On the Lebesgue constant for the Xu interpolation formula

Len Bos
Dept. of Mathematics and Statistics, University of Calgary (Canada)

Stefano De Marchi
Dept. of Computer Science, University of Verona (Italy)

Marco Vianello
Dept. of Pure and Applied Mathematics, University of Padova (Italy)

Corresponding author:
Stefano De Marchi,
Department of Computer Science, University of Verona
S.da Le Grazie 15, 37134 VERONA (ITALY)
email: `stefano.demarchi@univr.it`
tel: +39 0458027978; fax: +39 0458027068

Abstract

In the paper [8], the author introduced a set of Chebyshev-like points for polynomial interpolation (by a certain subspace of polynomials) in the square $[-1,1]^2$, and derived a compact form of the corresponding Lagrange interpolation formula. In [1] we gave an efficient implementation of the Xu interpolation formula and we studied numerically its Lebesgue constant, giving evidence that it grows like $\mathcal{O}((\log n)^2)$, n being the degree. The aim of the present paper is to provide an analytic proof that indeed the Lebesgue constant does have this order of growth.

1 Introduction

Suppose that $K \subset \mathbb{R}^d$ is a compact set with non-empty interior. Let V be a subspace of Π_n^d, the polynomials of degree n in d variables, of dimension $\dim(V) =: N$. Then given N points $X := \{\mathbf{x}_k\}_{k=1}^N \subset K$, the polynomial interpolation problem associated to V and X is the following: for each $f \in C(K)$, the space of continuous functions on the compact K, find a polynomial $p \in V$ such that

$$p(\mathbf{x}_k) = f(\mathbf{x}_k), \quad k = 1, \ldots, N.$$

If this is always possible the problem is said to be unisolvent. And if this is indeed the case we may construct the so-called Lagrange fundamental polynomials $\ell_j(\mathbf{x})$ with the property that

$$\ell_j(\mathbf{x}_k) = \delta_{jk},$$

the Kronecker delta. Further, the interpolant itself may be written as

$$(Lf)(\mathbf{x}) = \sum_{k=1}^{N} f(\mathbf{x}_k)\ell_k(\mathbf{x}).$$

The mapping $f \to Lf$ may be regarded as an operator from $C(K)$ (equipped with the uniform norm) to itself, and as such has an operator norm $\|L\|$. Classically, when $K = [-1, 1]$ and $V = \Pi_n^1$, $\dim(V) = n + 1$, this norm is known as the Lebesgue constant and it is known that then $\|L\| \geq C \log n$ and that this minimal order of growth is attained, for example, by the Chebyshev points (see e.g. [2]).

In the multivariate case much less is known. From Berman's Theorem (cf. [6, Theorems 6.4 and 6.5]) it follows that for $K = B^d$, the unit ball in $\mathbb{R}^d$, $d \geq 2$, and $V = \Pi_n^d$, the Lebesgue constant has a minimal rate of growth of $\mathcal{O}(n^{(d-1)/2})$.

In the tensor product case, when $K = [-1, 1]^d$ and $V = \bigotimes_{k=1}^d \Pi_n^1$, then $\|L\| \geq C(\log n)^d$ and this minimal rate of growth is attained for the tensor product of the univariate Chebyshev points. However, even for the cube and the polynomials of total degree n, i.e., for $K = [-1, 1]^d$ and $V = \Pi_n^d$, the minimal rate of growth is not known.

Recently Xu [7, 8] introduced a set of Chebyshev-like points for $K = [-1, 1]^2$, the square, and he also provided a compact Lagrange interpolation formula based on these points, establishing the connection between the so-called minimal cubature formulas and his Lagrange interpolation formula. We recall that a N-point cubature formula of degree $2n - 1$, has to satisfy

$$N \geq \dim(\Pi_{n-1}^2) + \lfloor \frac{n}{2} \rfloor , \tag{1}$$

and is called *minimal* when the lower bound is attained (see the original paper [4] by Möller). For n even, the minimal cubature formula corresponding to these Chebyshev-like points was introduced by Morrow and Patterson in [5], and later on extended for n odd by Xu in [8]. The connection with Lagrange interpolation was studied by Xu in [8], by introducing a certain subspace of polynomials, $V = V_n$, with the property that

