Nonsingularity of unsymmetric Kansa matrices: random collocation by MultiQuadrics and Inverse MultiQuadrics

R. Cavoretto[1], A. De Rossi[2], F. Dell'Accio[3], A. Sommariva[4], and M. Vianello[5]

[1,2]University of Torino, Italy
[3]University of Calabria, Rende (CS), Italy
[4,5]University of Padova, Italy

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Abstract

Unisolvence of unsymmetric Kansa collocation is still a substantially open problem. We prove that Kansa matrices with MultiQuadrics and Inverse MultiQuadrics for the Dirichlet problem of the Poisson equation are almost surely nonsingular, when the collocation points are chosen by any continuous random distribution in the domain interior and arbitrarily on its boundary.

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1 Introduction

Unsymmetric Kansa collocation has become over the years one of the most adopted meshless methods by RBF (Radial Basis Functions) for the numerical solution of PDEs (Partial Differential Equations) in a variety of engineering and scientific problems; cf., e.g., [13, 14] and [2, 4, 17, 22] with the references therein. On the other hand, in the popular textbook [9] one reads: "*Since the numerical experiments by Hon and Schaback show that Kansa's method cannot be well-posed for arbitrary center locations, it is now an open question to find* *sufficient conditions on the center locations that guarantee invertibility of the Kansa matrix*". Indeed, Hon and Schaback [12] proved that there are "rare" point configurations that make Kansa matrices singular. To overcome this problem, greedy as well as overtesting collocation techniques have been developed and successfully applied, cf. e.g. [5, 17, 20, 21, 22]. Nevertheless, theoretical invertibility of unsymmetric Kansa collocation matrices has remained a substantially open problem.

In a quite recent paper [7], unisolvence of *random* Kansa collocation by Thin-Plate Splines (TPS) has been proved for the Poisson equation, on 2-dimensional domains whose boundary curve has an analytic parametrization. The key properties in the proof are radiality of the Laplacian and the fact that the TPS basis functions are analytic up to their center, the latter used also in other recent papers on interpolation unisolvence of TPS without polynomial addition, cf. [1, 6, 23]. Though this result represents a first step towards unisolvence, it has a number of restrictions, besides the fact that the differential operator is the pure Laplacian: the RBF kind (indeed, the most usual approach to Kansa method is with MultiQuadrics), the dimension, the boundary regularity.

In the present paper, still resorting to the key property of *analiticity* of the basis, but this time with the presence of *complex singularities*, we prove that unsymmetric Kansa matrices with MultiQuadrics (MQ) and Inverse MultiQuadrics (IMQ) for the Dirichlet problem of the Poisson equation are almost surely invertible (in *any dimension*), when the collocation points are chosen by *any continuous random distribution* in the domain interior and *arbitrarily on its boundary*. We stress that, differently from [7], we do not make here any restrictive assumption on the boundary, except for the usual ones that guarantee well-posedness and regularity of the solution to the differential problem.

2 Unisolvence of random MQ and IMQ Kansa collocation

In this paper we study unisolvence of Kansa collocation by (scaled) MultiQuadrics (MQ)

$$\phi(r) = \phi_\varepsilon(r) = \sqrt{1 + (\varepsilon r)^2}\ , \tag{1}$$

and Inverse MultiQuadrics (IMQ)

$$\phi(r) = \phi_\varepsilon(r) = \frac{1}{\sqrt{1 + (\varepsilon r)^2}}\ , \tag{2}$$

which are both analytic in $\mathbb{R}$. The scale $\varepsilon > 0$ represents the so-called *shape parameter* associated with RBF [9, 16]. We consider the Poisson equation with Dirichlet boundary conditions (cf. e.g. [8])

