Least-squares polynomial approximation on weakly admissible meshes: disk and triangle *

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Abstract

We construct symmetric polar WAMs (Weakly Admissible Meshes) with low cardinality for least-squares polynomial approximation on the disk. These are then mapped to an arbitrary triangle. Numerical tests show that the growth of the least-squares projection uniform norm is much slower than the theoretical bound, and even slower than that of the Lebesgue constant of the best known interpolation points for the triangle. As opposed to good interpolation points, such meshes are straightforward to compute for any degree. The construction can be extended to polygons by triangulation.

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1 Introduction.

Locating good points for multivariate polynomial approximation, in particular polynomial interpolation, is an open challenging problem, even in standard domains like disks and triangles. The geometry of a discrete model of a compact set has a strong influence on the quality of interpolation and approximation based on it, even in one dimension, see e.g. [6, §7]. A new insight has been recently given by the theory of "admissible meshes" of Calvi and Levenberg [5], which are nearly optimal for least-squares approximation, and contain interpolation sets that distribute asymptotically as Fekete points of the domain [2].

In this note, we construct low-cardinality weakly admissible meshes on the disk and the simplex, improving the results of [2]. These meshes, that are

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essentially transformations of Chebyshev-Lobatto grids, have about n^2 points for least-squares approximation of degree n, to be compared with the approximately $n^2/2$ points used in polynomial interpolation. The theoretical bound for the uniform norm of the corresponding least-squares projection operator is $\mathcal{O}(n \log^2 n)$, but numerical tests show a much slower growth, even slower than that of the Lebesgue constant of the best known interpolation points for the triangle. Moreover, as opposed to good interpolation points, such weakly admissible meshes are straightforward to compute for any degree. By standard triangulation algorithms, we can compute WAMs for least-squares approximation over general polygons.

2 Weakly Admissible Meshes (WAMs).

Consider a compact set $K \subset \mathbb{R}^d$ (or $K \subset \mathbb{C}^d$) which is polynomial determining, i.e. polynomials vanishing on K vanish everywhere. We adopt the following notation:

$$\|f\|_X := \sup_{x \in X} |f(x)|$$

where f is any bounded function on the set X. Moreover we shall denote by $\mathbb{P}_n^d$ the space of polynomials of degree not larger than n, and by N its dimension

$$N := \dim \left(\mathbb{P}_n^d\right) = \binom{n+d}{d} \sim \frac{n^d}{d!}$$

We define a Weakly Admissible Mesh (WAM) to be a sequence of discrete subsets $\mathcal{A}_n \subset K$ such that

$$\|p\|_K \leq C(\mathcal{A}_n) \|p\|_{\mathcal{A}_n} \ , \quad \forall p \in \mathbb{P}_n^d \tag{1}$$

where both $\mathrm{card}(\mathcal{A}_n) \geq N$ and $C(\mathcal{A}_n)$ grow at most *polynomially* in n. When $C(\mathcal{A}_n)$ is bounded we speak of an Admissible Mesh (AM). We sketch below the main features of WAMs in terms of ten properties (cf. [2, 5]):

P1: if α is an affine mapping and $\mathcal{A}_n$ a WAM for K, then $\alpha(\mathcal{A}_n)$ is a WAM for $\alpha(K)$ with the same constant $C(\mathcal{A}_n)$

P2: any sequence of unisolvent interpolation sets whose Lebesgue constant grows at most polynomially with n is a WAM, $C(\mathcal{A}_n)$ being the Lebesgue constant itself

P3: any sequence of supersets of a WAM whose cardinalities grow polynomially with n is a WAM with the same constant $C(\mathcal{A}_n)$

P4: a finite union of WAMs is a WAM for the corresponding union of compacts, $C(\mathcal{A}_n)$ being the maximum of the corresponding constants

P5: a finite cartesian product of WAMs is a WAM for the corresponding product of compacts, $C(\mathcal{A}_n)$ being the product of the corresponding constants

P6: in $\mathbb{C}^d$ a WAM for the boundary ∂K is a WAM for K (by the maximum principle)

P7: given a polynomial mapping π_s of degree s, then $\pi_s(\mathcal{A}_{ns})$ is a WAM for $\pi_s(K)$ with constant $C(\mathcal{A}_{ns})$ (cf. [2, Prop.2])

P8: any K satisfying a Markov polynomial inequality of the form $\|\nabla p\|_K \leq Mn^r \|p\|_K$ has an AM with $\mathcal{O}(n^{rd})$ points (cf. [5, Thm.5])

