,

# On the generation of symmetric Lebesgue-like points in the triangle


Francesca Rapetti[a,*], Alvise Sommariva[b], Marco Vianello[b]

[a]*Department of Mathematics "J.-A. Dieudonné", CNRS & Univ. de Nice (France)*
[b]*Department of Pure and Applied Mathematics, University of Padova (Italy)*



## Abstract

We compute point sets on the triangle that have low Lebesgue constant, with sixfold symmetries and Gauss-Legendre-Lobatto distribution on the sides, up to interpolation degree 18. Such points have the best Lebesgue constants among the families of symmetric points used so far in the framework of triangular spectral elements.



*Keywords:* Bivariate polynomial interpolation, Triangle, Lebesgue constant, Lebesgue-type points, Symmetry, Triangular spectral elements
*2000 MSC:* 65D05, 41A63, 65L60


## 1. Introduction

Spectral element variational methods are high-order finite element techniques where the discrete space is constructed by introducing a partition of the domain into elements, by using polynomial basis functions to represent the solution element-wise and by stitching together local representations to approximate the global solution of a given differential problem. They can improve the accuracy of the approximated solution by increasing the polynomial degree of the basis functions as well as the number of mesh elements. Differently to *hp*-finite elements which are based on hierarchical non-nodal basis functions (see e.g. [32]), quadrangle-based spectral elements adopt tensorial nodal bases constructed as characteristic Lagrange


☆Work supported by the "ex-60%" funds and by the project "Interpolation and Extrapolation: new algorithms and applications" of the University of Padova, and by the INdAM GNCS.



*Corresponding author. Address: Univ. de Nice, Parc Valrose, 06108 Nice cedex 02, France.
*Email address:* `frapetti@math.unice.fr` (Francesca Rapetti)

bases with respect to the Gauss-Lobatto-Legendre (GLL) interpolation/quadrature nodes (see e.g. [1, 14, 17, 18, 23]). But using quadrangular/hexahedral mesh elements may be quite a restriction to handle problems in complex geometries, thus there have been a number of recent developments to define spectral methods for triangles/tetrahedra either conforming or non-conforming (see for example [26] and [20], respectively, and the references therein).

The question of how to distribute nodes in a triangle or tetrahedron which are suitable for high-order polynomial interpolation is still a somewhat open question. Two factors figure out prominently in the quality of high-order polynomial interpolation, namely, the smoothness of the function to be interpolated, and the location of the interpolation points. Interpolations using uniformly distributed points yield undesirable behavior (oscillations) even for smooth functions as soon as the interpolation degree increases (Runge-like phenomena).

Good distributions are the *Fekete points* [5, 33], that maximize the absolute value of the Vandermonde determinant and thus ensure that the Lebesgue constant is bounded (even if it's numerically much smaller) by the dimension of the polynomial space, and the *Lebesgue points*, that directly minimize the Lebesgue constant [2, 10, 22]. Actually, the maximum and the minimum are reached up to the machine precision, we thus use the terminology of *Lebesgue-type* (resp. *approximated Fekete*) points for those points that minimize (resp. maximize), up to the machine precision, the Lebesgue constant (resp. the absolute value of the Vandermonde determinant).

Indeed, we recall that the Lebesgue constant of a unisolvent interpolation array $\xi = \{P_1, \dots, P_m\}$ in a compact $K \subset \mathbb{R}^d$ is defined as

$$\Lambda_n(\xi) = \max_{P \in K} \lambda_n(P;\xi) \ , \quad \lambda_n(P;\xi) = \sum_{k=1}^{m} |\ell_i(P)| \ , \tag{1}$$

$$\ell_i(P) = \frac{\det(\mathrm{Vand}_n(P_1, \dots, P_{i-1}, P, P_{i+1}, \dots, P_m))}{\det(\mathrm{Vand}_n(\xi))} \ , \tag{2}$$

where $m = \binom{n+d}{d}$ is the dimension of the space of $d$-variate polynomials of degree $\leq n$ defined on $K$, $\ell_i(P)$ the Lagrange polynomial associated to the point $P_i$ (such that $\ell_i(P_j) = \delta_{ij}$, with $\delta_{..}$ the Kronecker symbol) and $\mathrm{Vand}_n$ the Vandermonde matrix of the points built on a chosen basis $\{\psi_k\}_{k=1,m}$ of the space of $d$-variate polynomials of degree $\leq n$ defined on $K$ (thus $(\mathrm{Vand}_n)_{ij} = \psi_i(P_j)$, $i, j = 1, m$). The Lagrangian polynomials $\ell_i$ and the basis functions $\psi_j$ are linked through the Vandermonde matrix by the relation $(\mathrm{Vand}_n)_{ij}\ell_j = \psi_i$.

Notice that, whereas the existence of Fekete points for a given compact set $K$ is trivial, since $\det(\mathrm{Vand}_n(\xi))$ is a polynomial in $\xi \in K^m$, the problem is more subtle concerning Lebesgue points. Indeed, the Lebesgue constant $\Lambda_n(\xi)$ is not continuous on the whole $K^m$, since the denominator of the Lagrange polynomials vanishes on a subset of $K^m$ which is an algebraic variety. For completeness, it is worth stating and proving a basic result concerning the existence of Lebesgue points.

