Low cardinality admissible meshes on quadrangles, triangles and disks *

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Abstract

Using classical univariate polynomial inequalities (Ehlich and Zeller, 1964), we show that there exist admissible meshes with $\mathcal{O}(n^2)$ points for total degree bivariate polynomials of degree n on convex quadrangles, triangles and disks. Higher-dimensional extensions are also briefly discussed.

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1 Introduction.

A new insight in the theoretical and computational framework of multivariate polynomial approximation, has been recently given by the theory of "admissible meshes" of Calvi and Levenberg [8]. These are sequences of finite discrete subsets $\mathcal{A}_n$ of a d-dimensional real (or complex) compact set K, where the polynomial inequality

$$\|p\|_K \le C\|p\|_{\mathcal{A}_n} \;, \quad \forall p \in \mathbb{P}_n^d(K) \tag{1}$$

holds for some $C > 0$ (with $\mathrm{card}(\mathcal{A}_n)$ that grows at most polynomially in n). Here and below, $\|f\|_X = \sup_{x\in X} |f(x)|$ for f bounded on X, and $\mathbb{P}_n^d(K)$ denotes the space of d-variate polynomials of total degree at most n, restricted to K. We note that in the literature discrete sets with this property are

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also known as "norming sets" and also "Marcinkiewicz-Zygmund arrays" (especially for the sphere, cf. [12, 14]). Such meshes are nearly optimal for least-squares approximation, and contain interpolation sets (discrete extremal sets) that distribute asymptotically as Fekete points of the domain and can be computed by basic numerical linear algebra [3, 4, 6, 16].

In principle, as is shown in [8, Thm.5], it is always possible to construct an admissible mesh with $\mathcal{O}(n^{rd})$ points on any real compact set satysfying a Markov polynomial inequality with exponent r. The mesh is obtained simply by intersecting the domain with a uniform grid having $\mathcal{O}(n^{-r})$ spacing. However, the Markov exponent exponent is typically $r = 2$ on real compacts, for example on convex compacts [17]. This means that in dimension 2 we can easily construct admissible meshes with $\mathcal{O}(n^4)$ points, but these become computationally intractable already for moderate values of n.

In order to circumvent this difficulty a weaker structure has been exploited, called a "weakly admissible mesh" (again introduced in [8]), i.e., a mesh where $C = C_n$ depends on n in the inequality above, but grows at most algebraically with n; cf. [3, 4, 7]. In such a way we can obtain for example meshes with approximately n^2 points and $C_n = \mathcal{O}(\log^2 n)$ for standard bidimensional compacts like the square, the disk, the triangle. We refer the reader to [5] for a survey on the properties of (weakly) admissible meshes and their discrete extremal sets.

A basic question, however, remains open: do admissible meshes with cardinality $\mathcal{O}(n^d)$ exist? A deep result by Coppersmith and Rivlin [9] shows that already in dimension 1 (the interval) we should look for nonuniformly distributed points, since admissible meshes with equally spaced points require a $\mathcal{O}(n^{-2})$ spacing. Indeed, classical inequalities for univariate polynomials by Ehlich and Zeller [10] show that admissible meshes with $\mathcal{O}(n)$ cardinality, based on Chebyshev points, exist in the interval.

In this note, using such inequalities and suitable geometric transformations, we construct admissible meshes with $\mathcal{O}(n^2)$ cardinality on convex quadrangles (with triangles as a degenerate case) and disks. Some higher-dimensional extensions are also sketched.

2 Univariate admissible meshes with $\mathcal{O}(n)$ points.

First, we recall a classical inequality for univariate trigonometric polynomials, from which an inequality for univariate algebraic polynomials follows; cf. [10].

Theorem 1 *Let t be a trigonometric polynomial of degree at most n. Then for every $m > n$ and for every $\alpha \in \mathbb{R}$ the following inequality holds:*

$$\|t\|_{[0,2\pi]} \leq \frac{1}{\cos(n\pi/2m)} \|t\|_{\Theta_m} \qquad (2)$$

where $\Theta_m = \Theta_m(\alpha) = \{\alpha + \pi k/m\,,\ k = 0, 1, \ldots, 2m-1\}$ *are the* $2m$ *equally spaced angles in* $[\alpha, \alpha + 2\pi)$.

For the reader's convenience we give the complete proof of this Theorem which is written in German in the original paper.

Proof of Theorem 1. Set $\theta_k = \alpha + \pi k/m\,,\ k = 0, 1, \ldots, 2m-1$. We prove that if $|t(\theta_k)| \leq 1$ for all k then $|t(\theta)| \leq 1/\cos(\pi n/2m)$ for all $\theta \in [0, 2\pi]$. First, observe that if it's true for a particular α then it's true for all α by a translation of θ. Take $\alpha = -\pi/2m$.