P9: least-squares polynomial approximation of $f \in C(K)$ (cf. [5, Thm.1]): the least-squares polynomial $\mathcal{L}_{\mathcal{A}_n} f$ on a WAM is such that

$$\|f - \mathcal{L}_{\mathcal{A}_n} f\|_K \lessapprox C(\mathcal{A}_n)\sqrt{\mathrm{card}(\mathcal{A}_n)}\, \min\{\|f - p\|_K,\, p \in \mathbb{P}_n^d\}$$

P10: Fekete points: the Lebesgue constant of Fekete points extracted from a WAM can be bounded like $\Lambda_n \leq NC(\mathcal{A}_n)$ (that is the elementary classical bound of the continuum Fekete points times a factor $C(\mathcal{A}_n)$)

The properties above give the basic tools for the construction and application of WAMs in the framework of polynomial interpolation and approximation. We focus now on the real bivariate case, i.e. $K \subset \mathbb{R}^2$. Property P8, applied to convex compacts like the disk or the triangle where a Markov inequality with exponent $r = 2$ holds, says that it is always possible to obtain an Admissible Mesh with $\mathcal{O}(n^4)$ points. In order to avoid such a large cardinality, which has severe computational drawbacks for example in least-squares approximation, we can turn to WAMs, which can have a much lower cardinality, typically $\mathcal{O}(n^2)$ points.

In [2] a WAM on the disk with about $2n^2$ points and $C(\mathcal{A}_n) = \mathcal{O}(\log^2 n)$ has been constructed with standard polar coordinates, using essentially property P2 for univariate Chebyshev and trigonometric interpolation. Moreover, using property P2 and P7, WAMs for the triangle and for linear trapezoids, again with about $2n^2$ points and $C(\mathcal{A}_n) = \mathcal{O}(\log^2 n)$, have been obtained simply by mapping the so-called Padua points of degree $2n$ from the square with standard quadratic transformations. We recall that the Padua points are the first known optimal points for bivariate polynomial interpolation, with a Lebesgue constant growing like log-squared of the degree (cf. [1, 4]).

In the following section, improving the result of [2], we construct a symmetric polar WAM on the unit disk with about n^2 points. In Table 1 below (§2.3), we compare using the old WAM for the disk of [2] with the new WAM constructed in this paper. We will see that the norms of the corresponding least-squares projection operators are very similar, but with the new WAM requiring about half the number of points.

Then, property P7 allows to obtain a WAM with about n^2 points on the unit simplex, via the standard quadratic mapping $(u, v) \mapsto (u^2, v^2)$, and thus to have a WAM on any triangle by property P1. In Table 2, we show that the growth of the norm of the corresponding least-squares projection operator is slower than that of the Lebesgue constant of the best known interpolation points for the triangle.

2.1 WAMs on the disk.

A symmetric WAM with about n^2 points on the unit disk, $K = \{x = (x_1, x_2) : x_1^2 + x_2^2 \leq 1\}$, can be obtained by working with *symmetric polar coordinates*, i.e.

$$(x_1, x_2) = (r\cos\theta, r\sin\theta)\ , \quad -1 \leq r \leq 1\ , \quad 0 \leq \theta < \pi \tag{2}$$

as is stated in the following

Proposition 1 *The sequence of symmetric polar grids*

$$\mathcal{A}_n = \{(r_j \cos\theta_k, r_j \sin\theta_k)\} \tag{3}$$

$$\{(r_j, \theta_k)\}_{j,k} = \left\{\cos\frac{j\pi}{n}\,,\, 0 \le j \le n\right\} \times \left\{\frac{k\pi}{n+1}\,,\, 0 \le k \le n\right\}$$

is a WAM of the unit disk with $C(\mathcal{A}_n) = \mathcal{O}(\log^2 n)$*,* $\mathrm{card}(\mathcal{A}_n) = n^2 + n + 1$ *for* n *even, and* $\mathrm{card}(\mathcal{A}_n) = n^2 + 2n + 1$ *for* n *odd.*

Proof. The restriction of a polynomial $p \in \mathbb{P}_n^2$ to the disk in the symmetric polar coordinates (2), $q(r,\theta) = p(r\cos\theta, r\sin\theta)$, becomes a polynomial of degree n in r for any fixed value of θ, and a trigonometric polynomial of degree n in θ for any fixed value of r. Observe that we can take $\theta \in [0, 2\pi]$ for such trigonometric polynomials, since the range of coordinates remains exactly the same (the whole disk). Similarly, the symmetric polar grid does not change taking $\theta_k \in \{2k\pi/(2n+2),\, 0 \le k \le 2n+1\}$ in (3), namely $2n+2$ equally spaced points on the circle. Now, for every $p \in \mathbb{P}_n^2$ we can write