**Proposition 1.** *Let $K$ be a polynomial determining compact subset of $\mathbb{R}^d$. For any degree $n > 0$ the Lebesgue constant $\Lambda_n(\xi)$, cf. (1), attains a global minimum in at least one unisolvent interpolation array $\xi \in K^m$ (such $\xi$ is termed an array of Lebesgue points).*

**Proof.** Since $K$ is polynomial determining, that is polynomials vanishing there vanish everywhere (this is true for example whenever $K$ has internal points), such is $K^m$ and thus there are points in $K^m$ where $\det(\mathrm{Vand}_n(\cdot))$, which is a nonzero polynomial, does not vanish. Let us introduce $\Gamma$ to denote the algebraic variety $\Gamma = \{z \in K^m : \det(\mathrm{Vand}_n(z)) = 0\}$. The Lebesgue constant $\Lambda_n(\xi)$ is continuous at every point $\xi \in K^m \setminus \Gamma$. Indeed, since $\det(\mathrm{Vand}_n(\xi)) \neq 0$, there exists a compact neighborhood of $\xi$, say $U \subset K^m \setminus \Gamma$, such that the Lebesgue function $\lambda_n(P; u)$ is continuous and thus uniformly continuous in $K \times U$. Then, for any $\varepsilon > 0$ there exists $\delta(\varepsilon)$ such that for all $u \in U$ with $\|\xi - u\|_2 < \delta(\varepsilon)$

$$|\Lambda_n(\xi) - \Lambda_n(u)| = |\,\|\lambda_n(\cdot,\xi)\|_K - \|\lambda_n(\cdot,u)\|_K\,| \le \|\lambda_n(\cdot,\xi) - \lambda_n(\cdot,u)\|_K < \varepsilon\,,$$

i.e., the maximum of the Lebesgue function in $K$ is continuous at $\xi$ (here and below we use the notation $\|f\|_X = \sup_{y\in X} |f(y)|$ for a function $f$ bounded in the set $X$).

Moreover, for every fixed $\eta = (Q_1, \dots, Q_m) \in \Gamma$, $\Lambda_n(\xi) \to +\infty$ as $\xi = (P_1, \dots, P_m) \to \eta$ in $K^m \setminus \Gamma$. In fact, the Vandermonde matrix being singular at $\eta$, there exists a nonzero polynomial, say $\hat{p}$, such that $\hat{p}(Q_j) = 0$, $j = 1, \dots, m$. Now, it is easy to prove that

$$\Lambda_n(\xi) = \sup_{p\neq 0,\, deg(p)\le n} \frac{\|p\|_K}{\|p\|_{\{P_1,\dots,P_m\}}}\,,$$

and hence $\Lambda_n(\xi) \ge \|\hat{p}\|_K / \|\hat{p}\|_{\{P_1,\dots,P_m\}} \to +\infty$ as $\xi \to \eta$ in $K^m \setminus \Gamma$.

Now, defining $\Lambda_n(z) = +\infty$ for $z \in \Gamma$, the Lebesgue constant becomes lower semicontinuous in the compact $K^m$, and thus by the generalized extreme value theorem it attains a global minimum (cf. e.g. [16]), in at least one $\xi$ that clearly belongs to $K^m \setminus \Gamma$. $\square$

We recall that Fekete and Lebesgue points are invariant under change of polynomial basis, and their Lebesgue constant is invariant under affine mapping of the domain.

These two families of points have to be computed numerically for the triangle. Indeed, Fekete points are known explicitly only in very few cases (the interval, the complex circle, and the cube for tensor-product polynomials) [12, 9], whereas Lebesgue points are not even known in one dimension. There is a specific literature on this numerical optimization problem, that becomes rapidly large-scale increasing the interpolation degree, and more generally on the search for good nodal sets for the triangle; cf., e.g., [15, 22, 33, 34] and references therein.

In the recent literature, as in [10, 22], two different approaches have been introduced to achieve Lebesgue-like point sets. In [22] this result have been obtained via unconstrained minimization according to a detailed pseudo-algorithm (but no code is available). In [10] the authors consider a method of multigrid nature, based on Matlab optimization routines. For numerical routines see [11].

In this paper we compute Lebesgue-like points for the equilateral triangle, that we term LEBGLS points, by routines of the Matlab Optimization Toolbox [25], with the constraints that the set has *sixfold symmetry* and the *GLL distribution* on the sides (which is the most usual to obtain conforming triangular spectral elements). Though it is not known whether "true" Lebesgue points for the triangle are symmetric (and computational results seem to say they are not, cf. [11, 22]), symmetry is a reasonable property: besides being a key requirement on the approximation point distribution on the mesh edges (resp. edges and faces) in 2D (resp. in 3D) if one adopts conforming variational methods (cf., e.g., [27]), it would allow to use the new nodal sets as constrained distributions on the faces of a tetrahedron, in view of computing optimal nodal sets for tetrahedral spectral elements.

In Section 2, we discuss the computational procedure adopted to obtain the LEBGLS points, and we compare them with other known interpolation sets on the triangle. In particular, they turn out to have the best Lebesgue constants among the families of symmetric points which are used nowadays in the framework of triangular spectral elements. In Section 3 by presenting some numerical results obtained by adopting the considered sets of points as interpolation points for triangle-based spectral element methods ($T$SEM). Section 4 concludes with a few general remarks and outlook toward future work.

## 2. Computational aspects

Let $K = \mathcal{T}$ be the equilateral triangle whose vertices are $V_1 = (-1, 0)$, $V_2 = (1, 0)$ and $V_3 = (0, \sqrt{3})$. The purpose of this section is to show how to compute, for a fixed degree $n$,

$$\text{LEBGLS} = \text{argmin}\left\{\Lambda_n(\xi)\,,\ \xi \in \mathcal{T}\,,\ \xi \text{ is 6-symmetric}\,,\ \xi\big|_{\text{side}} = \text{GLL}\right\} \qquad (3)$$

i.e., a *sixfold symmetric* set of points $\xi = \{P_1, \dots, P_m\} \subset \mathcal{T}$ with a minimal Lebesgue constant (1), and a fixed distribution on the sides (e.g. Gauss-Legendre-Lobatto points). Lebesgue-like points with no symmetries or with only the GLL distribution constraint on the sides have been computed in [22, 28] and more recently in [11]; the families of [11] have been termed LEB (no symmetry) and LEBGL (GLL distribution on the sides).