$$\Pi_{n-1}^2 \subset V_n \subset \Pi_n^2 ,$$

and dimension $N = \dim(\Pi^2_{n-1}) + \frac{n}{2} = n(n+2)/2$ for n even, $N = \dim(\Pi^2_{n-1}) + \lfloor \frac{n}{2} \rfloor + 1 = (n+1)^2/2$ for n odd. It should be remarked that V_n, although not a total degree space of polynomials, is much closer to Π^2_{n-1} than to the corresponding tensor-product space $\bigotimes_{k=1}^2 \Pi^1_{n-1}$ which has dimension n^2.

The numerical experiments of [1] gave us good evidence that the Lebesgue constant of Xu-like interpolation has growth of the order $(\log n)^2$ (just as in the tensor product case, and in contrast to the case of the ball where the minimal growth would be of order $\sqrt{n}$) and it is the purpose of this note to prove that this is indeed the case.

From this we may conclude that the points studied by Xu in [7, 8], are excellent points for practical polynomial interpolation. Moreover, our result also gives strong evidence that the minimal rate of growth for the Lebesgue constant for interpolation of polynomials of total degree n on a square is of order $(\log n)^2$. This indicates a fundamental difference between a square and a disk, where the minimal growth is of order $\sqrt{n}$, which perhaps surprising. We also remark that there has recently been introduced a set of points in the square, the so-called Padua points (cf. [3]) for which $V = \Pi^2_n$, that are another Chebyshev-like family, and for which numerical experiments indicate that the Lebesgue constant has this minimal $\mathcal{O}((\log n)^2)$ growth.

2 The Xu polynomial interpolation formula

We start by recalling briefly the construction of the Xu interpolation formula of degree n on the square $[-1, 1]^2$. In what follows we restrict, for simplicity's sake to even degrees n. Starting from the Chebyshev-Lobatto points on the interval $[-1, 1]$, that is

$$z_k = z_{k,n} = \cos \frac{k\pi}{n}, \quad k = 0, \ldots, n\,, \quad n = 2m, \tag{2}$$

the interpolation points on the square studied by Xu, are defined as the two dimensional array $X_N = \{\mathbf{x}_{r,s}\}$ of cardinality $N = n(n+2)/2$,

$$\mathbf{x}_{2i,2j+1} = (z_{2i}, z_{2j+1}), \quad 0 \le i \le m\,,\ 0 \le j \le m-1 \tag{3}$$

$$\mathbf{x}_{2i+1,2j} = (z_{2i+1}, z_{2j}), \quad 0 \le i \le m-1\,,\ 0 \le j \le m\ . \tag{4}$$

The Xu interpolant in Lagrange form of a given function f on the square is

$$L_n^{Xu} f(\mathbf{x}) = \sum_{\mathbf{x}_{k,l} \in X_N} f(\mathbf{x}_{k,l}) \frac{K_n^*(\mathbf{x}, \mathbf{x}_{k,l})}{K_n^*(\mathbf{x}_{k,l}, \mathbf{x}_{k,l})}\ , \tag{5}$$

where the polynomials $K_n^*(\cdot, \mathbf{x}_{k,l})$ are given by

$$K_n^*(\mathbf{x}, \mathbf{x}_{k,l}) = \frac{1}{2}\left(K_{n+1}(\mathbf{x}, \mathbf{x}_{k,l}) + K_n(\mathbf{x}, \mathbf{x}_{k,l})\right) - \frac{1}{2}(-1)^k \cdot (T_n(x) - T_n(y))\ . \tag{6}$$

Here x, y are the coordinates of the generic point $\mathbf{x}$ and T_n is the Chebyshev polynomial of the first kind of degree n, $T_n(x) = \cos(n \arccos x)$.