Let $K = \max_{0 \leq \theta \leq 2\pi} |t(\theta)|$. Without loss of generality we may assume that $K = t(\eta)$, $0 \leq \eta \leq \pi/2m$. Consider the auxiliary trigonometric polynomial $f(\theta) = K\cos(n(\theta - \eta)) - t(\theta)$. We have that $f(\eta) = f'(\eta) = 0$, and $f''(\eta) = -Kn^2 - t''(\eta) \leq 0$ by the Bernstein inequality for trigonometric polynomials [2, Thm. 5.1.4], which asserts that $|t''(\theta)| \leq Kn^2$. Now, $t''(\eta) \leq 0$, and $t''(\eta) = -Kn^2$ if and only if $t(\theta) = K\cos(n(\theta - \eta))$, i.e., $f \equiv 0$. When $t''(\eta) > -Kn^2$ we have that $f''(\eta) < 0$ and thus η is a strict point of maximum of f. By Riesz's Lemma (cf. [11]), which says that if η is a point of maximum of a (nonzero) trigonometric polynomial f of degree not greater than n, then $f(\theta)$ cannot vanish for $|\theta - \eta| \leq \pi/2n$, we get that $f(\theta) < 0$ for $|\theta - \eta| \leq \pi/2n$.

These considerations show that in any case $f(\theta) \leq 0$, and thus $t(\theta) \geq K\cos(n(\theta - \eta))$, for $|\theta - \eta| \leq \pi/2n$. Take now $\theta = \theta_1 = \alpha + \pi/m = -\pi/2m + \pi/m = \pi/2m$. Then, $1 \geq t(\theta_1) \geq K\cos(n(\pi/2m - \eta))$, from which it follows that $K \leq 1/\cos(\pi n/2m)$. $\square$

From this Theorem, setting $x = \cos\theta$, $0 \leq \theta \leq \pi$, and $\alpha = \pi/2m$, we get immediately

Theorem 2 *Let* p *be any univariate polynomial of degree not greater than* n. *Then for every* $m > n$ *the following inequality holds*

$$\|p\|_{[-1,1]} \leq \frac{1}{\cos(n\pi/2m)}\,\|p\|_{X_m} \tag{3}$$

where $X_m = \{\cos((2j-1)\pi/2m)\,,\ j = 1, \ldots, m\}$ *are the Gauss-Chebyshev points for degree* $m-1$, *i.e., the zeros of* $T_m(x) = \cos(m \arccos(x))$.

We note that there is a slightly weaker version of this result in [15]. In view of (1), X_m is an admissible mesh in $[-1,1]$ for $m = \lceil \mu n \rceil$, where $\mu > 1$, with constant $C = 1/\cos(\pi/2\mu)$ and cardinality $\lceil \mu n \rceil$. For example, taking $\mu = 2$ we get an admissible mesh in $[-1,1]$ with $2n$ points and $C = 1/\cos(\pi/4) = \sqrt{2}$. Since admissible meshes are preserved by affine transformations, we then have an admissible mesh with $\mathcal{O}(n)$ cardinality on any interval $[a,b]$. By extension, we can say that $\Theta_{\lceil \mu n \rceil}(\alpha)$ is a "trigonometric admissible mesh" for $[\alpha, \alpha + 2\pi]$.

Remark 1 It is worth noticing that inequality (3) can be obtained also with $X_m = \{\cos(j\pi/m)\,,\ j = 0, \dots, m\}$ (taking $\alpha = 0$ in Theorem 1), i.e, the $m + 1$ Chebyshev-Lobatto points for degree m. In this case, $X_{\lceil \mu n \rceil}$ is an admissible mesh in $[-1, 1]$, with cardinality $\lceil \mu n \rceil + 1$ and constant $C = 1/\cos(\pi/2\mu)$. On the other hand, it is easy to check that also $X_{\lceil \mu n \rceil - 1}$ is an admissible mesh in $[-1, 1]$ for $\mu > 3/2$ and $n \geq 2$, with cardinality $\lceil \mu n \rceil$ and constant $C = 1/\cos(\pi/(2\mu - 1))$. For example, taking $\mu = 2$ we get admissible meshes including the endpoints in $[-1, 1]$, with $2n+1$ points and $C = 1/\cos(\pi/4) = \sqrt{2}$, or $2n$ points and $C = 1/\cos(\pi/3) = 2$.