The Lebesgue constant is defined only in terms of the Lagrange polynomials $\ell_i$ which in turn are a function of the nodal positions $P_j$, regardless of the choice for the basis functions $\{\psi_j\}$. If one wishes to limit $\Lambda_n$, one has to optimize the placement of the $P_j$ in the triangle, that's our concern. However, the choice of the basis $\{\psi_j\}$ is numerically crucial, as it influences the conditioning of the Vandermonde matrix. A well-conditioned Vandermonde matrix is essential for the computation of the Lagrange polynomials. A generally satisfying choice is the adoption for $\{\psi_j\}$ of the *Koornwinder-Dubiner polynomial basis* [19] which is an orthonormal basis in the $L^2(T)$-scalar product.

It will be useful to determine each point $P \in \mathcal{T}$ via its barycentric coordinates $(\lambda_1, \lambda_2, \lambda_3)$, i.e., $P = \lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3$ with $\sum_{i=1}^{3} \lambda_i = 1$. We require that if a point $P = (\lambda_1, \lambda_2, \lambda_3)$ belongs to the point-set $\xi$ then all the points $Q$ whose barycentric coordinates are permutations of $(\lambda_1, \lambda_2, \lambda_3)$ (sometimes called the *orbits* of $P$) also belong to $\xi$. This property is equivalent to say that the set $\xi$ possesses all the six symmetries of the equilateral triangle $\mathcal{T}$.

Depending on the barycentric coordinates, we distinguish three cases:

1. All the barycentric coordinates are equal. This family $O_1$ is usually named of *orbit 1 type*. Since $\sum_k \lambda_k = 1$, it includes only the barycenter of the equilateral triangle $\mathcal{T}$, $C = (1/3, 1/3, 1/3)$.
2. Only two of the barycentric coordinates are equal. This family $O_3$ is usually named of *orbit 3 type*, since if $P_1 \in \xi$ also its 2 different orbits $P_2$ and $P_3$ belong to $\xi$.
3. All barycentric coordinates are different. This family $O_6$ is usually named of *orbit 6 type*, since if $P_1 \in \xi$ also its 5 different orbits $P_2, \dots, P_6$ belong to $\xi$.

A general sixfold symmetric point-set $\xi$ will consist of $n_1 \leq 1$ points of orbit 1 type, $3n_3$ points of orbit 3 type and $6n_6$ points of orbit 6 type so that $m = (n+1)(n+2)/2 = n_1 + 3\,n_3 + 6\,n_6$ (it is not too difficult to prove that for any $m = (n+1)(n+2)/2$ such a problem has at least one solution $n_1$, $n_3$, $n_6$).

From the point of view of spectral elements approximation, it is important that if $X_{GLL}^{(n+1)} = \{x_1, \dots, x_{n+1}\}$ are the Gauss-Legendre-Lobatto points of degree $n+1$ scaled in the interval $[0, 1]$ (cf. [21]), then all the $3n$ points whose barycentric coordinates are permutations of $(x_i, 1 - x_i, 0)$ also belong to $\xi$. A straightforward investigation shows that all these points lie on the sides of $\mathcal{T}$ since they are of the

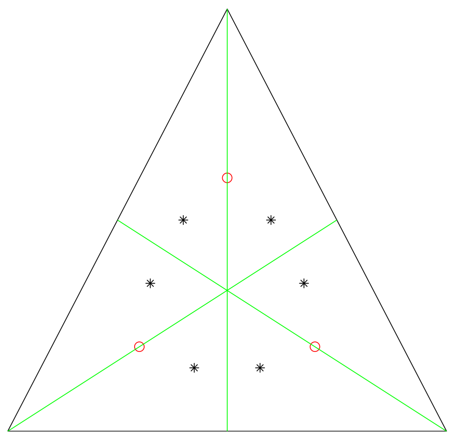

Figure 1: Orbit 3 type points (∘) and orbit 6 type points (∗).

form $x_i V_j + (1 - x_i)V_k$ (with $j \neq k,\ j,k = 1,2,3$) and that $0, 1 \in X_{GLL}^{(n+1)}$, implies that the vertices $V_1$, $V_2$ and $V_3$ are in $\xi$. Moreover the distribution of points on each side is obviously symmetric w.r.t. its midpoint being this one a property of $X_{GLL}^{(n+1)}$. The symmetry of points on the triangle sides is a fundamental property to build up a conforming Galerkin approach to the solution of a given PDE, and thus reconstruct a continuous function over a simplicial triangulation of the computational domain from local polynomial interpolants defined on the mesh simplices. If the points were not symmetrically disposed on the triangle sides, a non-conforming Galerkin method, such as the well-known Discontinuous Galerkin approach, would be necessarily adopted to approximate the PDE solution. Finally, we observe that if $n$ is even, then $1/2 \in X_{GLL}^{(n+1)}$ that easily implies that all the midpoints of a side $\overline{V_j V_k}$ are also in $\xi$.

We discuss now in detail the minimization procedure. Let be $m = n_1 + 3\,n_3 + 6\,n_6$ for some nonnegative integer $n_1$, $n_3$, $n_6$. It is straightforward to see that the barycenter is an element of $\xi$ if and only if rem$(m, 3)=$ 1, that is $n$ is a multiple of 3. A more careful analysis must be done for the $3\,n_3$ points of $O_3$ and the $6\,n_6$ points of $O_6$. Since if $P \in \xi$ also its orbits are in $\xi$ and one of them must be in the triangle $\hat{\mathcal{T}} \subset \mathcal{T}$ whose vertices are $H = (0,0)$, $V_2 = (1,0)$ and $C = (0, \sqrt{3}/2)$, we decided to compute $n_3$ points of $O_3$ and $n_6$ of $O_6$ in $\hat{\mathcal{T}}$ and then determine all their orbits in $\mathcal{T}$ by permutation of their barycentric coordinates. Since we have already assigned some points on the sides (i.e., those following a Gauss-Legendre-Lobatto distribution), some points of $O_3$ and $O_6$ are already given and one has only to provide, say, $\hat{n}_3$ points of $O_3$ and $\hat{n}_6$ of $O_6$. It is easy to see that $n_3 = \hat{n}_3 + 1 + rem(n+1, 2)$ and that $n_6 = \hat{n}_6 + \frac{(n-1-rem(n+1,2))}{2}$.