$$\begin{cases} \Delta u(P) = f(P)\ ,\ P \in \Omega\ , \\ u(P) = g(P)\ ,\ P \in \partial\Omega\ , \end{cases} \tag{3}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain (connected open set), $P = (x_1, \ldots, x_d)$ and $\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_d^2$ is the Laplacian. Differently from [7], where we studied Kansa discretization by TPS in $\mathbb{R}^2$ and we assumed that the boundary is a curve possessing an analytic parametrization, here we do not make any restrictive assumption on $\partial\Omega$, except for the usual ones that guarantee well-posedness and regularity of the solution (like e.g. that the boundary is Lipschitz, cf. e.g. [19] with the references therein). The main reason is that for the discretization of the boundary conditions, with MQ we can resort to a classical result by Micchelli [11] on interpolation unisolvence by any set of distinct points, result achieved also with IMQ since they are strictly positive definite.

Unsymmetric Kansa collocation (see e.g. [3, 9, 12, 13, 17, 22, 24]) consists in seeking a function

$$u_N(P) = \sum_{j=1}^{n} c_j\, \phi_j(P) + \sum_{k=1}^{m} d_k\, \psi_k(P)\ ,\quad N = n + m\ , \tag{4}$$

where

$$\phi_j(P) = \phi(\|P - P_j\|_2)\ ,\quad \{P_1, \ldots, P_n\} \subset \Omega\ , \tag{5}$$

$$\psi_k(P) = \phi(\|P - Q_k\|_2)\ ,\quad \{Q_1, \ldots, Q_m\} \subset \partial\Omega\ , \tag{6}$$

such that

$$\begin{cases} \Delta u_N(P_i) = f(P_i)\ ,\ i = 1, \ldots, n \\ u_N(Q_h) = g(Q_h)\ ,\ h = 1, \ldots, m\ . \end{cases} \tag{7}$$

The following properties will be used below. Defining $\phi_A(P) = \phi(\|P - A\|)$, we have $\phi_A(A) = 1$ and $\phi_A(B) = \phi_B(A)$. Moreover $\Delta\phi_A(B) = \Delta\phi_B(A)$ and $\Delta\phi_A(A) = \pm\varepsilon^2 d$. Indeed, the Laplacian in d-dimensional spherical coordinates centered at A (cf. e.g. [8, Ch.2], [9, App.D]) is the radial function

$$\Delta\phi_A = \phi''(r) + \frac{d-1}{r}\,\phi'(r) \tag{8}$$

and thus for $\phi(r) = (1 + (\varepsilon r)^2)^s$, $s \in \mathbb{R} \setminus \{0\}$, we get

$$\Delta\phi_A = 2\varepsilon^2 s\,(1 + (\varepsilon r)^2)^{s-2}\, p(r; d, \varepsilon, s) \tag{9}$$

where

$$p(r; d, \varepsilon, s) = d + (d + 2(s-1))(\varepsilon r)^2\ , \tag{10}$$

is a second-degree polynomial in r.

Kansa collocation can be rewritten in matrix form as

$$\begin{pmatrix} \Delta\Phi & \Delta\Psi \\ \Phi & \Psi \end{pmatrix} \begin{pmatrix} \mathbf{c} \\ \mathbf{d} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{g} \end{pmatrix} \tag{11}$$

where the block matrix is

$$K_n = K_{n,m}(\{P_i\}, \{Q_h\}) = \begin{pmatrix} \Delta\Phi & \Delta\Psi \\ \Phi & \Psi \end{pmatrix}$$

$$
= \begin{pmatrix}
c & \cdots & \Delta\phi_n(P_1) & \Delta\psi_1(P_1) & \cdots & \Delta\psi_m(P_1) \\
\vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\
\Delta\phi_1(P_n) & \cdots & c & \Delta\psi_1(P_n) & \cdots & \Delta\psi_m(P_n) \\
\phi_1(Q_1) & \cdots & \phi_n(Q_1) & 1 & \cdots & \psi_m(Q_1) \\
\vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\
\phi_1(Q_m) & \cdots & \phi_n(Q_m) & \psi_1(Q_m) & \cdots & 1
\end{pmatrix}
$$

with $c = \varepsilon^2 d$ for MQ and $c = -\varepsilon^2 d$ for IMQ, and $\mathbf{f} = \{f(P_i)\}_{i=1,\dots,n}$, $\mathbf{g} = \{g(Q_h)\}_{h=1,\dots,m}$. We are now ready to state and prove our main result.