$$|p(x_1, x_2)| = |q(r, \theta)| = |p(r\cos\theta, r\sin\theta)| \le c_1 \log n \, \max_j |q(r_j, \theta)|$$

where c_1 is independent of θ, since the $\{r_j\}$ are $n+1$ Chebyshev-Lobatto points in $[-1, 1]$; cf. [3]. Further

$$|q(r_j, \theta)| \le c_2 \log n \, \max_k |q(r_j, \theta_k)|$$

where c_2 is independent of j, since the $\{\theta_k\}$ correspond to $2n+2$ equally spaced points in $[0, 2\pi]$; cf. [10]. Thus

$$|p(x_1, x_2)| \le c_1 c_2 \log^2 n \, \max_{j,k} |q(r_j, \theta_k)| = c_1 c_2 \log^2 n \, \|p\|_{\mathcal{A}_n}$$

for every point (x_1, x_2) of the disk, i.e., $\mathcal{A}_n$ is a WAM of the disk with $C(\mathcal{A}_n) = O(\log^2 n)$. We conclude by observing that the number of distinct points of the symmetric polar grid is $(n+1) \times (n+1)$ for n odd, whereas for n even subtracting the repetitions of the center, it is $(n+1) \times (n+1) - n = n^2 + n + 1$. $\square$

Remark 1 The WAM (3) is symmetric with respect to rotations by an angle $\pi/(n+1)$, and hence, in particular a rotation by an angle π. Observe that we have to fix at least $n+2$ equally spaced points on the upper semicircle, since to determine and bound a trigonometric polynomial of degree n we need at least $2n+1$ equally spaced points on the whole circle. In any case we get an improvement with respect to the nonsymmetric polar WAMs given in [2], since here we have constants of the same order but roughly half the number of points.

Remark 2 Observe that the WAM of Proposition 1 contains the Chebyshev-Lobatto points of the vertical diameter $\theta = \pi/2$ only for n odd (whereas it always contains the Chebyshev-Lobatto points of the horizontal diameter $\theta = 0$), and thus is not invariant under rotations by an angle $\pi/2$.

In order to always have in general the Chebyshev-Lobatto points of the diameter $\theta = \pi/2$ in the mesh, we should take

$$\{(r_j, \theta_k)\}_{j,k} = \left\{\cos\frac{j\pi}{n}\,,\ 0 \le j \le n\right\} \times \left\{\frac{k\pi}{n+2}\,,\ 0 \le k \le n+1\right\}\,,\ n \text{ even} \quad (4)$$

i.e., $2n + 4$ equally spaced points on the circle. The cardinalities of this new WAM is then, subtracting the repetitions of the center, $\text{card}(\mathcal{A}_n) = (n+1) \times (n+2) - (n+1) = n^2 + 2n + 1$ also for n even. This WAM is now invariant under rotations by an angle $\pi/2$ (since, for $n = 2s$, and $k = s + 1$, $k\pi/(n + 2) = \pi/2$; see Figure 1).

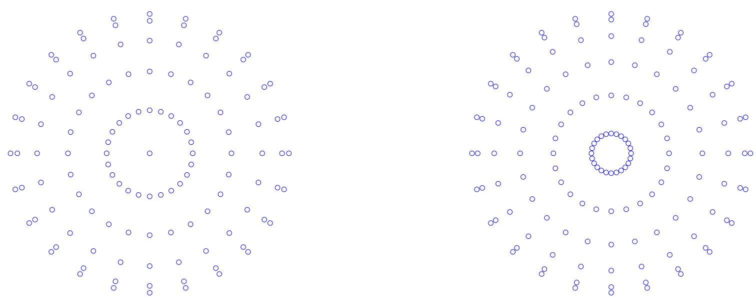

Figure 1: the symmetric polar WAMs of the disk for degree $n = 10$ (left) and $n = 11$ (right).

Proposition 2 *Consider the subspace of even polynomials, i.e. polynomials of even degree $m = 2n$, $n = 0, 1, 2, \ldots$, of the form $p(x^2, y^2)$, $p \in \mathbb{P}^2_n$. The sequence of polar grids on the first quadrant of the unit disk*

$$\mathcal{B}_m = \{(r_j \cos\theta_k, r_j \sin\theta_k)\} \quad (5)$$

$$\{(r_j, \theta_k)\}_{j,k} = \left\{\cos\frac{j\pi}{m}\,,\ 0 \le j \le n\right\} \times \left\{\frac{k\pi}{m}\,,\ 0 \le k \le n\right\}$$

is a WAM for even polynomials on (the first quadrant of) the disk, with $C(\mathcal{B}_m) = \mathcal{O}(\log^2 m)$, $\text{card}(\mathcal{B}_m) = n^2 + n + 1$.