We observe now that, with the exception of the barycenter $C$, the $O_3$ points of $\hat{\mathcal{T}}$ are all the points that lie in the segment $\overline{CH}$ or $\overline{CV_2}$ while all the other ones are in $O_6$. We parameterized the segment $\overline{CH}$ as $(1-\tau)C + \tau H$ for $\tau \in [0, 1]$, and $\overline{CV_2}$ as $(1+\tau)C - \tau H$ for $\tau \in [-1, 0]$ so that $\overline{CH} \cup \overline{CV_2}$ can be described by only one real variable, while all the other points $P$ of $O_6$ can be determined by the first two barycentric coordinates $\hat{\lambda}_1$, $\hat{\lambda}_2$ of the representative $P$ in the triangle $\hat{\mathcal{T}}$, i.e. $P = \hat{\lambda}_1 H + \hat{\lambda}_2 V_2 + \hat{\lambda}_3 C$ with $\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3 \geq 0$ and $\sum_{k=1}^{3} \hat{\lambda}_k = 1$. If some of the elements of $\xi$ do not belong to $\hat{\mathcal{T}}$, then we set the target function value in $\xi$ equal to $10^{20}$.

This discussion shows that we have reduced the problem to a minimization in $\hat{n}_3$ variables in the interval $[-1, 1]$ and $2\hat{n}_6$ variables that correspond to the barycentric coordinates $\hat{\lambda}_1$, $\hat{\lambda}_2$ of points of $\hat{\mathcal{T}}$ (the third coordinate $\hat{\lambda}_3$ is not involved since $\sum_k \hat{\lambda}_k = 1$). Any feasible combination of these $\hat{n}_3 + 2\hat{n}_6$ variables determines $\hat{n}_3 + \hat{n}_6$ points of $\hat{\mathcal{T}}$ and by permutation of the respective barycentric coordinates, $3\hat{n}_3$ points of $O_3$ and $6\hat{n}_6$ of $O_6$, that added to the assigned points on the sides having GLL distribution and possibly the barycenter, determine the set of symmetric points $\xi$ to be analyzed.

Consequently, setting $s = \hat{n}_3 + 2\hat{n}_6$, one can define in Matlab a target function that from a feasible $s$-array first determines a set $\xi$ of symmetric points of $\mathcal{T}$, and then computes an approximation of its Lebesgue constant $\Lambda_n(\xi)$, defined as $\max_{P \in \mathcal{T}} \sum_{i=1}^{m} |\ell_i(P)|$, by testing the Lebesgue function $\sum_{i=1}^{m} |\ell_i(P)|$ on a large control set $\mathcal{Y} \subset \mathcal{T}$ and then taking its maximum.

Once a good initial set is at hand, we used the same approach of [10]. We fix a sequence of positive integers $m_0 < m_1 < \ldots < m_k = 250$, and start the minimization process by evaluating the Lebesgue constant on a coarse mesh, namely a Weakly Admissible Mesh $Y_{m_0}$ of degree $m_0$ [8, 6, 13], with a fixed number of iterations, say 50. When the approximate solution is at hand, in a multigrid fashion, we restart the process evaluating the Lebesgue constant on a finer mesh, in our case $Y_{m_j}$ with $m_j > m_{j-1}$, $j = 0, \ldots, k$. After this first stage, we restart the algorithm from the initial $m_0$ and the point set, say $\xi^{(1)}$, just obtained, computing more stages. We repeat the process until there is no reasonable reduction between two subsequent stages $\xi^{(s)}$, $\xi^{(s+1)}$.

Concerning the Matlab routines, we noticed the good performance of the *active-set* algorithm, that is called by `fmincon` when the preference `'Algorithm'` is put as `'active-set'` in the optimizer variable `option`. As for the post-processing,

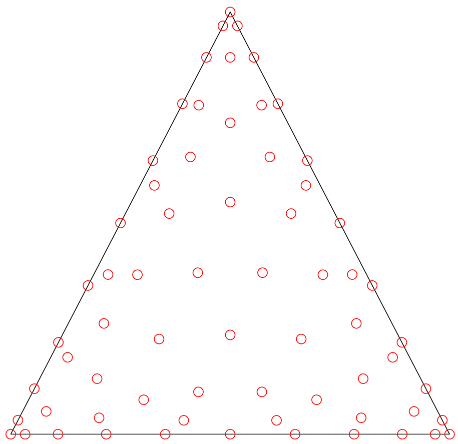

Figure 2: $N = 66$ (quasi-)Lebesgue points ($\circ$) for $n = 10$.

some improvements have been obtained performing additional stages with the Matlab built-in function `fminsearch`. Also in the case of symmetric points with assigned distribution on the sides, as one can expect, depending from the degree $n$ the CPU time ranges from some minutes to several hours. Some words are needed about the routines that we have used. The function `fmincon` relies on a method that solves a Quadratic Programming subproblem at each iteration. Furthermore, it updates an estimate of the Hessian of the Lagrangian at each iteration by the BFGS formula (see the `fmincon` function reference in [25]). The routine `fminsearch` uses the simplex search method of [24], a direct search algorithm that does not resort to numerical or analytic gradients (see the `fminsearch` function reference in [25]).

We have improved the numerical results by the optimization algorithm known as *Differential Evolution* that has the property to overcome local minima to reach lower values of the target function [29]. Its usage has been originally suggested in [28, p.56], using the factor $F$ equal to $1/2$ and the crossover parameter $C = 9/10$, performing 100 iterations. The Matlab software that we have used is a minor modification of the codes provided in [30], adapted to our instances. All the tests were run in Matlab 7.6.0, on a 2.13 GHz Intel Core 2 Duo with 4 GB of RAM.