3 Bivariate admissible meshes with $\mathcal{O}(n^2)$ points.

In view of Theorem 2 it folows immediately that for tensor product polynomials we have the inequality

$$\|p\|_{[-1,1]^d} \leq \frac{1}{\cos^d(n\pi/2m)}\, \|p\|_{(X_m)^d}\ ,\quad \forall p \in \bigotimes_{j=1}^{d} \mathbb{P}_n^1 \tag{4}$$

which in particular holds for total degree polynomials $p \in \mathbb{P}_n^d \subset \bigotimes_{j=1}^{d} \mathbb{P}_n^1$. This means that we have at hand admissible meshes $(X_{\lceil \mu n \rceil})^d$, even for tensor product polynomials, in d-dimensional cubes (and parallelepipeds by affine mapping) with cardinality $\lceil \mu n \rceil^d$.

Let us now focus on the 2-dimensional case. It is well-known that any convex quadrangle with vertices $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4$ is the image of a *bilinear transformation* of the square, namely

$$\boldsymbol{x} = (x_1, x_2) = \sigma(u, v) = \frac{1}{4}\ (\boldsymbol{a}_1(1-u)(1-v) + \boldsymbol{a}_2(1+u)(1-v)$$

$$+\boldsymbol{a}_3(1+u)(1+v) + \boldsymbol{a}_4(1-u)(1+v))\ ,\quad (u, v) \in [-1, 1]^2 \tag{5}$$

with a triangle, e.g. $\boldsymbol{a}_3 = \boldsymbol{a}_4$, as a special degenerate case. We can then prove the following

Proposition 1 *For every fixed $\mu > 1$, the sequence of "oblique" Gauss-Chebyshev grids*

$$\mathcal{A}_n = \{\sigma(\xi_j, \xi_k)\,,\ 1 \leq j, k \leq \lceil \mu n \rceil\}\ ,\quad \xi_s = \cos\frac{(2s-1)\pi}{2\lceil \mu n \rceil} \tag{6}$$

is an admissible mesh of the convex quadrangle $K = \{\boldsymbol{x} = \sum c_i \boldsymbol{a}_i\,,\ c_i \geq 0\,,\ \sum c_i = 1\,,\ 1 \leq i \leq 4\}$, cf. (5), with constant $C = 1/\cos^2(\pi/2\mu)$ and cardinality $\lceil \mu n \rceil^2$.

Proof. It is sufficient to observe that, for any polynomial $p \in \mathbb{P}_n^2$, $q(u, v) = p(\sigma(u, v))$ is a tensor product polynomial in $[-1, 1]^2$, since the transformation

σ is bilinear. Then we conclude by applying (4) to q with $m = \lceil \mu n \rceil$, and using the fact that σ is surjective, even in the degenerate case of a triangle. Since the transformation is bilinear, any segment $u = const$ or $v = const$ in $[-1,1]^2$ is mapped into a segment of the quadrangle. This entails that the admissible mesh is an "oblique" grid and that the points on each segment are exactly its Gauss-Chebyshev points. Concerning the cardinality, observe that the transformation is bijective when restricted to the interiors, even in the degenerate case of a triangle, and that the admissible mesh is made of interior points (the image of tensor product Gauss-Chebyshev points: see Figure 1 left). $\square$

A similar result holds for the disk, as is shown by the following

Proposition 2 *For every fixed $\mu > 1$, the sequence of symmetric polar grids*

$$\mathcal{A}_n = \{(r_j \cos\theta_k, r_j \sin\theta_k)\} \tag{7}$$

$$\{(r_j,\theta_k)\}_{j,k} = \left\{\cos\frac{(2j-1)\pi}{2\lceil \mu n \rceil}\,,\ 1 \le j \le \lceil \mu n \rceil\right\} \times \left\{\frac{k\pi}{\lceil \mu n \rceil}\,,\ 0 \le k \le \lceil \mu n \rceil - 1\right\}$$

is an admissible mesh of the unit disk $K = \{\boldsymbol{x} : x_1^2 + x_2^2 \le 1\}$, with constant $C = 1/\cos^2(\pi/2\mu)$ and cardinality $\lceil \mu n \rceil^2$ for $\lceil \mu n \rceil$ even, and $\lceil \mu n \rceil^2 - \lceil \mu n \rceil + 1$ for $\lceil \mu n \rceil$ odd.

