1)/10\right) \\ & + 0.5\, \exp\left(-(9x-7)^2/4 - (9y-3)^2/4 - (9z-5)^2/4\right) \\ & - 0.2\, \exp\left(-(9x-4)^2 - (9y-7)^2 - (9z-5)^2\right) \end{aligned} \tag{6.1}$$

and

$$f_2(x,y,z) = \left(1 + \tanh\left(-9x - 9y + 9z\right)\right)/9. \tag{6.2}$$

The value of If_1 computed by Maple to 20 significant digits is

$$If_1 = \int_{\mathbb{S}^2} f_1(\mathbf{x})\, d\mu(\mathbf{x}) \approx 6.6961822200736179523, \tag{6.3}$$

while

$$If_2 = \int_{\mathbb{S}^2} f_2(\mathbf{x})\, d\mu(\mathbf{x}) = \frac{4\pi}{9}. \tag{6.4}$$

We estimate the interpolation errors as follows. First we generate 3000 random points X_R with a uniform probability distribution and evaluate the RBF/SRBF interpolant s_X as well as the test functions f_1, f_2, on X_R and then we compute the relative errors

$$e_2(f) := \frac{\|s_X|_{X_R} - f|_{X_R}\|_2}{\|f|_{X_R}\|_2}, \qquad e_\infty(f) := \frac{\|s_X|_{X_R} - f|_{X_R}\|_\infty}{\|f|_{X_R}\|_\infty}. \tag{6.5}$$

Second example: the Franke function f_2

RBF	σ	e_2	e_∞	e_I	$\kappa(V)$	$\|V^{-1}\|_2$
MQ	0.775	1-E01	2-E02	1-E04	2+E15	1+E12
G	0.225	2-E01	3-E02	8-E06	3+E10	2+E10
IMQ	0.925	2-E01	2-E02	9-E05	5+E15	4+E12
Wend. C^2	1.600	9-E02	7-E03	5-E04	1+E05	2+E03
TPS	1.000	9-E02	9-E03	4-E04	3+E05	4+E02

SRBF	h	e_2	e_∞	e_I	$\kappa(V)$	$\|V^{-1}\|_2$
Abel-Poisson spline	0.350	2-E02	2-E01	5-E05	5+E15	5+E12
Log. spline	0.450	2-E02	1-E02	2-E04	3+E15	3+E12
Sph. Rec. Multiq.	0.400	2-E02	2-E02	9-E05	5+E15	5+E12

TABLE 6.2
Numerical results for the Franke function f_2.

The relative quadrature error is

$$e_I(f) = \frac{|\int_{\mathbb{S}^2} f(\mathbf{x})\, d\mu(\mathbf{x}) - Q^{(\mathrm{int})} f|}{|\int_{\mathbb{S}^2} f(\mathbf{x})\, d\mu(\mathbf{x})|}. \tag{6.6}$$

where $Q^{(int)} f$ is one of the rules introduced in the previous sections. The $2-$norm condition number of the interpolation matrix V is denoted by $\kappa(V)$. Values of $\kappa(V)$ larger than one over the machine precision, that is larger than 10^{15} must be treated with care.

In Tables 6.1 and 6.2, we consider only nearly optimal values of the scaling/localization parameters without showing what happens for general σ (or h). As is well-known for interpolation by RBF [25], a bad choice of σ may produce poor numerical results. For example, in the case of the function f_1 and $\sigma = 2$, radial basis functions *MQ*, *G* and *IMQ* provide results whose errors e_2, e_∞, e_I are bigger than 10^{-1}. Numerical experiments show that in general the TPS and the RBF with compact support (e.g. Wendland and the Buhmann) are more robust, since they provide good results even for non optimal σ (typically in parallel with not too high condition numbers). Table 6.2 also reports the values of $\|V^{-1}\|_2$, a fundamental parameter for analysing the sensitivity to perturbations of the RBF/SRBF process (see [30]).

In Tables 6.3, we report the performance of the Multilevel scheme (5.14) when it is used to approximate If_2 by the strategy described in (5.17). To this purpose, we consider interpolants s_ℓ (with $\ell = 1, \dots L$) that are linear combinations of the (possibly scaled) Wendland function $\phi(r) = (1-r)_+^4(4r+1)$. Furthermore, we take as centers the first 200, 400, 600, 800 and 1000 points of the thinned set X, while for the sparsity of the interpolation matrix V, we compute the percentage of nonzero entries. In particular we illustrate the differences between the stationary case, in which the ratio between the mesh-norm $h_{X,\mathbb{S}^2}$ and scaling the parameter σ is constant (in our tests we set $\sigma = 4\, h_{X,\mathbb{S}^2}$), and the non-stationary case in which we have choosen $\sigma = 1$.

The numerical results show the main advantages of the multilevel scheme in the stationary case not only for interpolation but also for cubature. With the exception of the first level, the linear systems that arise in (5.3) are sparser and also have better condition numbers, without affecting after these improvements the cubature error e_I.