At this point some issues must still be discussed. We must determine between all the $n_1 \in \{0, 1\}$, $n_3$, $n_6$ such that $m = (n + 1)(n + 2)/2 = n_1 + 3n_3 + 6n_6$ what is the choice *preferred* by the Lebesgue points. To this purpose we observe that there is numerical evidence that Lebesgue points have high absolute values of the Vandermonde matrix $\mathrm{Vand}_n(\xi)$ determinant (w.r.t. the orthonormal Dubiner basis). So we compute, via numerical optimization, for all the combinations of $n_1$, $n_3$, $n_6$ for which $m = n_1 + 3n_3 + 6n_6$, several sets $\xi$ having $n_1$ points in $O_1$, $3n_3$ points in $O_3$ and $6n_6$ points in $O_6$ with GLL distribution on the sides and that are providing high values of $|\det(\mathrm{Vand}_n(\xi))|$. The computation of the aforementioned determinants is less time consuming than the approximation of the Lebesgue constant and the convergence of the optimization algorithm rather fast. Up to degree $n = 18$, *high* values of $n_3$ were providing sets $\xi$ nearly singular, and in general only few were giving high $|\det(\mathrm{Vand}_n(\xi))|$, always with only one choice of $n_1$, $n_3$ and $n_6$ with much greater magnitude w.r.t. the remaining competitors.

We observe that in [28, p.14] the author asserts that if $n$ is the degree and $m = (n + 1)(n + 2)/2 = n_1 + 3n_3 + 6n_6$, there is numerical evidence that $n_1 + n_3 = \mathrm{ceil}((n+1)/2)$, and since $n_1$ is known for any $n$, so are $n_3$ and $n_6 = (m-n_1-3n_3)/6$. Our tests confirm this statement up to degree $n = 18$. Once the right $n_1$, $n_3$, $n_6$ and a set $\xi$ with particularly high $|\det(\mathrm{Vand}_n(\xi))|$ are at hand, we start the numerical process for computing the (quasi-)Lebesgue points.

In [10], we have computed some non-symmetric point-sets on the triangle $\mathcal{T}$, with low Lebesgue constants, i.e. LEB and LEBGL, the first one without any constraint and the latter with assigned GLL distribution on the sides.

In Table 1 we compare these new sets LEBGLS with LEB and LEBGL, the Taylor-Wingate-Vincent sets named TWV [33] that approximately maximize the Vandermonde determinant, and the Warburton sets shortened as WB [34], that for low degrees have good Lebesgue constant and are available as soon as a one variable optimization process is performed. We point out that the two latter sets are symmetric with Gauss-Legendre distribution on the sides. Furthermore, since not all the sets of TWV are available we follow the results obtained [28] about the Lebesgue constants, while the conditioning and the maximum absolute value of the cardinal functions $L_i$ are computed only for the sets provided in [33]. In [28], the author computes sets that we will cite as ROTH, improving the Lebesgue constant of [33] and still preserving side distribution and the sixfold symmetries. The sets are not available, but for completeness sake we report the results.

Since LEBGLS has more constraints, it is natural to expect that we obtain a worst Lebesgue constant w.r.t. LEB and LEBGL, nonetheless the results are still good, not too far from those of LEBGL for $n \leq 15$. Furthermore, it improves the Lebesgue constants of the previous known sixfold symmetric sets with assigned Gauss-Legendre-Lobatto distribution on the sides.

In [27] and [28] it has been considered the maximum of the Lagrange poly-

Table 1: Lebesgue constants $\Lambda_n$ of some sets in the triangle $\mathcal{T}$.

| deg | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| LEB | 1.00 | 1.49 | 1.97 | 2.42 | 2.90 | 3.39 | 3.94 | 4.55 | 5.28 |
| LEBGL | 1.00 | 1.67 | 2.11 | 2.59 | 3.08 | 3.59 | 4.14 | 5.21 | 5.51 |
| LEBGLS | 1.00 | 1.66 | 2.11 | 2.59 | 3.08 | 3.59 | 4.14 | 4.77 | 5.49 |
| ROTH | 1.00 | 1.66 | 2.11 | 2.72 | 3.61 | 4.17 | 4.92 | 5.90 | 6.80 |
| TWV | 1.00 | 1.66 | 2.11 | 2.72 | 3.61 | 4.17 | 4.92 | 5.90 | 6.80 |
| WB | 1.00 | 1.66 | 2.11 | 2.66 | 3.12 | 3.70 | 4.27 | 4.96 | 5.73 |

| deg | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
|---|---|---|---|---|---|---|---|---|---|
| LEB | 5.63 | 6.45 | 6.90 | 7.59 | 8.31 | 9.07 | 8.58 | 9.12 | 9.88 |
| LEBGL | 5.93 | 6.56 | 7.13 | 7.74 | 8.31 | 9.07 | 8.58 | 9.12 | 9.88 |
| LEBGLS | 6.29 | 7.00 | 7.26 | 8.58 | 8.83 | 8.91 | 10.66 | 11.41 | 12.69 |
| ROTH | 7.85 | 7.91 | 8.47 | 9.28 | 9.96 | 10.02 | 10.69 | 11.53 | 13.13 |
| TWV | 7.88 | 7.91 | 8.47 | 9.28 | 9.96 | 10.02 | 12.19 | 13.88 | 14.74 |
| WB | 6.67 | 7.90 | 9.36 | 11.46 | 13.97 | 17.64 | 22.22 | 28.76 | 36.76 |

Table 2: Conditioning of the Vandermonde matrix w.r.t. the orthonormal Koornwinder-Dubiner basis.

| deg | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| LEBGLS | 3.7 | 10 | 20 | 40 | 52 | 69 | 94 | 102 | 148 |
| TWV | – | – | 15 | – | – | 70 | – | – | 141 |
| WB | 3.7 | 10 | 14 | 39 | 53 | 69 | 76 | 119 | 143 |

| deg | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
|---|---|---|---|---|---|---|---|---|---|
| LEBGLS | 156 | 277 | 228 | 332 | 480 | 397 | 422 | 569 | 370 |
| TWV | – | – | 235 | – | – | 328 | – | – | 425 |
| WB | 172 | 190 | 209 | 272 | 246 | 349 | 553 | 796 | 1150 |