Proof. The restriction of an even polynomial p of degree $m = 2n$ to the disk in the symmetric polar coordinates becomes a polynomial of degree n in r^2 for any fixed value of θ, and a polynomial of degree n in $\cos^2\theta$ for any fixed value of r, say $g(r^2, \cos^2\theta) = p(r\cos\theta, r\sin\theta)$. Now, the range of g is completely determined by its values for $r \in [0, 1]$, $\theta \in [0, \pi/2]$ (the first quadrant of the disk). Recalling that $\cos^2 t = (1 + \cos 2t)/2$, we see that $\{r_j^2\}$ are exactly the Chebyshev-Lobatto points of degree n for $[0, 1]$, as are $\{\cos^2\theta_k\}$. Then, given any even polynomial p of degree $m = 2n$, proceeding as in the proof of Proposition 1 we can write

$$|p(x_1, x_2)| = |g(r^2, \cos^2\theta)| = \mathcal{O}(\log^2 m)\ \max_{j,k} |g(r_j^2, \cos^2\theta_k)| = \mathcal{O}(\log^2 m)\ \|p\|_{\mathcal{B}_m}$$

for every point (x_1, x_2) of the disk, with a constant of the $\mathcal{O}(\cdot)$ symbol independent of (x_1, x_2). We conclude by observing that the number of distinct points of $\mathcal{B}_m$, subtracting the repetitions of the center, is $(n+1)\times(n+1)-n = n^2+n+1$. $\square$

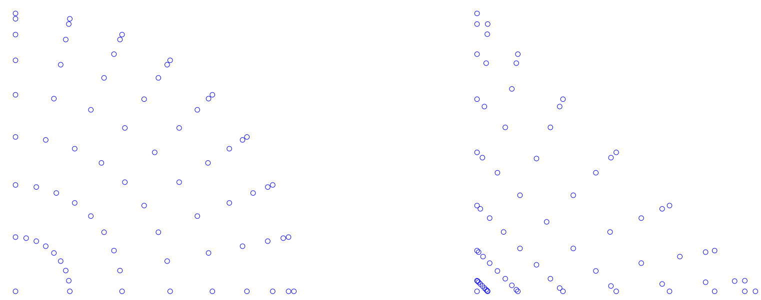

Figure 2: the WAM (5) of the quadrant for even polynomials of degree $n = 16$ (left), and the corresponding WAM (6) of the simplex for degree $n = 8$ (right).

2.2 Mapping to the simplex.

Using the results of the previous section, we now show how to construct a WAM with approximately n^2 points in the unit simplex. The basic tool is the standard quadratic transformation

$$\{u^2 + v^2 \leq 1,\, u \geq 0,\, v \geq 0\} = Q \to K = \{x_1 + x_2 \leq 1,\, x_1 \geq 0,\, x_2 \geq 0\}$$

$$(u, v) \mapsto (u^2, v^2)$$

of the first quadrant of the disk into the unit simplex.

Proposition 3 *The sequence of trapezoidal Chebyshev-Lobatto grids*

$$\mathcal{A}_n = \{(r_j^2 \cos^2\theta_k, r_j^2 \sin^2\theta_k)\} \tag{6}$$

where

$$\{(r_j, \theta_k)\}_{j,k} = \left\{\cos\frac{j\pi}{2n},\, 0 \leq j \leq n\right\} \times \left\{\frac{k\pi}{2n},\, 0 \leq k \leq n\right\}$$

is a WAM of the unit simplex with $C(\mathcal{A}_n) = \mathcal{O}(\log^2 n)$, $\mathrm{card}(\mathcal{A}_n) = n^2+n+1$. *In particular, the mesh points on the sides are the corresponding* $n+1$ *Chebyshev-Lobatto points of degree* n.