The polynomials $K_n(\mathbf{x},\mathbf{y})$ can be represented in the form

$$
\begin{aligned}
K_n(\mathbf{x},\mathbf{y}) &= D_n(\theta_1+\phi_1,\theta_2+\phi_2)+D_n(\theta_1+\phi_1,\theta_2-\phi_2) \\
&+ D_n(\theta_1-\phi_1,\theta_2+\phi_2)+D_n(\theta_1-\phi_1,\theta_2-\phi_2)\ , \\
\mathbf{x} &= (\cos\theta_1,\cos\theta_2), \quad \mathbf{y}=(\cos\phi_1,\cos\phi_2)\ ,
\end{aligned} \tag{7}
$$

where the function D_n is defined by

$$
D_n(\alpha,\beta)=\frac{1}{2}\,\frac{\cos((n-1/2)\alpha)\ \cos(\alpha/2)-\cos((n-1/2)\beta)\ \cos(\beta/2)}{\cos\alpha-\cos\beta}\ . \tag{8}
$$

As shown in [8] the values $K_n^*(\mathbf{x}_{k,l},\mathbf{x}_{k,l})$ are explicitly known in terms of the degree n, that is

$$
K_n^*(\mathbf{x}_{k,l},\mathbf{x}_{k,l})=\begin{cases} n^2 & k=0 \text{ or } k=n,\ l\,\text{odd} \\ & l=0 \text{ or } l=n,\ k\,\text{odd} \\ & \\ n^2/2 & \text{in all other cases}\end{cases}\ . \tag{9}
$$

Observe that this *constructive* approach yields immediately unisolvence of the interpolation problem, since for any given basis of V_n the corresponding Vandermonde system has a solution for every vector $\{f(\mathbf{x}_{k,l})\}$, and thus the Vandermonde matrix is invertible.

3 The Lebesgue constant of the Xu points

We will show that

Theorem 1 *The Lebesgue constant of the Xu points Λ_n^{Xu}, is bounded by*

$$
\Lambda_n^{Xu}\le 8\left(\frac{2}{\pi}\log n+5\right)^2+4\ . \tag{10}
$$

The proof will follow from a sequence of technical lemmas.

Lemma 1 *The function $D_n(\alpha,\beta)$ can be written as*

$$
D_n(\alpha,\beta)=\frac{1}{4}\left\{\frac{\sin n\left(\frac{\alpha+\beta}{2}\right)\sin n\left(\frac{\alpha-\beta}{2}\right)}{\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)}+\frac{\sin(n-1)\left(\frac{\alpha+\beta}{2}\right)\sin(n-1)\left(\frac{\alpha-\beta}{2}\right)}{\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)}\right\}\ . \tag{11}
$$

Proof. The proof is obtained by simple trigonometric manipulations. Indeed, using the identity

$$
\cos(A)\cos(B)=\frac{\cos(A+B)+\cos(A-B)}{2},
$$

we obtain

$$
\begin{aligned}
D_n(\alpha,\beta) &= \frac{1}{4}\frac{\cos(n\alpha)+\cos((n-1)\alpha)-[\cos(n\beta)-\cos((n-1)\beta)]}{\cos\alpha-\cos\beta} \\
&= \frac{1}{4}\frac{\cos(n\alpha)-\cos(n\beta)+[\cos((n-1)\alpha)-\cos((n-1)\beta)]}{\cos\alpha-\cos\beta}.
\end{aligned}
$$

Then, by the fact that $\cos(A)-\cos(B)=2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$, the result follows. $\square$

Now, for the points $\mathbf{x}=(\cos\theta_1,\cos\theta_2)$ and $\mathbf{x}_{k,l}=(\cos\phi_1,\cos\phi_2)$, by using (6) and (7), we have

$$
\begin{aligned}
K_n^*(\mathbf{x},\mathbf{x}_{k,l}) &= \frac{1}{2}\{(D_n+D_{n+1})(\theta_1+\phi_1,\theta_2+\phi_2)+(D_n+D_{n+1})(\theta_1+\phi_1,\theta_2-\phi_2) \\
&+ (D_n+D_{n+1})(\theta_1-\phi_1,\theta_2+\phi_2)+(D_n+D_{n+1})(\theta_1-\phi_1,\theta_2-\phi_2)\} \\
&- \frac{1}{2}(-1)^k(\cos(n\theta_1)-\cos(n\theta_2)) \qquad (12)
\end{aligned}
$$