Table 3: Maximal values of the cardinal functions $L_i$ on the reference triangle $\mathcal{T}$ computed on a fine grid.

| deg | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| LEBGLS | 1.00 | 1.00 | 1.00 | 1.03 | 1.05 | 1.03 | 1.03 | 1.04 | 1.04 |
| TWV | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| WB | 1.00 | 1.00 | 1.00 | 1.00 | 1.01 | 1.01 | 1.01 | 1.01 | 1.01 |

| deg | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
|---|---|---|---|---|---|---|---|---|---|
| LEBGLS | 1.08 | 1.18 | 1.04 | 1.10 | 1.19 | 1.10 | 1.19 | 1.09 | 1.07 |
| TWV | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| WB | 1.01 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.07 | 1.28 | 1.81 |

Table 4: Orbits $\hat{n}_1$, $\hat{n}_3$, $\hat{n}_6$ of the Lebesgue set LEBGLS.

| deg | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| $\hat{n}_1$ | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 |
| $\hat{n}_3$ | 0 | 0 | 0 | 1 | 2 | 1 | 3 | 3 | 3 |
| $\hat{n}_6$ | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 2 | 3 |

| deg | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
|---|---|---|---|---|---|---|---|---|---|
| $\hat{n}_1$ | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 |
| $\hat{n}_3$ | 4 | 5 | 4 | 6 | 6 | 6 | 7 | 8 | 7 |
| $\hat{n}_6$ | 4 | 5 | 7 | 8 | 10 | 12 | 14 | 16 | 19 |

nomials as one of the measures of the influence of the nodes in the interpolation process. It is pointed out that these values must not be too large, better if close to 1. In Table 3 we show that all the sixfold symmetric sets LEBGLS, TWM and WB enjoy this property, though for $n > 17$ the values of WB tend to increase too much. Finally, in Table 4 we list the orbits $\hat{n}_1$, $\hat{n}_3$, $\hat{n}_6$ of the set LEBGLS, from which one can derive also the number of variables $\hat{n}_3 + 2\hat{n}_6$ involved in the optimization process.

## 3. An application to solve a PDE

To further compare the LEBGLS with the LEBGL, we have carried out two convergence tests for the triangle-based spectral element method ($T$SEM) defined in [27] applied to the equation $-\mathrm{div}(D\mathrm{grad}u) + \alpha u = f$ in $\Omega$, with mixed Dirichlet-Neumann boundary conditions on $\partial\Omega$, where $D$ is a suitable $2\times 2$ symmetric matrix and $\alpha \geq 0$ a given real. We recall that the rate of convergence of the $T$SEM solution $u_n$ to the real one $u_{exact}$ with respect to $n$ is virtually bounded only by $s$, the smoothness degree of the real solution $u_{exact}$. Thus for $u_{exact} \in H^s(\Omega)$, one can expect the optimal error estimate

$$||u_{exact} - u_n||_{L^2(\Omega)} = O(n^{-s}). \tag{4}$$

Here $u_{exact}$ is chosen to be analytical, we then expect to obtain the so-called spectral accuracy, *i.e.*, an exponentially decreasing error as a function of $n$.

### 3.1. Results on one triangle

We firstly consider the reference triangle $T = \{(x, y);\ x, y \geq -1,\ x + y \geq 0\}$ as computational domain $\Omega$. Concerning the equation data $(D, \alpha)$ we set

$$D = \begin{pmatrix} y^2 + \epsilon x^2 & -(1-\epsilon)xy \\ -(1-\epsilon)xy & x^2 + \epsilon y^2 \end{pmatrix}, \quad \alpha = 0,$$

with $\epsilon \in \{1, 10^{-1}, 10^{-2}, 10^{-4}\}$. The right-hand side $f$ of the equation and the Dirichlet boundary condition on $\partial\Omega$ are compatible with the analytical solution $u_{exact} = \sin(\pi\ x)\sin(\pi\ y)$. The discretization of the diffusive term with highly anisotropic diffusion can lead to numerical instabilities. In Table 5, we detail the $L^\infty$-norm of the approximation error $(u_n - u_{exact})$ computed by the $T$SEM with different sets of interpolation points in the triangle $T$ and by varying the value of $\epsilon$. We can remark that the method's spectral precision is maintained despite the ratio of anisotropy between the two $x$ and $y$ directions varies from 1 to $10^4$ and the number of degrees of freedom $ndof$ is only $(n+1)(n+2)/2$, with $n \in \{9, 12, 15, 18\}$.

Table 5: $L^\infty$-norm of the approximation error with some sets in the triangle $\mathcal{T}$.

| | deg | $\epsilon = 1$ | $\epsilon = 10^{-1}$ | $\epsilon = 10^{-2}$ | $\epsilon = 10^{-4}$ |
|---|---|---|---|---|---|
| TWV | 9 | 5.8551e-2 | 1.6146e-2 | 6.8236e-2 | 7.1981e-2 |
| | 12 | 6.1157e-4 | 7.2309e-5 | 1.9696e-4 | 2.6727e-4 |
| | 15 | 6.3740e-6 | 3.1264e-6 | 4.5819e-6 | 5.3509e-6 |
| | 18 | 6.2835e-8 | 2.0731e-8 | 5.5160e-8 | 1.9918e-7 |
| WB | 9 | 5.5463e-2 | 1.7371e-2 | 7.1970e-2 | 1.4273e-2 |
| | 12 | 3.2353e-4 | 1.4579e-4 | 1.4536e-4 | 1.9091e-4 |
| | 15 | 5.0802e-6 | 3.5157e-6 | 3.5585e-6 | 3.5743e-6 |
| | 18 | 1.8770e-7 | 6.6611e-8 | 6.8238e-8 | 1.7519e-8 |
| LEBGL | 9 | 8.9436e-2 | 1.1839e-2 | 1.0375e-1 | 1.5888e-1 |
| | 12 | 3.9209e-4 | 7.7523e-5 | 3.3503e-4 | 3.9538e-4 |
| | 15 | 4.4529e-6 | 6.1957e-6 | 8.1674e-6 | 1.2272e-5 |
| | 18 | 7.9921e-8 | 2.4293e-8 | 5.1246e-8 | 9.8565e-8 |
| LEBGLS | 9 | 5.2980e-2 | 1.9349e-2 | 7.8346e-2 | 1.5080e-1 |
| | 12 | 7.3061e-4 | 9.6412e-5 | 2.8924e-4 | 3.7571e-4 |
| | 15 | 5.1188e-6 | 3.3400e-6 | 4.6035e-6 | 5.2235e-6 |
| | 18 | 1.2909e-7 | 2.0160e-7 | 2.6482e-7 | 2.3948e-7 |