Proof. It is sufficient to observe that the restriction of a polynomial $p \in \mathbb{P}_n^2$ to the disk in the symmetric polar coordinates $(x_1, x_2) = (r\cos\theta, r\sin\theta)$, $-1 \le r \le 1$, $0 \le \theta < \pi$, becomes a polynomial $q(r,\theta) = p(r\cos\theta, r\sin\theta)$ of degree n in r for any fixed value of θ, and a trigonometric polynomial of degree n in θ for any fixed value of r. Observe that we can take $\theta \in [0, 2\pi]$ for such trigonometric polynomials, since the range of coordinates remains exactly the same (the whole disk). We can conclude by applying Theorems 1 and 2 with $m = \lceil \mu n \rceil$, and observing that the discrete subset is centrally symmetric (see Figure 1 right). To compute the cardinality, we have to subtract the repetitions of the center of the disk for $\lceil \mu n \rceil$ odd. $\square$

We stress that if we had used the standard polar coordinates (see the proof of Proposition 2), we would have obtained an admissible mesh with twice the number of points and an artificial clustering at the center of the disk. On the other hand, using standard polar coordinates and a similar reasoning, it is easy to construct an admissible mesh with the same constant and $2\lceil \mu n \rceil^2$ points in any annulus $K = \{\boldsymbol{x} : \rho_1 \le x_1^2 + x_2^2 \le \rho_2\,,\ 0 < \rho_1 < \rho_2\}$. In this case the $\{r_j\}$ are the Gauss-Chebyshev points of the interval $[\rho_1, \rho_2]$.

Remark 2 In view of Remark 1, it is clear that the constructions above can be based also on Chebyshev-Lobatto points. In this case we get a feature that could be useful in applications, namely that some points of the

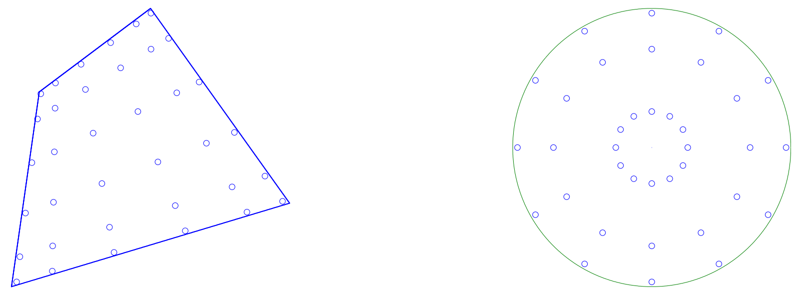

Figure 1: Admissible meshes for degree $n = 3$ and $\mu = 2$ (36 points) in a quadrangle and in a disk.

admissible mesh lie on the boundary of K. For example, with a quadrangle the side points are exactly the corresponding Chebyshev-Lobatto points.

Remark 3 It is interesting to recall that, in view of [8, Thm.1], the uniform error of the least-squares polynomial approximants at the admissible meshes above is (at most) $\mathcal{O}(nE_n(f;K))$, where $E_n(f;K)$ is the error of the best polynomial approximation in $\mathbb{P}_n^2$ to $f \in C(K)$.

Remark 4 *(Higher-dimensional extensions).* It is easy to see that the Propositions above can be easily extended to d-dimensional instances. In particular, we can construct an admissible mesh with (at most) $\lceil \mu n \rceil^d$ cardinality in any compact set which is a d-linear transformation of $[-1,1]^d$, e.g. in the 3-dimensional *simplex* (and thus in any *tetrahedron* by affine mapping).

More generally, we can extend the constructions to any compact set such that, by a suitable change of variables/coordinates, a polynomial on it becomes a (tensor product) algebraic polynomial in a cube with respect to a group of variables, and a (tensor product) trigonometric polynomial in the other variables, where each angle ranges in a whole interval of periodicity.

In thisway, we obtain for example admissible meshes with $\mathcal{O}(n^2)$ points on the *sphere* and the *torus* (where only trigonometric polynomials are involved in spherical and surface toroidal coordinates, respectively), as well as admissible meshes with $\mathcal{O}(n^3)$ points in the *ball*, the *cylinder* and the *solid torus* (where the polynomial becomes mixed algebraic/trigonometric in polar, cylindrical and toroidal coordinates, respectively). We recall that in the case of the sphere, existence of admissible meshes (norming sets) with $\mathcal{O}(n^2)$ cardinality was proved in [12] and also [14], using other methods.

Finally, we mention that Kroó [13] has recently shown that there exist low cardinality admissible meshes with $\mathcal{O}(n^d)$ points (optimal meshes) for a wide variety of compacts in $\mathbb{R}^d$. In the complex case, we mention only that it is easy to construct an admissible mesh with $\mathcal{O}(n)$ points for any compact set $K \subset \mathbb{C}^1$ satisfying a Markov inequality of exponent 1, and boundary given by a C^1 parametric curve, cf. [1, Prop.17].

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