Polynomial approximation on Lissajous curves in the d-cube

L. Bos[a,*], S. De Marchi[b], M. Vianello[c]

[a]*Department of Computer Science, University of Verona, Verona, Italy*
[b]*Department of Mathematics, University of Padova, Padova, Italy*
[c]*Department of Mathematics, University of Padova, Padova, Italy*

Abstract

For $\mathbf{a} \in \mathbb{Z}^d_{>0}$ we let $\ell_{\mathbf{a}}(t) := (\cos(a_1 t), \cos(a_2 t), \cdots, \cos(a_d t))$ denote an associated Lissajous curve. We study such Lissajous curves which have the quadrature property for the cube $[-1,1]^d$ that

$$\int_{[-1,1]^d} p(x)d\mu_d(x) = \frac{1}{\pi}\int_0^\pi p(\ell_{\mathbf{a}}(t))dt$$

for all polynomials $p(x) \in V$ where V is either the space of d-variate polynomials of degree at most m or else the d-fold tensor product of univarite polynomials of degree at most m. Here $d\mu_d$ is the product Chebyshev measure (also the pluripotential equilibrium measure for the cube). Among such Lissajous curves with this property we study the ones for which $\max_{p\in V} \deg(p(\ell_{\mathbf{a}}(t)))$ is as small as possible. In the tensor product case we show that this is uniquely minimized by $\mathbf{g} := (1, (m+1), (m+1)^2, \cdots, (m+1)^{d-1})$. In the case of $m = 2n$ we construct discrete hyperinterpolation formulas which are easily evaluated with, for example, the Chebfun system ([7]).

Keywords: Lissajous curves, Lissajous sampling, multivariate polynomial interpolation, hyperinterpolation
2000 MSC: 41A05, 41A10, 41A63, 65D05

*Corresponding author
Email addresses: leonardpeter.bos@univr.it (L. Bos), demarchi@math.unipd.it (S. De Marchi), marcov@math.unipd.it (M. Vianello)

1. Introduction

Given a compact set $K \subset \mathbb{R}^d$ two classical problems of numerical analysis are

(a) to find good quadrature sets for K, i.e., to find a *discrete* subset $X \subset K$ such that

$$\sum_{x \in X} w_x p(x) = \int_K p(x) d\mu(x)$$

holds for polynomials $p(x)$ of as high degree as possible. Here $d\mu(x)$ is some given measure of interest on K and the w_x are the so-called quadrature weights, and need to be determined along with X.

(b) to find good sets for polynomial interpolation, i.e., to find for a given degree n a discrete set $X_n \subset K$ (of a certain determined cardinality) such that for any vector of values $\mathbf{z} \in \mathbb{R}^{X_n}$ (real vectors indexed by the elements of X_n), there exists a unique polynomial p, of degree at most n, such that $p(x) = z_x$, $x \in X_n$, with X_n (nearly) satisfying some optimality property. Typically one asks to minimize the so-called Lebesgue constant, or else maximize the determinant of the associated Vandermonde matrix (the so-called Fekete points).

Both of these may be regarded as an attempt to replace the "complicated" continuum K by the much simpler discrete subset X_n, that acts as a proxy for K for a particular purpose. There is an intermediate possibility. Instead of directly seeking a 0-dimensional proxy set X_n, one may first seek a one-dimensional curve $(t, \gamma_n(t))$ that can act as a proxy for K for a given purpose. A particularly successful example of such an approach is the so-called Padua points for $[-1,1]^2$ ([1, 3, 5, 6]) where one first shows that for the Lissajous curve, $\gamma_n(t) := (\cos(nt), \cos((n+1)t))$, there is a good quadrature formula, i.e.,

$$\frac{1}{\pi^2} \int_{[-1,1]^2} p(x,y) \frac{dx}{\sqrt{1-x^2}} \frac{dy}{\sqrt{1-y^2}} = \frac{1}{\pi} \int_0^\pi p(\gamma_n(t)) dt$$

for at least all polynomials of degree at most $2n-1$. Then on this curve, one selects a special subset, equally spaced in the parameter t, which turns out to be provably good for polynomial interpolation.

Of special importance is the fact that in the square, the Lissajous curves $\gamma_n(t)$ have self-intersections. This in no longer the case for $d \geq 3$. However, one may still apply the same procedure and obtain a discrete set, with cardinality strictly greater than the dimension of the $d-$variate polynomials, on which one may construct a good *hyper*interpolation formula. This was done in the important special case of $d = 3$ in [2], where the reader may find a more detailed discussion of numerical procedures and possible applications.