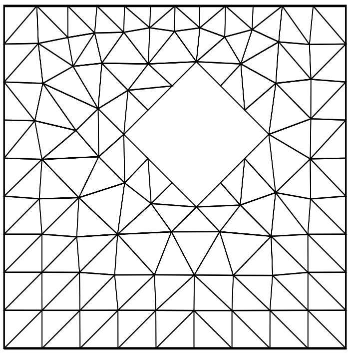

Figure 3: Unstructured simplicial mesh (created by Triangle, a free-charge 2D mesh generator) for the $T$SEM convergence tests. Despite the weak number of elements (163) of the mesh, the number of degree of freedom (ndof) may be important, namely 12042 with $N = 12$ and 26865 with $N = 18$.

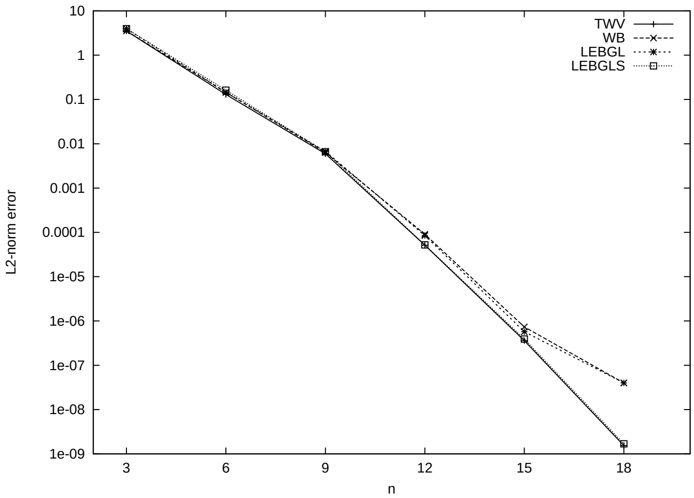


Figure 4: Semi-logarithmic plot of the $L^2$-error versus the polynomial order $n$ for different distributions of interpolation points in the mesh triangles.

### *3.2. Results on a mesh*

We now consider $\Omega = (-10, 10)^2 \setminus \mathcal{H}$, where $\mathcal{H}$ is a square hole, and $\Omega$ is discretized with the unstructured mesh presented in Fig. 3. For the problem equation, we set $D = I_2$ (the $2 \times 2$ identity matrix) and $\alpha = 1$. The analytical solution is chosen to be $u_{exact} = \sin(2x + y) \sin(x + 1) \sin(1 - y)$ and thus the source term $f$ and the values for the Dirichlet conditions on the outer boundary and Neumann conditions on the (interior) hole boundary are set accordingly.

By looking at Fig. 4 (Table 6) we can remark that the spectral accuracy is achieved in the $T$SEM with the considered nodal sets. Moreover, concerning the $L^2$-norm of the approximation error, we remark that the LEBGLS points show a better behavior than the LEBGL ones, and an equally good behavior as the TWV ones. We note that this ranking is in the opposite order of their Lebesgue constants.

## 4. Conclusions

We have presented an optimization algorithm to distribute, symmetrically in the triangle, nodes that minimize the Lebesgue constant. Indeed, the LEBGLS nodes were shown to have Lebesgue constants which are better than or comparable with all existing node sets up to at least interpolation degree 18. We showed test interpolation results that give further confidence on the quality of the generated nodes in the frame of high-order triangle-based spectral element methods. LEBGLS points

Table 6: $L^2$-norm of the approximation error for different nodal sets.

| deg | ndof | TWV | WB | LEBGL | LEBGLS |
|---|---|---|---|---|---|
| 3 | 810 | 3.5267e-0 | 3.5267e-0 | 3.9673e-0 | 3.9673e-0 |
| 6 | 3087 | 0.1297e-0 | 0.1457e-0 | 0.1447e-0 | 0.1621e-0 |
| 9 | 6831 | 5.9582e-3 | 6.3634e-3 | 6.7333e-3 | 6.5863e-3 |
| 12 | 12042 | 5.1437e-5 | 9.0467e-5 | 8.4629e-5 | 5.2218e-5 |
| 15 | 18720 | 3.6078e-7 | 7.3335e-7 | 5.7335e-7 | 3.9009e-7 |
| 18 | 26865 | 1.5364e-9 | 3.9966e-8 | 3.9897e-8 | 1.6872e-9 |

could be next used to constrain the construction (and the optimization process) of symmetric distributions in the tetrahedron. Other three-dimensional shapes could be analyzed (prisms and pyramids) in order to rely on so-called hybrid meshes for performing approximations of PDE solutions by the $T$SEM in three-dimensions. In [35], the reader can download Matlab files containing the LEBGLS points on the unit simplex of vertices $(0,0)$, $(1,0)$, $(0,1)$ and the other set of points that have been used to make comparative tests. For the WB points, generation Matlab files are given in [34].

## References


[1] C. Bernardi and Y. Maday, *Spectral element methods*, in Handbook of Numerical Analysis Tome 5, P. G. Ciarlet and J. L. Lions editors, North Holland, Amsterdam (1996), 209–486.

[2] L. Bos, *Bounding the Lebesgue function for Lagrange interpolation in a simplex*, J. Approx. Theory 38 (1983), 43–59.