Proof. By Proposition 2, the polar grid $\mathcal{B}_{2n} = \{(r_j \cos\theta_k, r_j \sin\theta_k)\}$, with $\{(r_j, \theta_k)\}$ as in (5), is a WAM for even polynomials on the first quadrant of the unit disk. Now, the quadratic transformation $\pi_2 : (u, v) \mapsto (u^2, v^2)$ from the first quadrant onto the unit simplex is invertible, and by a slight extension of property P7 of WAMs (actually we need to identify only WAMs for polynomials of the form $p \circ \pi_2$) we have that $\mathcal{A}_n = \pi_2(\mathcal{B}_{2n})$ is a WAM of the unit simplex. Moreover, $\mathrm{card}(\mathcal{A}_n) = \mathrm{card}(\mathcal{B}_{2n}) = n^2 + n + 1$, and

$C(\mathcal{A}_n) = C(\mathcal{B}_{2n}) = \mathcal{O}(\log^2 2n) = \mathcal{O}(\log^2 n)$. Observing as in the proof of Proposition 2 that $\{r_j^2\}$ are the Chebyshev-Lobatto points of degree n for $[0,1]$, we see that the mesh points on the legs of the simplex are exactly their Chebyshev-Lobatto points. On the other hand, observing that also $\{\cos^2\theta_k\}$ are the Chebyshev-Lobatto points of degree n for $[0,1]$, we see that the mesh points on the hypothenuse, namely $\{(\cos^2\theta_k, 1-\cos^2\theta_k)\}$, are exactly its Chebyshev-Lobatto points. Moreover, the points of the WAM lie on a grid of intersecting straight lines, namely a pencil from the origin cut by a pencil parallel to the hypothenuse, obtained by the quadratic transformation from a grid of intersecting rays and circular arcs of the quadrant; see Figure 2. Indeed, any ray $v = ku$, $k > 0$, is mapped onto the ray $y = k^2 x$ (and $u = 0$ onto $x = 0$), while any arc $u^2 + v^2 = c$, $0 < c \leq 1$, is mapped onto the segment $x + y = c$, $x, y \geq 0$. Such a grid splits the simplex into the union of small trapezoids, degenerating into triangles at the origin. The fact that the grid points on each segment of the pencils are exactly its Chebyshev-Lobatto points, is an immediate consequence of elementary geometry, namely of the "intercept theorem" by Thales of Miletus. □

Remark 3 Once we have a WAM of the unit simplex, we have also a WAM of any triangle with the same constants and cardinalities, by property P1 of WAMs. Indeed, it is sufficient to map the points by the standard affine transformation between triangles.

2.3 Discrete least-squares approximation.

Consider a WAM $\{\mathcal{A}_n\}$ of a polynomial determining compact set $K \subset \mathbb{R}^d$ (or $K \subset \mathbb{C}^d$)

$$\mathcal{A}_n = \{a_1, \dots, a_M\}\ ,\ \ M \geq N = \dim(\mathbb{P}_n^d) \tag{7}$$

and the associated rectangular Vandermonde-like matrix

$$V(\boldsymbol{a};\boldsymbol{p}) := [p_j(a_i)]\ ,\ \ 1 \leq i \leq M\ ,\ 1 \leq j \leq N \tag{8}$$

where $\boldsymbol{a} = (a_i)$, and $\boldsymbol{p} = (p_j)$ is a given basis of $\mathbb{P}_n^d$. For convenience, we shall consider $\boldsymbol{p}$ as a column vector

$$\boldsymbol{p} = (p_1, \dots, p_N)^t$$

The least-squares projection operator at the WAM can be constructed by the following algorithm

iterated orthogonalization:

- (i) compute the QR factorization $V(\boldsymbol{a};\boldsymbol{p}) = Q_1 R_1$
- (ii) compute a second QR factorization $Q_1 = Q_2 R_2$
- (iii) set $Q = Q_2$ and $T = R_1^{-1} R_2^{-1}$

which amounts to a change of basis from $\boldsymbol{p}$ to the discrete orthonormal basis

$$\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_N)^t = T^t \boldsymbol{p} \tag{9}$$

with respect to the inner product

$$\langle f, g \rangle = \sum_{i=1}^{M} f(a_i)\overline{g(a_i)} \tag{10}$$

(we use here the QR factorization with Q rectangular $M \times N$ and R upper triangular $N \times N$). Observe that the Vandermonde matrix in the new basis

$$V(\boldsymbol{a}; \boldsymbol{\varphi}) = V(\boldsymbol{a}; \boldsymbol{p})T = Q$$

is a numerically orthogonal (unitary) matrix, i.e. $\overline{Q^t}Q = I$. The reason for iterating the QR factorization is to cope with ill-conditioning which is typical of Vandermonde-like matrices. Two orthogonalization iterations generally suffice, unless the original matrix $V(\boldsymbol{a}; \boldsymbol{p})$ is so severely ill-conditioned (rule of thumb: condition number greater than the reciprocal of machine precision) that the algorithm fails. This well-known phenomenon of "twice is enough" in numerical Gram-Schmidt orthogonalization, has been deeply studied and explained in [11].