Theorem 1 *Let K_n be the MQ or IMQ Kansa collocation matrix defined above, where $\{Q_h\}$ is any fixed set of m distinct points on $\partial\Omega$, and $\{P_i\}$ is a sequence of i.i.d. (independent and identically distributed) random points in Ω with respect to any probability density $\sigma \in L^1_+(\Omega)$. Then for every $m \geq 1$ and for every $n \geq 0$ the matrix K_n is a.s. (almost surely) nonsingular.*

Before proving the theorem, we recall that the construction of i.i.d. random sequences with respect to any probability density can be accomplished (via the uniform distribution) by the well-known acceptance-rejection method, cf. e.g. [10]. Though uniform random points could be the most natural choice for collocation, the possibility of adopting other distributions could be interesting whenever it is known that the solution has steep gradients, or other regions where it is useful to increase the discretization density.

For the reader's convenience, we begin by stating and proving a Lemma which concerns the induction base.

Lemma 1 *The assertion of Theorem 1 holds true for $n = 0$ and $n = 1$.*

Proof. For $n = 0$ the collocation matrix coincides with the $m \times m$ interpolation matrix on the boundary discretization points, which is (deterministically) nonsingular. For IMQ this is a consequence of their positive definiteness (cf. e.g. [9, 24]), while for MQ this comes from a classical result by Micchelli on conditionally positive definite RBF of order 1, cf. [11].

For $n = 1$, consider the augmented matrix

$$\tilde{K}(P) = \begin{pmatrix} c & \Delta\psi_1(P) & \cdots & \Delta\psi_m(P) \\ \psi_1(P) & 1 & \cdots & \psi_m(Q_1) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_m(P) & \psi_1(Q_m) & \cdots & 1 \end{pmatrix}$$

and first observe that $\tilde{K}(P_1) = K_1$ since $\psi_k(P_1) = \phi_1(Q_k)$. For convenience, let us denote by $\Psi(Q_1, \ldots, Q_m)$ the interpolation square block on the boundary points. Developing the determinant of $\tilde{K}(P)$ by Laplace's rule on the first row, we can expand the determinant as a function of $\psi_1(P)$ and $\Delta\psi_1(P)$ and characterize the coefficient of each combination term, obtaining the representation

$$\delta(P) = \det(\tilde{K}(P)) = \lambda\psi_1(P)\Delta\psi_1(P) + \alpha(P)\Delta\psi_1(P) + \beta(P)\psi_1(P) + \gamma(P) \;, \quad (12)$$

where

$$\alpha \in \mathrm{span}\{\psi_k \,,\, 2 \le k \le m\} \;,\; \beta \in \mathrm{span}\{\Delta\psi_k \,,\, 2 \le k \le m\} \;, \quad (13)$$
$$\gamma \in \mathrm{span}\{1, \psi_h\Delta\psi_k \,,\, 2 \le h, k \le m\} \;,$$

with $\lambda = -\det(\Psi(Q_2, \ldots, Q_m)) \neq 0$ (by the same interpolation results quoted above in the case $n = 0$).