In general, for $\mathbf{a} \in \mathbb{Z}^d_{>0}$ we let $\ell_{\mathbf{a}}(t) := (\cos(a_1 t), \cos(a_2 t), \cdots, \cos(a_d t))$ denote an associated Lissajous curve. In Theorem 1 we characterize such Lissajous curves which have the quadrature property for the cube $[-1,1]^d$ that

$$\int_{[-1,1]^d} p(x) d\mu_d(x) = \frac{1}{\pi} \int_0^\pi p(\ell_{\mathbf{a}}(t)) dt$$

for all polynomials $p(x)$ in certain spaces $V = \Pi^r_m$ (see (2) which include the special cases of V the space of d-variate polynomials of degree at most m (Π^1_m) and the d-fold tensor product of univarite polynomials of degree at most m (Π^∞_m). Here $d\mu_d$ is the product Chebyshev measure (also the pluripotential equilibrium measure for the cube), i.e., $d\mu_d(x) = \frac{1}{\pi^d} \prod_{j=1}^d \frac{1}{\sqrt{1-x_j^2}} dx_j$.

It then becomes important to select among these curves one which is optimal in the sense that $\max\limits_{p \in V} \deg(p(\ell_{\mathbf{a}}(t)))$ is as small as possible. In [2] we studied the case of $V = \Pi^1_m$, the space of d-variate polynomials of total degree at most m (and $d = 3$). We conjectured a formula for such optimal tuples, but in higher dimensions things seem to be rather more complicated. In this work, we consider the case of $V = \Pi^\infty_m$, the d-fold tensor product of univariate polynomials of degree at most m. It turns out that then, in any dimension d, there is a simple formula for the unique optimal tuple (Theorem 2).

For the case of $m = 2n$ one may easily construct hyperinterpolation formulas based on discrete sets, as explained in §3.

2. Quadrature Formulas on Lissajous Curves

For $\mathbf{x} \in \mathbb{R}^d$ we let $\mathbb{R}[\mathbf{x}]$ denote the d-variate polynomials with coefficients in $\mathbb{R}$. Writing $p \in \mathbb{R}[\mathbf{x}]$ as $p(\mathbf{x}) = \sum\limits_{\alpha \in \mathbb{Z}^d_{\geq 0}} a_\alpha x^\alpha$ we may follow Trefethen[8] and consider, for $1 \leq r \leq \infty$, the generalized degrees

$$\deg_r(p) := \max_{a_\alpha \neq 0} \|\alpha\|_r \tag{1}$$

where, for $1 \le r < \infty$, $\|\alpha\|_r := \left\{\sum_{j=1}^d |\alpha_j|^r\right\}^{1/r}$ and, for $r = \infty$, $\|\alpha\|_\infty := \max_{1\le j\le d} |\alpha_j|$.

We then have natural spaces of d-variate polynomials of "degree" at most m :

$$\Pi_m^r := \{p \in \mathbb{R}[\mathbf{x}] \,:\, \deg_r(p) \le m\}. \tag{2}$$

Note that for $r = 1$, Π_m^1 is the usual total-degree subspace of polynomials of degree m while for $r = \infty$, Π_m^∞ is the tensor product space of univariate polynomials of degree at most m.

Now, for $\mathbf{a} \in \mathbb{Z}_{>0}^d$ we let

$$\ell_{\mathbf{a}}(t) := (\cos(a_1 t), \cos(a_2 t), \cdots, \cos(a_d t)) \tag{3}$$

denote an associated Lissajous curve.

Definition 1. *Suppose that* $\mathbf{a} = (a_1, a_2, \cdots, a_d) \in \mathbb{Z}_{>0}^d$. *We say that* $\mathbf{a}$ *is* Π_m^r*-admissible if*

$$\nexists\ \mathbf{b} \in \mathbb{Z}^d,\ \mathbf{b} \ne 0,\ \|\mathbf{b}\|_r \le m$$

such that

$$\sum_{i=1}^d b_i a_i = 0.$$

□

We remark that $\mathbf{a}$ is Π_m^r-admissible iff there are no "small" (defined in a sense depending on Π_m^r) solutions $\mathbf{x} = \mathbf{b}$ of the homogeneous linear diophantine equation $\sum_{i=1}^d x_i a_i = 0$.