Since we want to bound $\Lambda_n^{\mathrm{X_u}}$, we start by finding an upper bound for $|K_n^*(\mathbf{x},\mathbf{x}_{k,l})|$. First we observe that from Lemma 1

$$
|D_n(\alpha,\beta)+D_{n+1}(\alpha,\beta)| \le \frac{1}{4}\left\{2\left|\frac{\sin n\left(\frac{\alpha+\beta}{2}\right)}{\sin\left(\frac{\alpha+\beta}{2}\right)}\cdot\frac{\sin n\left(\frac{\alpha-\beta}{2}\right)}{\sin\left(\frac{\alpha-\beta}{2}\right)}\right| + \right.
$$
$$
\left.\left|\frac{\sin(n+1)\left(\frac{\alpha+\beta}{2}\right)}{\sin\left(\frac{\alpha+\beta}{2}\right)}\cdot\frac{\sin(n+1)\left(\frac{\alpha-\beta}{2}\right)}{\sin\left(\frac{\alpha-\beta}{2}\right)}+\frac{\sin(n-1)\left(\frac{\alpha+\beta}{2}\right)}{\sin\left(\frac{\alpha+\beta}{2}\right)}\cdot\frac{\sin(n-1)\left(\frac{\alpha-\beta}{2}\right)}{\sin\left(\frac{\alpha-\beta}{2}\right)}\right|\right\}.
$$

Lemma 2

$$
\left|\frac{\sin(n+1)\left(\frac{\alpha+\beta}{2}\right)}{\sin\left(\frac{\alpha+\beta}{2}\right)}\cdot\frac{\sin(n+1)\left(\frac{\alpha-\beta}{2}\right)}{\sin\left(\frac{\alpha-\beta}{2}\right)}+\frac{\sin(n-1)\left(\frac{\alpha+\beta}{2}\right)}{\sin\left(\frac{\alpha+\beta}{2}\right)}\cdot\frac{\sin(n-1)\left(\frac{\alpha-\beta}{2}\right)}{\sin\left(\frac{\alpha-\beta}{2}\right)}\right| \le
$$
$$
2\left|\frac{\sin n\left(\frac{\alpha+\beta}{2}\right)}{\sin\left(\frac{\alpha+\beta}{2}\right)}\cdot\frac{\sin n\left(\frac{\alpha-\beta}{2}\right)}{\sin\left(\frac{\alpha-\beta}{2}\right)}\right|+2\,.
$$

Proof. Let $\theta=\frac{\alpha+\beta}{2}$ and $\phi=\frac{\alpha-\beta}{2}$, then by using simple trigonometric identities, the numerator can be re-written as

$$
\begin{gathered}
(\sin n\theta\cos\theta+\sin\theta\cos n\theta)(\sin n\phi\cos\phi+\sin\phi\cos n\phi) \\
+(\sin n\theta\cos\theta-\sin\theta\cos n\theta)(\sin n\phi\cos\phi-\sin\phi\cos n\phi)
\end{gathered}
$$

Thus

$$
\begin{aligned}
&\left|\frac{\sin(n+1)\theta}{\sin\theta}\cdot\frac{\sin(n+1)\phi}{\sin\phi}+\frac{\sin(n-1)\theta}{\sin\theta}\cdot\frac{\sin(n-1)\phi}{\sin\phi}\right| \\
=\;& 2\left|\frac{\sin n\theta}{\sin\theta}\frac{\sin n\phi}{\sin\phi}\cos\theta\cos\phi+\frac{\sin\theta\sin\phi\cos n\theta\cos n\phi}{\sin\theta\sin\phi}\right| \\
\le\;& 2\left|\frac{\sin n\theta}{\sin\theta}\frac{\sin n\phi}{\sin\phi}\right|+2\,. \qquad \square
\end{aligned}
$$

By Lemma 2, the following upper bound for $|(D_n+D_{n+1})(\alpha,\beta)|$ holds.