Finally, in Table 6.4 we give information about the cubature weights $\mathbf{w}$. At this

Multilevel method and the Franke function f_2

Points	σ	$\kappa(V)$	*Sparsity*	e_I
200	1	3+E02	25%	8-E03
400	1	1+E03	25%	3-E03
600	1	4+E03	25%	2-E03
800	1	8+E03	25%	1-E03
1000	1	1+E04	25%	6-E04

Points	σ	$\kappa(V)$	*Sparsity*	e_I
200	1.06	3+E02	28%	8-E03
400	0.70	2+E02	13%	3-E03
600	0.61	4+E02	9%	2-E03
800	0.53	3+E02	7%	1-E03
1000	0.43	4+E02	6%	5-E04

TABLE 6.3
Stationary and non-stationary multilevel method applied to f_2.

stage we need to be careful, especially in the case of RBF/SRBF conditionally positive of order $M > 0$. We will use the bold face to represent vectors. With an obvious notation, the (possibly augmented) linear system (5.3) can be rewritten as

$$A^{(\mathrm{aug})}\mathbf{c}^{(\mathrm{aug})} = \mathbf{f}^{(\mathrm{aug})}. \tag{6.7}$$

On the other hand, knowing the weights $\mathbf{w}$ and denoting by $(\cdot,\cdot)_{\mathbb{R}^N}$ the scalar product in $\mathbb{R}^N$, one can write

$$If \approx (\mathbf{w}, \mathbf{f})_{\mathbb{R}^N}. \tag{6.8}$$

Now we have to compute the weights $\mathbf{w}$. If $\mathbf{I}^{(\mathrm{aug})}$ is the vector

$$\mathbf{I}^{(\mathrm{aug})} = \begin{pmatrix} I_{\phi,\sigma}\mathbf{e}_N \\ \mathbf{b} \end{pmatrix} \tag{6.9}$$

where $\mathbf{e}_N = (1,\ldots,1) \in \mathbb{R}^N$ and $\mathbf{b}$ is the vector whose entries are the integrals of the polynomials required by the conditionally positive RBF/SRBF, then for a certain $\mathbf{w}^{(\mathrm{aug})} \in \mathbb{R}^{N+M^2}$ we have

$$If \approx (\mathbf{c}^{(\mathrm{aug})}, \mathbf{I}^{(\mathrm{aug})})_{\mathbb{R}^{N+M^2}} = (\mathbf{w}^{(\mathrm{aug})}, \mathbf{f}^{(\mathrm{aug})})_{\mathbb{R}^{N+M^2}}. \tag{6.10}$$

Consequently, by (6.7) and the fact that $(A^{(\mathrm{aug})})^{-1}$ is symmetric, we get

$$\begin{aligned} (\mathbf{w}^{(\mathrm{aug})}, \mathbf{f}^{(\mathrm{aug})})_{\mathbb{R}^{N+M^2}} &= (\mathbf{c}^{(\mathrm{aug})}, \mathbf{I}^{(\mathrm{aug})})_{\mathbb{R}^{N+M^2}} \\ &= ((A^{(\mathrm{aug})})^{-1}\mathbf{f}^{(\mathrm{aug})}, \mathbf{I}^{(\mathrm{aug})})_{\mathbb{R}^{N+M^2}} \\ &= (\mathbf{f}^{(\mathrm{aug})}, (A^{(\mathrm{aug})})^{-1}\mathbf{I}^{(\mathrm{aug})})_{\mathbb{R}^{N+M^2}}. \end{aligned} \tag{6.11}$$

Since (6.11) holds for any $\mathbf{f}^{(\mathrm{aug})}$, we have

$$\mathbf{w}^{(\mathrm{aug})} = (A^{(\mathrm{aug})})^{-1}\mathbf{I}^{(\mathrm{aug})}. \tag{6.12}$$

Weights

RBF	*Min*	*Max*	%	$\sum(w_i)_+$	$\sum(w_i)_-$
MQ	−2-E02	5-E02	87%	13.5	−0.9
G	−2-E02	5-E02	88%	13.3	−0.7
IMQ	−3-E02	6-E02	86%	13.7	−1.1
Wend. C^2	+1-E03	3-E02	100%	12.6	0
TPS	+4-E03	3-E02	100%	12.6	0

SRBF	*Min*	*Max*	%	$\sum(w_i)_+$	$\sum(w_i)_-$
Abel-Poisson spline	−4-E02	7-E02	84%	13.9	−1.3
Log. spline	−3-E02	5-E02	87%	13.5	−1.0
Sph. Rec. Multiq.	−3-E02	6-E02	85%	13.8	−1.2

TABLE 6.4
Weights of the RBF/SRBF used in the numerical tests on the Franke function f_2.

As the last M^2 components of $\mathbf{f}^{(\mathrm{aug})}$ are zero, we deduce that only the first N terms of $\mathbf{w}^{(\mathrm{aug})}$ are relevant for the cubature. Thus the vector $\mathbf{w}$ of cubature weights for use in (6.8) is just the first N elements of the solution $\mathbf{w}^{(\mathrm{aug})}$ to $\mathbf{A}^{(\mathrm{aug})}\mathbf{w}^{(\mathrm{aug})} = \mathbf{I}^{(\mathrm{aug})}$.

The same results can alternatively be established working directly with the Riesz representer v_I of the linear functional I.

In Table 6.4, *Min*, *Max* are respectively the minimum and maximum weights, % is the percentage of positive terms in the vector $\mathbf{w}$, $\sum(w_i)_+$ and $\sum(w_i)_-$ are respectively the sum of positive and negative weights. One common stability criterion, $\sum(w_i)_+ \gg -\sum(w_i)_-$, is satisfied by all the RBF/SRBF. In particular the percentage of positive weights is particularly high for the Wendland C^2 and the Thin Plate Splines.

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