Theorem 1. *Suppose that* $\mathbf{a} \in \mathbb{Z}_{>0}^d$ *and that* $1 \le r \le \infty$. *We have, for* $\mathbf{x} = (x_1, \cdots, x_d) \in \mathbb{R}^d$,

$$\int_{[-1,1]^d} p(\mathbf{x}) d\mu_d(\mathbf{x}) = \frac{1}{\pi} \int_0^\pi p(\ell_{\mathbf{a}}(t)) dt \tag{4}$$

for all polynomials $p \in \Pi_m^r$ *if and only if* $\mathbf{a}$ *is* Π_m^r*-admissible.*

Proof. First suppose that $\mathbf{a}$ is Π_m^r-admissible. It suffices to prove (4) for a basis. Hence consider

$$p(x) = \prod_{j=1}^d T_{c_j}(x_j),\ c_j \in \mathbb{Z}_{\ge 0},\ \|\mathbf{c}\|_r \le m$$

where $T_k(x)$ denotes the classical Chebyshev polynomial of degree k. If $c_1 = c_2 = \cdots = c_d = 0$ then $p(x) = 1$ and the identity (4) is trivial as both measures are probability measures. Otherwise, suppose that not all the $c_j = 0$. Then the left hand side of (4) is 0 and we must prove that the right hand side is also 0, i.e., that

$$\frac{1}{\pi}\int_0^\pi \prod_{i=1}^d T_{c_j}(\cos(a_j\theta))d\theta = \frac{1}{\pi}\int_0^\pi \prod_{i=1}^d \cos(a_jc_j\theta)d\theta = 0.$$

But using the identity

$$\cos(A)\cos(B) = \frac{\cos(A+B)+\cos(A-B)}{2}$$

we may easily verify by induction on d that

$$\prod_{j=1}^d \cos(a_jc_j\theta) = 2^{-d}\sum_{\boldsymbol{\epsilon}\in\{-1,1\}^d} \cos\left(\left(\sum_{j=1}^d \epsilon_ja_jc_j\right)\theta\right).$$

For an $\boldsymbol{\epsilon} \in \{-1,+1\}^d$, setting $b_j := \epsilon_jc_j$ we have

$$\|\mathbf{b}\|_r = \|\mathbf{c}\|_r \le m$$

and so, by the assumption of admissibility, the sum $\sum_{j=1}^d \epsilon_ja_jc_j = \sum_{j=1}^d b_ja_j \neq 0$. Consequently, each of the terms

$$\int_0^\pi \cos\left(\left(\sum_{j=1}^d \epsilon_ja_jc_j\right)\theta\right)d\theta = 0.$$

Conversely, suppose that (4) holds for all polynomials $p \in \Pi_m^r$. Suppose also, for the sake of contradiction, that there exists $\mathbf{b} \in \mathbb{Z}^d$ with $\mathbf{b} \neq 0$ and $\|\mathbf{b}\|_r \le m$ such that

$$\sum_{j=1}^d a_jb_j = 0.$$

Let $c_j = |b_j|$, $1 \le j \le d$ and consider

$$p(x) := \prod_{j=1}^d T_{c_j}(x_j).$$

Then $\deg_r(p) = \|\mathbf{c}\|_r = \|\mathbf{b}\|_r \le m$, i.e., $p \in \Pi_m^r$. We claim that

$$\int_{[-1,1]^d} p(\mathbf{x}) d\mu_d(\mathbf{x}) = 0$$

while

$$\frac{1}{\pi}\int_0^\pi p(\ell_{\mathbf{a}}(\theta))d\theta \neq 0,$$

a contradiction. Indeed, $\displaystyle\int_{[-1,1]^d} p(\mathbf{x}) d\mu_d(\mathbf{x}) = 0$ as at least one of the $c_j \neq 0$, and

$$\begin{aligned}\int_0^\pi p(\ell_{\mathbf{a}}(\theta))d\theta &= \int_0^\pi \prod_{j=1}^d \cos(a_j c_j \theta) d\theta \\ &= 2^{-d}\int_0^\pi \sum_{\boldsymbol{\epsilon}\in\{-1,+1\}^d} \cos\left(\left(\sum_{j=1}^d \epsilon_j a_j c_j\right)\theta\right) d\theta.\end{aligned}$$