Denoting by $\mathcal{L}_{\mathcal{A}_n}$ the discrete least-squares projection operator, we can write

$$\mathcal{L}_{\mathcal{A}_n} f(x) = \sum_{j=1}^{N} \left(\sum_{i=1}^{M} f(a_i)\overline{\varphi_j(a_i)} \right) \varphi_j(x) = \sum_{i=1}^{M} f(a_i)\, g_i(x) \tag{11}$$

where

$$g_i(x) = K_n(x, a_i)\ ,\ i = 1, \ldots, M\ ;\ \ K_n(x, y) := \sum_{j=1}^{N} \varphi_j(x)\overline{\varphi_j(y)} \tag{12}$$

$K_n(x, y)$ being the *reproducing kernel* (cf. [9]) corresponding to the discrete inner product. In matrix terms, the relevant set of generators of $\mathbb{P}_n^d$ (which is not a basis when $M > N$), becomes simply

$$\boldsymbol{g} = (g_1, \ldots, g_M)^t = QT^t\boldsymbol{p} \tag{13}$$

where the transformation matrix T and the orthogonal (unitary) matrix Q are computed once and for all for a fixed mesh. Moreover, the norm of the least-squares operator as an operator on $C(K)$ is given by

$$\|\mathcal{L}_{\mathcal{A}_n}\| = \max_{x \in K} \sum_{i=1}^{M} |g_i(x)| = \max_{x \in K} \|QT^t\boldsymbol{p}(x)\|_1 \tag{14}$$

Property P9 ensures that the WAMs described in the previous sections can be directly used for least-squares approximation of continuous functions with an error which is near-optimal, up to a factor $\mathcal{O}(n \log^2 n)$. The latter, however, turns out to be a rough overestimate.

Below we report some numerical tests, all done using basic linear algebra functions of Matlab [14]. In Figure 3 we report the norms (14) for the WAMs of the disk and of the simplex, numerically evaluated by discrete maximization of $\|QT^t\boldsymbol{p}(x)\|_1$ using as control sets a sequence of WAMs $\mathcal{A}_{kn}$, $k = 2, 3, \ldots$, until we see a stabilization (a further discrete maximization on 5000 random points has then confirmed the results). We have used as $\{p_j\}$ the Koornwinder basis for the disk [13], and the Dubiner basis for the simplex [8]. Such bases, which are both orthogonal with respect to the so-called equilibrium measure of complex pluripotential theory for the relevant compact (for more details see e.g. [12]), give a not too ill-conditioned initial Vandermonde matrix; their computation, however, is not straightforward, see Remark 6. Recalling that, for any continuous function f we have

$$\|f - \mathcal{L}_{\mathcal{A}_n} f\|_K \leq (1 + \|\mathcal{L}_{\mathcal{A}_n}\|) \min \{\|f - p\|_K,\, p \in \mathbb{P}_n^d\} \tag{15}$$

since $\mathcal{L}_{\mathcal{A}_n}$ is a projection on $\mathbb{P}_n^d$, we see that the factor $\mathcal{O}(n \log^2 n)$ given by property P9 here heavily overestimates the actual operator norm given by (14). Notice that, as for Lebesgue constants for interpolation, the norms of polynomial least-squares operators are invariant under affine mapping, so Figure 3 gives an estimate of what happens in *any* disk and triangle.

In Table 1 we compare the uniform norms of least-squares projection operators on the old WAM in [2] with the new WAM (3)-(4) for the disk. Such norms are slightly smaller with the new WAM, that requires about half the number of points. The growth of the norms, at least in the range of degrees considered, is not far from the optimal one for polynomial projection in the disk, that is $\mathcal{O}(\sqrt{n})$ (cf. [15]).

A similar lower bound for the triangle does not seem to be theoretically known. It is also interesting to compare our results with the Lebesgue constant of the best known points for polynomial interpolation in the triangle, which have been obtained by various authors with different techniques in view of the relevance to spectral and high-order methods for PDEs. Such near-optimal points, however, have been computed only numerically up to degree $n = 19$, cf. [16, 17] and references therein, whereas the WAM (6) can be explicitly computed at any degree and used via the iterated orthogonalization process, provided that the Vandermonde conditioning is not too severe. The comparison with the best Lebesgue constants collected in [17] is reported in Table 2.

Remark 4 It is worth noticing that, if the compact K belongs to a family which is invariant under affine transformations, like disks or triangles, we can compute the matrices Q and T for a given degree once and for all, with a given basis $\boldsymbol{p}$ and a reference WAM, say $\mathcal{A}_n$, on a reference set (e.g., the unit disk and the unit simplex). Then, the least-squares polynomial approximation of degree n for a given function f in K can be computed as

$$\mathcal{L}_n f(x) = \sum_{i=1}^{M} f(\alpha(a_i))\, g_i(x) \ , \ \ \boldsymbol{g}(x) = QT^t \boldsymbol{p}(\alpha^{-1}(x))$$

where $x = \alpha(t) = At + b$ is the affine transformation from the reference compact of the family to K.