[3] L. Bos, *On certain configurations of points in $R^n$ which are uniresolvent for polynomial interpolation*, J. Approx. Theory 64 (1991), 271–280.

[4] L. Bos, J.-P. Calvi, N. Levenberg, A. Sommariva and M. Vianello, *Geometric weakly admissible meshes, discrete least squares approximation and approximate Fekete Points*, Math. Comp. 80 (2011), 1601–1621.

[5] L. Bos, S. De Marchi, A. Sommariva and M. Vianello, *Computing multivariate Fekete and Leja points by numerical linear algebra*, SIAM J. Numer. Anal. 48 (2010), 1984–1999.

[6] L. Bos, S. De Marchi, A. Sommariva and M. Vianello, *Weakly Admissible Meshes and Discrete Extremal Sets*, Numer. Math. Theory Methods Appl. 4 (2011), 1–12.

[7] L. Bos, N. Levenberg and S. Waldron, *On the spacing of Fekete points for a sphere, ball or simplex*, Indag. Math. 19 (2008), 163–176.

[8] L. Bos, A. Sommariva and M. Vianello, *Least-squares polynomial approximation on weakly admissible meshes: disk and triangle*, J. Comput. Appl. Math. 48 (2010), 1984–1999.

[9] L. Bos, M.A. Taylor and B.A. Wingate, *Tensor product Gauss-Lobatto points are Fekete points for the cube*, Math. Comput. 70 (2001), 1543–1547.

[10] M. Briani, A. Sommariva and M. Vianello, *Computing Fekete and Lebesgue points: simplex, square, disk*, preprint, available at http://www.math.unipd.it/~marcov/CAApubl.html.

[11] M. Briani, A. Sommariva and M. Vianello, *LEBFEK: Matlab codes and data sets for bivariate Lebesgue and Fekete interpolation points,* available online at http://www.math.unipd.it/~marcov/CAAsoft.html.

[12] J. P. Boyd, *A numerical comparison of seven grids for polynomial interpolation on the interval*, Comput. Math. Appl. 38 (1999), 35–50.

[13] J.-P. Calvi and N. Levenberg, *Uniform approximation by discrete least squares polynomials*, J. Approx. Theory 152 (2008), 82–110.

[14] C. Canuto, M. Y. Hussaini, A. Quarteroni and T. A. Zang, *Spectral methods. Evolution to complex domains and applications to fluid dynamics*, Springer, N.Y. (2007).

[15] Q. Chen and Ivo Babuska, *Approximate optimal points for polynomial interpolation of real functions in an interval and in a triangle*, Comput. Methods Appl. Mech. Engrg. 128 (1995), 405–417.

[16] G. Choquet, *Cours d'Analyse. Tome II, Topologie*, Masson, Paris (1964).

[17] M. O. Deville, P. F. Fischer and E. H. Mund, *High-order methods for incompressible fluid flow*, Cambridge Univ. Press, Cambridge (2002).

[18] P. Solin, K. Segeth and I. Dolezel, *Higher-Order Finite Element methods*, Chapman & Hall/CRC Press, Boca Raton (2004).

[19] M. Dubiner, *Spectral methods on triangles and other domains*, J. Sci. Computing 6 (1991), 345–390.

[20] G. J. Gassner, F. Lörcher, C.-D. Munz and J. S. Hesthaven, *Polymorphic nodal elements and their application in discontinuous Galerkin methods*, J. Comput. Phys. 228 (2009), 1573–1590.

[21] W. Gautschi, *Orthogonal polynomials: computation and approximation*, Num. Math. and Scient. Comput., Oxford Univ. Press, Oxford (2004).

[22] W. Heinrichs, *Improved Lebesgue constants on the triangle*, J. Comput. Phys. 207 (2005), 625–638.

[23] G. E. Karniadakis and S. J. Sherwin, *Spectral/hp element methods for computational fluid dynamics*, Oxford Univ. Press, Oxford (1999).

[24] J.C. Lagarias, J.A. Reeds, M.H. Wright, and P.E. Wright, *Convergence properties of the Nelder-Mead simplex method in low dimensions*, SIAM J. of Optimization 9 (1998), 112–147.

[25] MathWorks, *Matlab Optimization Toolbox*, Version 4.0, online description at http://www.mathworks.com/products/optimization.

[26] R. Pasquetti and F. Rapetti, *Spectral element methods on triangles and quadrilaterals: comparisons and applications*, J. Comput. Phys. 198 (2004), 349–362.

[27] R. Pasquetti and F. Rapetti, *Spectral element methods on unstructured meshes: which interpolation points?*, Numer. Algorithms 55 (2010), 349–366.

[28] M.J.Roth, *Nodal configurations and Voronoi tessellations for triangular spectral elements*, Ph.D. thesis, University of Victoria (2005), available online at https://dspace.library.uvic.ca:8443/bitstream/handle/1828/44/Roth.pdf.

[29] R. Storn and K. Price, *Differential evolution - a simple and efficient adaptive scheme for global optimization over continuous spaces*, J. Global Optimization 11 (1997), 341–359.

[30] R. Storn, *Differential Evolution for Continuous Function Optimization, Matlab codes*, available online at http://www.icsi.berkeley.edu/~storn/code.html.

[31] A. Sommariva and M. Vianello, *Computing approximate Fekete points by QR factorizations of Vandermonde matrices*, Comput. Math. Appl. 57 (2009), 1324–1336.

[32] I. Babuska, B. A. Szabó and I. N. Katz, *The $p$-version of the finite element method*, SIAM J. Numer. Anal. 18 (1981), 515–545.

[33] M. A. Taylor, B. A. Wingate and R. E. Vincent, *An algorithm for computing Fekete points in the triangle*, SIAM J. Numer. Anal. 38 (2000), 1707–1720.

[34] T. Warburton, *An explicit construction for interpolation nodes on the simplex*, J. Eng. Math. 56 (2006), 247–262.

[35] http://www.math.unipd.it/~alvise/LEBGLS/lebgls.html.