But

$$\int_0^\pi \cos(r\theta)d\theta = \begin{cases} 0 & \text{if } r \neq 0 \\ \pi & \text{if } r = 0\end{cases}$$

and hence

$$\begin{aligned}\int_0^\pi p(\ell_{\mathbf{a}}(\theta))d\theta = 0 &\iff \int_0^\pi \sum_{\epsilon\in\{-1,+1\}^d} \cos\left(\left(\sum_{j=1}^d \epsilon_j a_j c_j\right)\theta\right) d\theta = 0 \\ &\iff \int_0^\pi \cos\left(\left(\sum_{j=1}^d \epsilon_j a_j c_j\right)\theta\right) d\theta = 0\end{aligned}$$

for all sign choices $\boldsymbol{\epsilon} \in \{-1,+1\}^d$. But for the choice of $\epsilon_j = \text{sign}(b_j)$, $\epsilon_j c_j = b_j$ and hence, for this choice of signs, $\sum_{j=1}^d \epsilon_j a_j c_j = \sum_{j=1}^d a_j b_j = 0$ and

$$\int_0^\pi \cos\left(\left(\sum_{j=1}^d \epsilon_j a_j c_j\right)\theta\right) d\theta = \pi \neq 0.$$

$\square$

If we restrict a polynomial $p \in \Pi_m^r$ to the curve $\ell_{\mathbf{a}}(t)$ the resulting *univariate* (trigonometric) polynomial $q(t) := p(\ell_{\mathbf{a}}(t))$ has complexity bounded by its degree. Hence, it is of some interest to determine that Π_m^r-admissible tuple $\mathbf{a}\in\mathbb{Z}_{>0}^d$ for which the degree of $q(t)$ is as small as possible.

Now, as usual, for $1\le r\le\infty$ we let r' denote the dual index, i.e., that $1\le r'\le\infty$ so that

$$\frac{1}{r}+\frac{1}{r'}=1.$$

It is easy to see that, for $p\in\Pi_m^r$,

$$\deg(p(\ell_{\mathbf{a}}(t)))\le m\|\mathbf{a}\|_{r'}.$$

Hence it is natural to try to find,

$$\min_{\Pi_m^r-\text{admissible }\mathbf{a}\in\mathbb{Z}_{>0}^d}\|\mathbf{a}\|_{r'}. \tag{5}$$

In the total degree case, $r=1$, $r'=\infty$ and hence, in this case we seek

$$\min_{\Pi_m^1-\text{admissible }\mathbf{a}\in\mathbb{Z}_{>0}^d}\ \max_{1\le j\le d} a_j. \tag{6}$$

This is the problem considered in [2], primarily for the case $d=3$. In particular it is shown there [2, Lemma 2]) that the quantity (6) has growth at least of $O(m^{d-1})$. We give here a brief summary of what is known for the $d=3$ and $d=4$ cases.

First, for $d=3$ a computer search reveals the following optimal tuples listed in Table 1.

One notices immediately that there is not, in general, a unique optimal triple. In particular, for $m=7$, there are six of them. However, from this table, and more extensive calculations we conjecture (cf. [2, Conjecure 1]) that the following formulas provide *an* optimal tuple.

Conjecture 1. *For $m\equiv 0(4)$ let*

$$a_1=\frac{3m^2+4m}{16},\ a_2=\frac{3m^2+8m}{16},\ a_3=\frac{3m^2+12m+16}{16}.$$

For $m\equiv 1(4)$ let

$$a_1=\frac{3m^2+6m+7}{16},\ a_2=\frac{3m^2+10m+19}{16},\ a_3=\frac{3m^2+14m+15}{16}.$$

Table 1: Some Optimal Triples for $d = 3$

m			
2	1	2	3
3	1	3	5
	3	4	5
4	4	5	7
	4	6	7
5	7	8	10
	7	9	10
6	7	11	12
7	7	15	17
	9	11	17
	9	15	17
	10	16	17
	13	14	17
	13	16	17
8	14	16	19
	14	17	19
9	19	21	24
	19	22	24
10	19	26	27
11	24	33	34
12	30	33	37
	30	34	37
15	41	47	57
	49	52	57
	49	54	57
31	177	191	209
	177	195	209
	184	208	209
	193	200	209
	193	202	209