Lemma 3

$$
\begin{aligned}
|D_n(\alpha,\beta)+D_{n+1}(\alpha,\beta)| &\le \frac{1}{4}\left\{4\left|\frac{\sin n\left(\frac{\alpha+\beta}{2}\right)}{\sin\left(\frac{\alpha+\beta}{2}\right)}\cdot\frac{\sin n\left(\frac{\alpha-\beta}{2}\right)}{\sin\left(\frac{\alpha-\beta}{2}\right)}\right|+2\right\}\\
&= \left|\frac{\sin n\left(\frac{\alpha+\beta}{2}\right)}{\sin\left(\frac{\alpha+\beta}{2}\right)}\cdot\frac{\sin n\left(\frac{\alpha-\beta}{2}\right)}{\sin\left(\frac{\alpha-\beta}{2}\right)}\right|+\frac{1}{2}\,. \qquad (13)
\end{aligned}
$$

Now we consider $K_n^*(\mathbf{x},\mathbf{x}_{k,l})$. Letting $\mathbf{x}=(\cos\theta_1,\cos\theta_2)$ and $\mathbf{x}_{k,l}=(\cos\phi_1,\cos\phi_2)$, we know that $K_n^*(\mathbf{x},\mathbf{x}_{k,l})$ can be written as in (12). Thus, from Lemmas 1 and 2,

$$
|K_n^*(\mathbf{x},\mathbf{x}_{k,l})| \le \frac{1}{2}\left\{2+\left|\frac{\sin n\left(\frac{\theta_1+\theta_2+\phi_1+\phi_2}{2}\right)}{\sin\left(\frac{\theta_1+\theta_2+\phi_1+\phi_2}{2}\right)}\cdot\frac{\sin n\left(\frac{\theta_1+\phi_1-\theta_2-\phi_2}{2}\right)}{\sin\left(\frac{\theta_1+\phi_1-\theta_2-\phi_2}{2}\right)}\right|+3\text{ other terms }+2\right\} \qquad (14)
$$

and so for the Lagrange polynomials

$$
|L_{k,l}(\mathbf{x})| = \left|\frac{K_n^*(\mathbf{x},\mathbf{x}_{k,l})}{K_n^*(\mathbf{x}_{k,l},\mathbf{x}_{k,l})}\right| \le \frac{2}{n^2}|K_n^*(\mathbf{x},\mathbf{x}_{k,l})|\,,\ k=0,\dots,\frac{n}{2},\quad l=0,\dots,\frac{n}{2}-1\,. \qquad (15)
$$

The Xu points are of two types (cf. (3) and (4)) that for short we call `typeA` and `typeB`, that is

$$
\texttt{typeA}: \mathbf{x}_{2i,2j+1}=(z_{2i},z_{2j+1})
$$

and

$$
\texttt{typeB}: \mathbf{x}_{2i+1,2j}=(z_{2i+1},z_{2j})\,,
$$

where z_s are as in (2) and $i=0,\dots,\frac{n}{2},\ \ j=0,\dots,\frac{n}{2}-1$.

Consider the sum of the Lagrange polynomials for the points of `typeA`. In the bound of $K_n^*(\mathbf{x},\mathbf{x}_{k,l})$ (see above formula (14)), there are four terms plus a constant that sum up to 1 which does not contribute to the dominant growth of the Lebesgue constant. Hence, we need only to bound the four terms involving the sines. Indeed,