Notice that δ is a real analytic function in $\mathbb{R}^d$, because such are all the functions involved in its definition by linear combinations and products, and real analytic functions form a function algebra [15]. We claim that $\delta(P)$ is not identically zero in Ω. Indeed, if δ were identically zero in Ω, it would be identically zero also in $\mathbb{R}^d$, since the zero set of a not identically zero real analytic function must have null Lebesgue measure (cf. [18] for an elementary proof) whereas $meas(\Omega) > 0$. Then taking the line $P(t) = Q_1 + tv$ where $v = (v_1, \ldots, v_d)$ is a given unit vector, we obtain that the real univariate function $\delta(P(t))$ would be identically zero for $t \in \mathbb{R}$. Consequently, its analytic extension to the complex plane, say $\delta(P(z))$, would also be identically zero for $z \in \mathbb{C}$ (technically, such an extension is identically zero in the open set $\mathbb{C} \setminus \{branch\ cuts\}$ where the branch cuts corresponding to (22) below are a finite number of vertical half-lines).

Observe now that the complex functions $\psi_1(P(z)) = (1 + (\varepsilon z)^2)^s$, and in view of (9)-(10)

$$\Delta\psi_1(P(z)) = 2\varepsilon^2 s\,(1 + (\varepsilon z)^2)^{s-2} p(z; d, \varepsilon, s) \;,\; s = 1/2 \;\text{ or }\; s = -1/2 \;, \quad (14)$$

have two branching points in $z = \pm i/\varepsilon$, since $p(\pm i/\varepsilon; d, \varepsilon, s) = 2(1 - s) \neq 0$ for $s \neq 1$ (we take in (1)-(2) the branch of the square root that is positive on the positive reals). Moreover

$$\psi_1(P(z))\Delta\psi_1(P(z)) = 2\varepsilon^2 s\,(1 + (\varepsilon z)^2)^{2s-2} p(z; d, \varepsilon, s) \quad (15)$$

has a pole there, of order 1 for MQ and of order 3 for IMQ. On the other hand, the functions $\alpha(P(z))$, $\beta(P(z))$ and $\gamma(P(z))$ are analytic at $z = \pm i/\varepsilon$. In fact, if A is one of the boundary collocation points different from Q_1, that is $A \in \{Q_k,\, k \neq 1\} \subset \partial\Omega$, we first observe that $\|P(z) - A\|_2^2 = \|Q_1 + zv - A\|_2^2$ has to be seen as the complex extension of the corresponding real function, hence not the complex 2-norm but the sum of the squares of the complex components. Then the complex numbers

$$1 + \varepsilon^2 \|P(\pm i/\varepsilon) - A\|_2^2 = 1 + \varepsilon^2 \sum_{j=1}^{d} (Q_1 \pm iv/\varepsilon - A)_j^2$$

$$= 1 + \varepsilon^2 \sum_{j=1}^{d} [(Q_1 - A)_j^2 \pm 2i(P - A)_j v_j/\varepsilon - v_j^2/\varepsilon^2] \tag{16}$$

(recalling that $\|v\|_2 = 1$) have a.s. positive real part, namely $\|Q_1 - A\|_2^2 > 0$, since Q_1 is a.s. distinct from A, and thus the complex functions corresponding to the chosen branch of the complex square root

$$\left(1 + \varepsilon^2 \sum_{j=1}^{d} (Q_1 + zv - A)_j^2\right)^{\pm 1/2} \tag{17}$$

$A \in \{Q_k,\, k \neq 1\} \subset \partial\Omega$ are both analytic at $z = \pm i/\varepsilon$. This means that $\phi_A(P(z))$ and $\Delta\phi_A(P(z))$ are analytic at $z = \pm i/\varepsilon$, and so are $\alpha(P(z))$, $\beta(P(z))$ and $\gamma(P(z))$ in view of (13).