For $m \equiv 2(4)$ *let*

$$a_1 = \frac{3m^2+4}{16}, \ a_2 = \frac{3m^2+12m-4}{16}, \ a_3 = \frac{3m^2+12m+12}{16}.$$

For $m \equiv 3(4)$ let

$$a_1 = \begin{cases} \frac{3m^2+2m-1}{16} & m \equiv 3\,(8) \\ \frac{3m^2+6m+19}{16} & m \equiv 7\,(8), \end{cases} \quad a_2 = \begin{cases} \frac{3m^2+14m+11}{16} & m \equiv 3\,(8) \\ \frac{3m^2+10m+7}{16} & m \equiv 7\,(8), \end{cases} \quad a_3 = \frac{3m^2+14m+27}{16}.$$

We note that, as shown in [2, Appendix], the triple (a_1, a_2, a_3) is indeed Π_m^1-admissible. But moreover, we conjecture that the triple (a_1, a_2, a_3) is optimal in the sense that

$$a_3 = \max\{a_1, a_2, a_3\} = \min_{\Pi_m^1-\text{admissible } \mathbf{b}} \max_{1\le i\le d} b_i.$$

The case of $d = 4$ seems already to be more complicated. Table 2 gives some optimal 4-tuples found by exhaustive search. As the largest component of the tuple grows like $O(m^3)$ an exhaustive search quickly becomes very expensive. From the small number of searches that we were reasonably able to perform, it is difficult to discern a pattern. At this point we are not even able to say if there is an optimal tuple with entries that are polynomials in m, as seems to be the case for $d = 3$.

In the tensor-product case, $r = \infty$, $r' = 1$ and then for $p \in \Pi_m^\infty$,

$$\deg(p(\ell_{\mathbf{a}}(t)) \le m\|\mathbf{a}\|_1 = m\left(\sum_{i=1}^d a_i\right)$$

so that the minimization problem (5) becomes

$$\min_{\Pi_m^\infty-\text{admisible } \mathbf{a}\in\mathbb{Z}_{>0}^d} \sum_{i=1}^d a_i.$$

In contrast to the total degree case, it turns out this minimization problem has a simple *unique* solution.

Theorem 2. *Suppose that $r = \infty$. The tuple*

$$\mathbf{g} = (1, (m+1), (m+1)^2, \cdots, (m+1)^{d-1}) \text{ is } \Pi_m^\infty-\text{admissible}$$

and is the unique minimizer (up to permutation) of

$$\min_{\Pi_m^\infty-\text{admisible } \mathbf{a}\in\mathbb{Z}_{>0}^d} \sum_{i=1}^d a_i.$$

Table 2: Some Optimal Tuples for $d = 4$

m				
2	1	2	3	4
3	1	3	5	7
	4	5	6	7
4	5	9	11	12
5	5	13	17	19
6	11	24	27	28
	15	24	27	28
7	9	31	37	39
8	34	50	54	55
9	59	61	71	74
	59	62	72	74
10	59	90	95	96
	65	90	91	96
	53	89	90	96
11	77	89	119	121
	53	109	119	121
12	105	138	150	152
13	159	167	187	188
14	177	215	229	230
15	193	219	267	273
	199	215	271	273

Proof. Note that in this case $\|\mathbf{b}\|_r = \max_{1\le i\le d}|b_i|$ and hence we may write explicitly that $\mathbf{a} \in \mathbb{Z}^d_{>0}$ is Π^∞_m-admissible iff

$$\nexists\ \mathbf{b} \in \mathbb{Z}^d,\ \ \mathbf{b} \neq 0,\ \max_{1\le i\le d}|b_i| \le m$$

such that

$$\sum_{i=1}^{d} b_i a_i = 0.$$

We first show that $\mathbf{g}$ is Π^∞_m-admissible. Indeed, if this were *not* the case there would exist $\mathbf{b} \in \mathbb{Z}^d$, $\mathbf{b} \neq 0$, $\max_{1\le i\le d} |b_i| \le m$, such that

$$\sum_{i=1}^{d} b_i (m+1)^{i-1} = 0.$$

Separating the positive and negative b_i we would have

$$-\sum_{b_i\le 0} |b_i|(m+1)^{i-1} + \sum_{b_i>0} |b_i|(m+1)^{i-1} = 0.$$

i.e.,

$$\sum_{b_i<0} |b_i|(m+1)^{i-1} = \sum_{b_i>0} |b_i|(m+1)^{i-1}$$

with $|b_i| \le m$, integers, not all zero. This contradicts the uniqueness of the representation of integers with respect to the base $b = m+1$.