$$
\begin{aligned}
A_{\texttt{typeA}} &:= \sum_{i=0}^{n/2}\sum_{j=0}^{n/2-1}|L_{2i,2j+1}| \qquad (16)\\
&\le \frac{2}{n^2}\sum_{i=0}^{n/2}\sum_{j=0}^{n/2-1}|K_n^*(\mathbf{x},\mathbf{x}_{2i,2j+1})|\\
&\le \frac{2}{n^2}\sum_{i=0}^{n/2}\sum_{j=0}^{n/2-1}\left\{2+\frac{1}{2}\left|\frac{\sin n\left(\frac{\theta_1+\theta_2+\phi_1+\phi_2}{2}\right)}{\sin\left(\frac{\theta_1+\theta_2+\phi_1+\phi_2}{2}\right)}\cdot\frac{\sin n\left(\frac{\theta_1+\phi_1-\theta_2-\phi_2}{2}\right)}{\sin\left(\frac{\theta_1+\phi_1-\theta_2-\phi_2}{2}\right)}\right|+3\text{ other terms}\right\}\\
&= \frac{4}{n^2}\frac{n}{2}\left(\frac{n}{2}+1\right)+\frac{1}{n^2}\sum_{i=0}^{n/2}\sum_{j=0}^{n/2-1}\left\{\left|\frac{\sin n\left(\frac{\theta_1+\theta_2+\phi_1+\phi_2}{2}\right)}{\sin\left(\frac{\theta_1+\theta_2+\phi_1+\phi_2}{2}\right)}\cdot\frac{\sin n\left(\frac{\theta_1+\phi_1-\theta_2-\phi_2}{2}\right)}{\sin\left(\frac{\theta_1+\phi_1-\theta_2-\phi_2}{2}\right)}\right|+\dots\right\}.
\end{aligned}
$$

Let A_n be the first of these four terms. Since $\mathbf{x}_{2i,2j+1} = \left(\cos\frac{2i\pi}{n}, \cos\frac{(2j+1)\pi}{n}\right)$, we can write it as

$$A_n = \frac{1}{n^2}\sum_{i=0}^{n/2}\sum_{j=0}^{n/2-1}\left|\frac{\sin n\left(\frac{\theta_1+\theta_2+(2i+2j+1)\pi/n}{2}\right)}{\sin\left(\frac{\theta_1+\theta_2+(2i+2j+1)\pi/n}{2}\right)}\cdot\frac{\sin n\left(\frac{\theta_1-\theta_2+(2i-2j-1)\pi/n}{2}\right)}{\sin\left(\frac{\theta_1-\theta_2+(2i-2j-1)\pi/n}{2}\right)}\right|. \tag{17}$$

Now, change variables in the double sum. Letting $k = i + j$ and $m = i - j$. This is a 1-1 mapping between the pairs of integers (i, j), $\quad 0 \le i \le \frac{n}{2}$, $\quad 0 \le j \le \frac{n}{2} - 1$ in a subset of the pairs of integers (k, m), $\quad 0 \le k \le n-1$, $\quad -\frac{n}{2} \le m \le \frac{n}{2}$.

Hence,

$$\begin{aligned}
A_n &\le \frac{1}{n^2}\sum_{k=0}^{n-1}\sum_{m=-n/2}^{n/2}\left|\frac{\sin n\left(\frac{\theta_1+\theta_2+(2k+1)\pi/n}{2}\right)}{\sin\left(\frac{\theta_1+\theta_2+(2k+1)\pi/n}{2}\right)}\cdot\frac{\sin n\left(\frac{\theta_1-\theta_2+(2m-1)\pi/n}{2}\right)}{\sin\left(\frac{\theta_1-\theta_2+(2m-1)\pi/n}{2}\right)}\right| \\
&= \frac{1}{n^2}\sum_{k=0}^{n-1}\left|\frac{\sin n\left(\frac{\theta_1+\theta_2+(2k+1)\pi/n}{2}\right)}{\sin\left(\frac{\theta_1+\theta_2+(2k+1)\pi/n}{2}\right)}\right|\cdot\sum_{m=-n/2}^{n/2}\left|\frac{\sin n\left(\frac{\theta_1-\theta_2+(2m-1)\pi/n}{2}\right)}{\sin\left(\frac{\theta_1-\theta_2+(2m-1)\pi/n}{2}\right)}\right| \\
&= \left(\frac{1}{n}\sum_{k=0}^{n-1}\left|\frac{\sin n\left(\frac{\theta_1+\theta_2+(2k+1)\pi/n}{2}\right)}{\sin\left(\frac{\theta_1+\theta_2+(2k+1)\pi/n}{2}\right)}\right|\right)\cdot\left(\frac{1}{n}\sum_{m=-n/2}^{n/2}\left|\frac{\sin n\left(\frac{\theta_1-\theta_2+(2m-1)\pi/n}{2}\right)}{\sin\left(\frac{\theta_1-\theta_2+(2m-1)\pi/n}{2}\right)}\right|\right) \qquad (18)
\end{aligned}$$

The next step consists in bounding each factor separately. Start with the first in (18).