Since $\lambda \neq 0$, by $\delta(P(z)) \equiv 0$ in view of (12)-(15) we would get

$$[\lambda \psi_1(P(z)) \Delta\psi_1(P(z)) + \gamma(P(z))](1 + (\varepsilon z)^2)^{2-2s}$$

$$= 2\varepsilon^2 s\, \lambda\, p(z; d, \varepsilon, s) + \gamma(P(z))(1 + (\varepsilon z)^2)^{2-2s}$$

$$\equiv -[\alpha(P(z))\Delta\psi_1(P(z)) + \beta(P(z))\psi_1(P(z))](1 + (\varepsilon z)^2)^{2-2s}$$

$$= -2\varepsilon^2 s\, \alpha(P(z)) p(z; d, \varepsilon, s)(1 + (\varepsilon z)^2)^{-s} - \beta(P(z))(1 + (\varepsilon z)^2)^{2-s}$$

$$= [-2\varepsilon^2 s\, \alpha(P(z)) p(z; d, \varepsilon, s) - \beta(P(z))(1 + (\varepsilon z)^2)^2](1 + (\varepsilon z)^2)^{-s} \tag{18}$$

which for both MQ ($s = 1/2$) and IMQ ($s = -1/2$) gives a contradiction, because the first term in (18) is analytic at $z = \pm i/\varepsilon$, and nonvanishing because $p(\pm i/\varepsilon; d, \varepsilon, s) = 2(1 - s) \neq 0$ for $s \neq 1$, whereas the last term either vanishes or has a branching point there.

Then, $\det(K_1) = \delta(P_1)$ is a.s. nonzero, by the already quoted fundamental result that the zero set of a not identically zero real analytic function on an open connected set $\Omega \subset \mathbb{R}^d$ is a null set for the Lebesgue measure (and thus also for any probability measure with density $\sigma \in L_+^1(\Omega)$). $\square$

Proof of Theorem 1. The proof proceeds by (complete) induction on n. The induction base is given by Lemma 1. For the inductive step, we define the augmented matrix

$$\tilde{K}(P) = \begin{pmatrix} c & \cdots & \Delta\phi_n(P_1) & \Delta\phi_1(P) & \Delta\psi_1(P_1) & \cdots & \Delta\psi_m(P_1) \\ \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \Delta\phi_1(P_n) & \cdots & c & \Delta\phi_n(P) & \Delta\psi_1(P_n) & \cdots & \Delta\psi_m(P_n) \\ \Delta\phi_1(P) & \cdots & \Delta\phi_n(P) & c & \Delta\psi_1(P) & \cdots & \Delta\psi_m(P) \\ \phi_1(Q_1) & \cdots & \phi_n(Q_1) & \psi_1(P) & 1 & \cdots & \psi_m(Q_1) \\ \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_1(Q_m) & \cdots & \phi_n(Q_m) & \psi_m(P) & \psi_1(Q_m) & \cdots & 1 \end{pmatrix}$$

Observe that in this case $K_{n+1} = \tilde{K}(P_{n+1})$ since $\psi_k(P_{n+1}) = \phi_{n+1}(Q_k)$ and $\Delta\phi_j(P_i) = \Delta\phi_i(P_j)$.

Developing $\det(\tilde{K}(P))$ by Laplace's rule on the $(n+1)$-row, we can expand the determinant as a function of $\Delta\phi_n(P)$ and $(\Delta\phi_n(P))^2$ and characterize the coefficient of each combination term, obtaining

$$\delta(P) = \det(\tilde{K}(P)) = -\det(K_{n-1})(\Delta\phi_n(P))^2 + \alpha(P)\Delta\phi_n(P) + \beta(P) \tag{19}$$

where

$$\alpha \in \text{span}\{\Delta\phi_j, \psi_k, \Delta\psi_k\,;\, 1 \le j \le n-1\,,\, 1 \le k \le m\}\,, \tag{20}$$

$$\beta \in \text{span}\{1, \Delta\phi_i\Delta\phi_j, \Delta\phi_i\Delta\psi_h, \psi_k\Delta\phi_i, \psi_k\Delta\psi_h\,;\, 1 \le i,j \le n-1\,,\, 1 \le k,h \le m\}\,.$$