Next, we prove that $\mathbf{g}$ is optimal, i.e., that if $\mathbf{a} \in \mathbb{Z}^d_{>0}$ with

$$\sum_{i=1}^{d} a_i < \sum_{i=1}^{d} g_i = \sum_{i=1}^{d} (m+1)^{i-1} = \frac{(m+1)^d - 1}{m} =: N_m$$

then $\mathbf{a}$ is *not* Π_m^∞-admissible, i.e., $\exists\ \mathbf{b} \in \mathbb{Z}^d$, $\mathbf{b} \neq 0$, with $\max\limits_{1\le i\le m} |b_i| \le m$ such that

$$\sum_{i=1}^{d} b_i a_i = 0.$$

This is essentially a special case of the much more general Siegel's Lemma [9] on the number of small solutions of linear diophantine equations. We extract the part of its proof that suffices for our purposes.

First note that there are $(m+1)^d - 1$ integer tuples $0 \neq \mathbf{y} \in \mathbb{Z}^d_{\ge 0}$ such that $0 \le y_i \le m$, $1 \le i \le d$. Let S_m be the set of such tuples and consider the function

$$F(\mathbf{y}) := \sum_{i=1}^{d} y_i a_i.$$

Then $F : S_m \to [\min_{1\le i\le d} a_i, m\sum_{i=1}^{d} a_i]$ so that (as the $a_i \ge 1$ by assumption)

$$\#(F(S_m)) \le m \sum_{i=1}^{d} a_i < mN_m,$$

by hypothesis.

But

$$mN_m = (m+1)^d - 1 = \#(S_m),$$

i.e.,

$$\#((F(S_m)) < \#(S_m)$$

and by the Pigeon Hole Principle, there must exist $\mathbf{y}^{(1)} \neq \mathbf{y}^{(2)} \in S_m$ such that $F(\mathbf{y}^{(1)}) = F(\mathbf{y}^{(2)})$. Then

$$\mathbf{b} := \mathbf{y}^{(1)} - \mathbf{y}^{(2)}$$

has the stated properties.

Finally, we prove uniqueness (up to permutation), i.e., if $\mathbf{a} \in \mathbb{Z}^d_{>0}$ is Π^∞_m-admissible, $a_1 \leq a_2 \leq \cdots \leq a_d$, and

$$\sum_{j=1}^d a_j = \sum_{j=1}^d g_j = \sum_{j=1}^d (m+1)^{j-1} = \frac{(m+1)^d - 1}{m},$$

then $\mathbf{a} = \mathbf{g}$.

Indeed, we will show by induction on j that $a_j = g_j = (m+1)^{j-1}$, $j = 1, 2, \cdots, d$.

First note, however, that if for some $k < d$,

$$\sum_{i=1}^k a_i < \sum_{i=1}^k (m+1)^{i-1}$$

then by the optimality of $\mathbf{g}$ in dimension k there exist $b_1, \cdots, b_k \in \mathbb{Z}$, $|b_i| \leq m$, not all zero, such that

$$\sum_{i=1}^k b_i a_i = 0.$$

By taking $b_{k+1} = b_{k+2} = \cdots = b_d = 0$ we arrive at a $\mathbf{b} \in \mathbb{Z}^d$ (with the stated restrictions) such that $\sum_{i=1}^d b_i a_i = 0$. Hence such an $\mathbf{a}$ cannot be Π^∞_m-admissible.

We may therefore assume that for $k = 1, 2, \cdots, d$,

$$\sum_{i=1}^k a_i \geq \sum_{i=1}^k (m+1)^{i-1}.$$

We now proceed with the induction on j. Firstly, $a_1 = 1 = g_1$, for otherwise

$$2 \le a_1 \le a_2 \le \cdots \le a_d$$

and $1 \notin F(S_m)$. Consequently

$$F \,:\, S_m \to [2, mN_m]$$

and $\#(F(S_m)) \le mN_m - 1 < \#(S_m)$ and so, again by the Pigeon Hole Principle, we would have a $0 \ne \mathbf{b} \in \mathbb{Z}^d$, $\max_{1\le i\le d} |b_i| \le m$ such that

$$\sum_{i=1}^{d} b_i a_i = 0.$$

Suppose therefore that $a_i = g_i$, $1 \le i \le j$. We show that then $a_{j+1} = g_{j+1} = (m+1)^j$. Indeed, then

$$\begin{aligned} \sum_{i=1}^{j+1} a_i &\ge \sum_{i=1}^{j+1} g_i \\ \implies\ a_{j+1} + \sum_{i=1}^{j} g_j &\ge g_{j+1} + \sum_{i=1}^{j} g_i \\ \implies\ a_{j+1} &\ge g_{j+1} = (m+1)^j. \end{aligned}$$