Lemma 4 *Suppose that $\phi \in [-\pi, \pi]$, and set $\theta_k = \phi + \frac{(2k+1)\pi}{2n}$. Then,*

$$\frac{1}{n}\sum_{k=0}^{n-1}\left|\frac{\sin n\theta_k}{\sin\theta_k}\right| \le \frac{2}{\pi}\log n + 4\,. \tag{19}$$

Proof. Let $0 \le \phi_0 < \phi_1 < \cdots < \phi_{n-1} \le \pi$ be the set of angles $\{\theta_k\}_{k=0}^{n-1}$ taken modulo π. Then, the ϕ_j are equally spaced, i.e.,

$$\phi_j - \phi_{j-1} = \frac{\pi}{n}, \quad j = 1, \ldots, n-1\,.$$

Then, since $|\sin(\theta \pm \pi)| = |\sin\theta|$, we have

$$\begin{aligned}
\frac{1}{n}\sum_{k=0}^{n-1}\left|\frac{\sin n\theta_k}{\sin\theta_k}\right| &= \frac{1}{n}\sum_{k=0}^{n-1}\left|\frac{\sin n\phi_k}{\sin\phi_k}\right| \\
&\le 4 + \frac{1}{n}\sum_{k=2}^{n-3}\left|\frac{\sin n\phi_k}{\sin\phi_k}\right| \quad \left(\text{since each term} \le \tfrac{1}{n}\times n = 1\right) \\
&\le 4 + \frac{1}{n}\sum_{k=2}^{n-3}\left|\frac{1}{\sin\phi_k}\right| \le 4 + \frac{1}{\pi}\left(\frac{\pi}{n}\right)\sum_{k=2}^{n-3}|\csc\phi_k| \\
&\le 4 + \frac{1}{\pi}\int_{\pi/n}^{\pi-\pi/n}|\csc(\theta)|d\theta \quad \left(\text{by the convexity of } \csc\theta\right) \\
&= 4 + \frac{2}{\pi}\int_{\pi/n}^{\pi/2}\csc(\theta)d\theta = 4 + \frac{2}{\pi}\left\{-\log\left(\csc\theta + \cot\theta\right)\Big|_{\frac{\pi}{n}}^{\frac{\pi}{2}}\right\}
\end{aligned}$$

$$
\begin{aligned}
&= 4+\frac{2}{\pi}\log\left(\csc\frac{\pi}{n}+\cot\frac{\pi}{n}\right) \le 4+\frac{2}{\pi}\log\left(2\csc\frac{\pi}{n}\right) \\
&\le 4+\frac{2}{\pi}\log\left(2\frac{1}{\frac{2}{\pi}\cdot\frac{\pi}{n}}\right) = 4+\frac{2}{\pi}\log n\,. \quad \square
\end{aligned}
$$

For the second factor in (18) we have a similar result.

Lemma 5 *For n even and $\phi \in [-\pi, \pi]$, set $\theta_k = \phi + \frac{(2k+1)\pi}{2n}$. Then,*

$$
\frac{1}{n}\sum_{k=-n/2}^{n/2}\left|\frac{\sin n\theta_k}{\sin\theta_k}\right| \le \frac{2}{\pi}\log n + 5\,. \tag{20}
$$

Proof. The argument is the same as the previous Lemma, except there is one more term in the sum. But this term like all the others is bounded by 1. □.

Proof of the Main Theorem. It follows that $A_{\texttt{typeA}}$ is bounded by 2 plus four times the bound for A_n, i.e.,

$$
A_{\texttt{typeA}} \le 2+4\left(\frac{2}{\pi}\log n+5\right)^2.
$$

Then, including the same bound for the `typeB` points, we have

$$
\begin{aligned}
\Lambda_n^{Xu} &\le 2\left\{2+4\left(\frac{2}{\pi}\log n+5\right)^2\right\} \\
&= 8\left(\frac{2}{\pi}\log n+5\right)^2+4\,. \quad \square
\end{aligned}
$$

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