Reasoning as in Lemma 1 with ϕ_n substituting ψ_1, we can prove that $\delta(P)$ is almost surely not identically zero in Ω. Observe that in view of (9)-(10)

$$\Delta\phi_n(P(z)) = 2\varepsilon^2 s\,(1+(\varepsilon z)^2)^{s-2} p(z; d, \varepsilon, s)\;,\;\; s = 1/2 \;\text{ or }\; s = -1/2\,, \tag{21}$$

has two branching points in $z = \pm i/\varepsilon$ (we take in (1)-(2) the branch of the square root that is positive on the positive reals), and $(\Delta\phi_n(P(z)))^2$ has a pole there, of order 3 for MQ and of order 5 for IMQ. On the other hand, the functions $\alpha(P(z))$ and $\beta(P(z))$ are analytic at $z = \pm i/\varepsilon$. In fact, by the same considerations of (16), the complex functions

$$\left(1 + \varepsilon^2 \sum_{j=1}^{d} (P_n + zv - A)_j^2\right)^{\pm 1/2} \tag{22}$$

for $A \in \{Q_h\} \subset \partial\Omega$ or $A \in \{P_k,\, k \neq n\} \subset \Omega$ are both analytic at $z = \pm i/\varepsilon$. This means that $\phi_A(P(z))$ and $\Delta\phi_A(P(z))$ are analytic at $z = \pm i/\varepsilon$, and so are $\alpha(P(z))$ and $\beta(P(z))$ in view of (20).

Since by inductive hypothesis a.s. $\det(K_{n-1}) \neq 0$, by $\delta(P(z)) \equiv 0$ in view of (19)-(21) we would get

$$[-\det(K_{n-1})(\Delta\phi_n(P(z)))^2 + \beta(P(z))](1 + (\varepsilon z)^2)^{4-2s}$$

$$= -4\varepsilon^4 s^2 \det(K_{n-1}) p^2(z; d, \varepsilon, s) + \beta(P(z))(1 + (\varepsilon z)^2)^{4-2s}$$

$$\equiv -\alpha(P(z))\Delta\phi_n(P(z))(1 + (\varepsilon z)^2)^{4-2s}$$

$$= -2\varepsilon^2 s\, \alpha(P(z)) p(z; d, \varepsilon, s)(1 + (\varepsilon z)^2)^{2-s} \,, \tag{23}$$

which for both MQ ($s = 1/2$) and IMQ ($s = -1/2$) gives a contradiction, because the first term in (23) is analytic and nonvanishing at $z = \pm i/\varepsilon$, whereas the last term vanishes there (and has even a branching point if $\alpha(P(\pm i/\varepsilon)) \neq 0$).

Then, $\det(K_{n+1}) = \delta(P_{n+1})$ is a.s. nonzero, again by the basic fundamental result on the zero sets of nonzero analytic functions. Indeed, denoting by Z_δ the zero set of δ in Ω and recalling that $\det(K_{n-1}) \neq 0$ (which a.s. holds) implies $\delta \not\equiv 0$, taking the probability of the corresponding events we get

$$\text{prob}\{\det(K_{n+1}) = 0\} = \text{prob}\{\delta(P_{n+1}) = 0\}$$

$$= \text{prob}\{\delta \equiv 0\} + \text{prob}\{\delta \not\equiv 0 \;\&\; P_{n+1} \in Z_\delta\} = 0 + 0 = 0 \,,$$

and the inductive step is completed. □

Conclusions

We have made a further step within the open unisolvence problem of unsymmetric Kansa collocation by RBF. The case of MQ and IMQ is considered, proving almost sure invertibility of collocation matrices by random interior nodes and arbitrary boundary nodes, for the Laplace equation with Dirichlet boundary conditions on general domains. Analiticity of the involved RBF, up to complex singularities, plays a key role together with randomness.

On the other hand, random nodes are quite irregular, so the problem of extending the result to collocation nodes with a better layout, for example quasi-random such as low discrepancy points, is a research subject that will deserve further attention.

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