We need to show that, in fact, $a_{j+1} = (m+1)^j$. Indeed, if $a_{j+1} > (m+1)^j$ then we claim that $(m+1)^j \notin F(S_m)$. To see this, note that if for some i, $j+1 \le i \le d$, $y_i \ge 1$ then for $\mathbf{y} \in S_m$,

$$F(\mathbf{y}) = \sum_{i=1}^{d} y_i a_i \ge \sum_{i=j+1}^{d} y_i a_i \ge a_{j+1} > (m+1)^j$$

while if $y_i = 0$, $j+1 \le i \le d$, then

$$F(\mathbf{y}) = \sum_{i=1}^{j} y_i a_i \le \sum_{i=1}^{j} m(m+1)^{j-1} = (m+1)^j - 1 < (m+1)^j.$$

It follows that $\#(F(S_m)) < \#(S_m)$, and, just as in the proof of the optimality of $\mathbf{g}$, that $\mathbf{a}$ is *not* Π_m^∞-admissbile. Since we have assumed that $\mathbf{a}$ *is* admissbile, this is not possible, i.e., $a_{j+1} = g_{j+1}$, as claimed. $\square$

3. Hyperinterpolation

In this section we briefly outline a disctretization of Lissajous curves that allows one to construct a so-called Hyperinterpolation operator, based on a discrete set of points. Some practical applications of such a projection are discussed in [2].

Consider again $1 \le r \le \infty$. If we take $m = 2n$ then for Π_m^r-admissible $\mathbf{a} \in \mathbb{Z}_{>0}^d$ we have, by Theorem 1, the integration formula (4). Equivalently, for any $p, q \in \Pi_n^r$,

$$\int_{[-1,1]^d} p(x)q(x)d\mu_d(x) = \frac{1}{\pi}\int_0^\pi p(\ell_{\mathbf{a}}(t))q(\ell_{\mathbf{a}}(t))dt. \tag{7}$$

For $f \in C([-1,1]^d)$, its best least squares approximation in the space $L_2([-1,1]^d; d\mu_d)$ by polynomials $p \in \Pi_n^r$ is given by

$$\pi_n(f) = \sum_{\|\alpha\|_r \le n} \langle f, \hat{T}_\alpha \rangle \hat{T}_\alpha.$$

Here, $\langle f, g\rangle$ denotes the inner product

$$\langle f, g\rangle := \int_{[-1,1]^d} f(x)g(x)d\mu_d(x)$$

and

$$\hat{T}_\alpha(x) := c_\alpha \prod_{j=1}^d T_{\alpha_j}(x_j)$$

where (again) $T_{\alpha_j}(x_j)$ is the classical Chebyshev polynomial of degree α_j in the variable x_j and the constant c_α is chosen so that the norm (squared) $\langle \hat{T}_\alpha, \hat{T}_\alpha\rangle = 1$. We note that then the polynomials $\{\hat{T}_\alpha\}_{\|\alpha\|_r \le n}$ form an orthonormal basis for the space Π_n^r.

Now define

$$\pi_n^{\mathbf{a}} := \sum_{\|\alpha\|_r \le n} \langle f, \hat{T}_\alpha \rangle_{\mathbf{a}} \hat{T}_\alpha$$

where

$$\langle f, g\rangle_{\mathbf{a}} := \frac{1}{\pi}\int_0^\pi f(\ell_{\mathbf{a}}(t))g(\ell_{\mathbf{a}}(t))dt.$$

By the quadrature formula (7), we have

$$\pi_n^{\mathbf{a}}(p) = \pi_n(p) = p, \quad \forall p \in \Pi_n^r,$$

i.e, $\pi_n^{\mathbf{a}}$ is a projection onto Π_n^r.