Remark 5 In view of property P4, it is immediate to construct, by finite union, a WAM for a polygon from the WAM (6), as soon as we have at hand a triangulation of the polygon. The latter can be obtained by one of the polygon triangulation algorithms widely used in the framework of computational geometry (see e.g. [7]). The constant of such a WAM can be bounded by the maximum of the constants corresponding to the triangular elements, and thus is $\mathcal{O}(\log^2 n)$, irrespectively of the number of sides of the polygon, or of the fact that it is convex or concave. On the other hand, the cardinality of the WAM is approximately n^2 times the number of triangles. Hence, the theoretical bound for the norm of the corresponding least-squares projection operators given by property P9 is still $\|\mathcal{L}_{\mathcal{A}_n}\| = \mathcal{O}(n \log^2 n)$, where the constant of the $\mathcal{O}$-symbol is now proportional to the square root of the number of triangles. Observe that, by construction, the mesh points on each side of the polygon are exactly its Chebyshev-Lobatto points.

In Figure 4, we show two examples of WAM of a non regular convex hexagon for degree $n = 8$. The triangulation is that trivially generated by the barycenter of the hexagon. In the mesh on the left the point $(0, 0)$ of the simplex is mapped in the barycenter for each triangle, whereas in the mesh on the right it is mapped in a boundary vertex. Since the mesh on each triangle has been selected independently of the other triangles that make up the hexagon, we see some obvious over-accumulation of points along the internal edges. In Figure 5 we report the norms of the least-squares projection operators for the given hexagon up to degree 20. We have used here as polynomial basis $\boldsymbol{p}$ the product Chebyshev basis of the minimal rectangle containing the hexagon. The values of the norm are slightly higher than those for the triangle, but still much below the theoretical bound.

Remark 6 We recall that the Koornwinder polynomials for the unit disk, orthogonal with respect to the equilibrium measure $d\mu = dx_1\, dx_2/\sqrt{1 - x_1^2 - x_2^2}$, are given by

$$\phi_{h,k}(x_1, x_2) = (1 - x_1^2)^{k/2}\, P_k^{(-1/2,-1/2)}\left(\frac{x_2}{\sqrt{1 - x_1^2}}\right)\, P_{h-k}^{(k,k)}(x_1)\ ,\ 0 \le h + k \le n \tag{16}$$

where $P_j^{(a,b)}(t)$, $t \in [-1, 1]$, is the Jacobi polynomial of degree j with parameters (a, b) (cf., e.g., [9]). The Dubiner polynomials for the unit simplex, orthogonal with respect to the equilibrium measure $d\mu = dx_1\, dx_2/\sqrt{x_1 x_2(1 - x_1 - x_2)}$, are

$$\psi_{h,k}(x_1, x_2) = (1 - x_2)^h\, \hat{P}_h^{(-1/2,-1/2)}\left(\frac{x_1}{1 - x_2}\right)\, \hat{P}_k^{(2h,-1/2)}(x_2)\ ,\ 0 \le h + k \le n \tag{17}$$

where $\hat{P}_j^{(a,b)}(u) = P_j^{(a,b)}(2u - 1)$, $u \in [0, 1]$.

Observe that both (16) and (17) require computing scaled polynomials of the form

$$q_j(t) = c^j \pi_j(t/c)$$

in a stable manner for $t \in [-1, 1]$, where $\{\pi_j,\ j \ge 0\}$ is a set of orthogonal polynomials (on $[-1, 1]$) satisfying the three-term recurrence relation

$$\begin{aligned} \pi_{j+1}(s) = (s - \alpha_j)\pi_j(s) - \beta_j \pi_{j-1}(s), \quad j = 0, 1, 2, \ldots, \\ \pi_{-1} := 0,\ \pi_0 := 1 \end{aligned} \tag{18}$$

for some scalars $\alpha_j, \beta_j \in \mathbb{R}$. Each π_j is of degree j and is monic.

If $|c| \ge 1$ then $t/c \in [-1, 1]$ for $t \in [-1, 1]$ and the polynomial $q_j(t)$ can be computed by the recurrence relation without any difficulty. However, if $|c| < 1$ then t/c may be large (in absolute value) and the recurrence relation may become unstable. It is this case our approach is to just note that $q_j(t)$ also satisfies a stable three term recurrence. In fact

$$\begin{aligned} q_{j+1}(t) &= c^{j+1}\pi_{j+1}(t/c) \\ &= c^{j+1}\left\{(t/c-\alpha_j)\pi_j(t/c)-\beta_j\pi_{j-1}(t/c)\right\} \\ &= (t-c\alpha_j)c^j\pi_j(t/c)-(c^2\beta_j)c^{j-1}\pi_{j-1}(t/c) \\ &= (t-c\alpha_j)q_j(t)-(c^2\beta_j)q_{j-1}(t) \\ &= (t-\alpha_j')q_j(t)-\beta_j'q_{j-1}(t) \end{aligned}$$

where

$$\alpha_j' := c\alpha_j \text{ and } \beta_j' := c^2\beta_j \ .$$

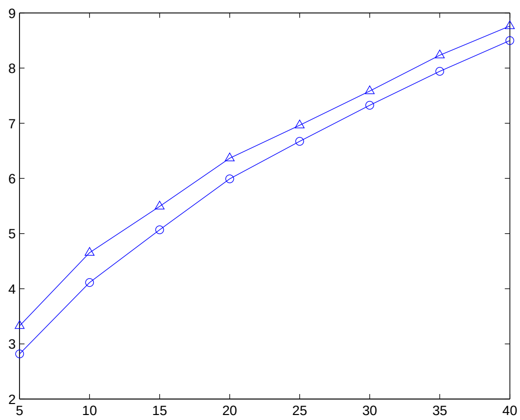

Figure 3: numerically evaluated norms of the discrete least-squares operators for the WAMs of the disk ($\circ$) and of the simplex ($\triangle$), for $n = 5, 10\dots, 40$.

Table 1: Comparison of the uniform norms of least-squares projection operators at the old WAM in [2] with the new WAM (3)-(4) for the disk.

degree	5	10	15	20	25	30	35	40
old WAM	3.7	4.9	5.9	6.8	7.6	8.4	9.0	9.5
new WAM	2.8	4.1	5.1	6.0	6.7	7.3	8.0	8.5

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Table 2: Comparison of the Lebesgue constants of the best known points for interpolation in the triangle with the uniform norms of least-squares projection operators at the WAM (6).

degree	5	6	7	8	9	10	11	12	13	14	15
intp	3.1	3.7	4.3	5.0	5.7	6.7	7.3	7.6	9.3	9.0	9.3
LS	3.3	3.5	3.9	4.2	4.4	4.7	4.9	5.0	5.2	5.3	5.5

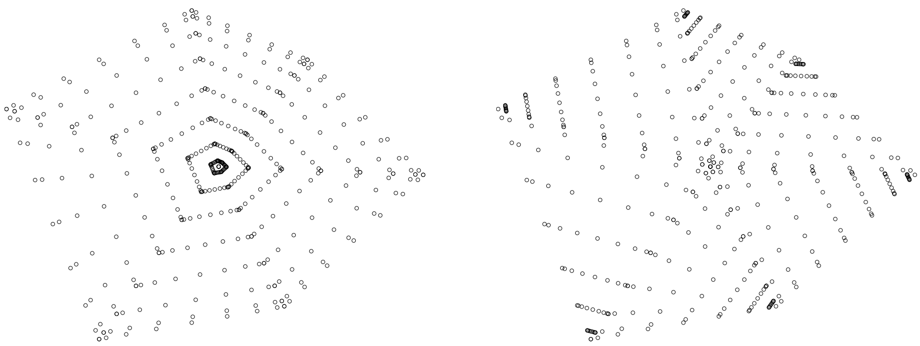

Figure 4: two WAMs of a non regular convex hexagon for degree $n = 8$.

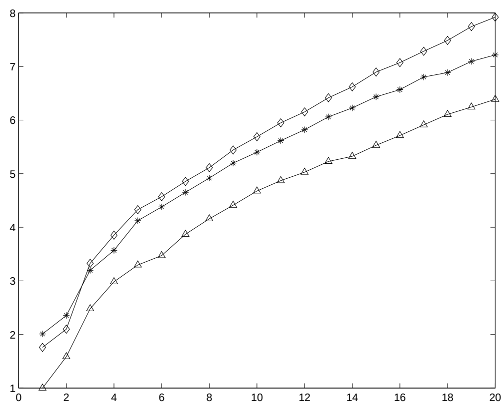

Figure 5: numerically evaluated norms of the discrete least-squares operators at the WAMs of the simplex ($\triangle$) and of the hexagon of Figure 4-left ($*$) and Figure 4-right ($\diamond$), for $n = 1\ldots,20$.

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