In words, $\pi_n^{\mathbf{a}}$ is the least squares projection on the curve $\ell_{\mathbf{a}}(t)$ approximating (well) π_n, the least squares projection in the space $L_2([-1,1]^d; d\mu_d)$.

It turns out that we may easily discretize the operator $\pi_n^{\mathbf{a}}$. In fact, we have the classical quadrature formula

$$\frac{1}{\pi}\int_0^\pi t(\theta)d\theta = \frac{1}{N}\left\{\frac{1}{2}t(\theta_0) + \sum_{k=1}^{N-1} t(\theta_k) + \frac{1}{2}t(\theta_N)\right\}$$

for (at least) all *even* trigonometric polynomials of degree at most $2N-1$. Here $\theta_k := k\pi/N$, $0 \le k \le N$, are the equally spaced angles. But for $p \in \Pi_{2n}^r$, and Π_{2n}^r-admissible $\mathbf{a} \in \mathbb{Z}_{>0}^d$, $p(\ell_{\mathbf{a}})$ is an even trigonometric polynomial of degree at most $2n\|a\|_{r'}$. Hence, taking

$$N \ge 1 + n\|\mathbf{a}\|_{r'}$$

and letting

$$x_k := \ell_{\mathbf{a}}(\theta_k), \quad w_0 := \frac{1}{2N},\ w_k := \frac{1}{N},\ 1 \le k \le N-1,\ w_N = \frac{1}{2N},$$

we have

$$\langle p, q\rangle_{\mathbf{a}} = \sum_{k=0}^{N} w_k p(x_k) q(x_k) \tag{8}$$

for all $p, q \in \Pi_n^r$.

Computing $\pi_n^{\mathbf{a}}$ by means of (8) gives a projection based on discrete points and is known as a *hyperinterpolation operator*, whose uniform norm for the Chebyshev measure in the total degree case ($r = 1$) on the d-cube is $O(\log^d(n))$, as proved in [10].

4. Conclusions and Future Work

We have given a formula for optimal tuples in the tensor-product case. The total degree case remains substantially open. Conjecture 1 suggests optimal tuples for dimension $d = 3$. It would be very interesting to also determine formulas for optimal total degree tuples in higher dimensions (cf. Table 2). Trefethen[8] has noted that other values of r, in particular $r = 2$, are of considerable interest. We do not know of any results for optimal tuples, even in this case.

A general open question is how to discretize, in a similar manner, sets other than the cube. The real ball would already be an interesting and important problem.

[1] L. Bos, M. Caliari, S. De Marchi, M. Vianello and Y. Xu, Bivariate Lagrange interpolation at the Padua points: the generating curve approach, J. Approx. Theory 143 (2006), 15–25.

[2] L. Bos, S. De Marchi and M. Vianello, Trivariate Polynomial Approximation on Lissajous Curves, IMA J. of Numer. Anal., online (2016), doi: 10.1093/imanum/drw013.

[3] L. Bos, S. De Marchi, M. Vianello and Y. Xu, Bivariate Lagrange interpolation at the Padua points: the ideal approach, Numer. Math. 108 (2007), 43–57.

[4] L. Bos, S. De Marchi, A. Sommariva and M. Vianello, Computing multivariate Fekete and Leja points by numerical linear algebra, SIAM J. Numer. Anal. 48 (2010), 1984–1999.

[5] M. Caliari, S. De Marchi, A. Sommariva and M. Vianello, Padua2DM: fast interpolation and cubature at the Padua points in Matlab/Octave, Numer. Algorithms 56 (2011), 45–60.

[6] M. Caliari, S. De Marchi and M. Vianello, Bivariate polynomial interpolation on the square at new nodal sets, Appl. Math. Comput. 165/2 (2005), 261–274 .

[7] T.A. Driscoll, N. Hale, and L.N. Trefethen, editors, Chebfun Guide, Pafnuty Publications, Oxford, 2014.

[8] N. Trefethen, Multivariate polynomial approximation in the hypercube, to appear in Proc. Amer. Math. Soc..

[9] P. Vojta, Applications of arithmetic algebraic geometry to Diophantine approximations, Arithmetic algebraic geometry (Trento, 1991), 164–208, Lecture Notes in Math., 1553, Springer, Berlin, 1993.

[10] H. Wang, K. Wang and X. Wang, On the norm of the hyperinterpolation operator on the d-dimensional cube, Comput. Math. Appl. 68 (2014), 632–638.