

Asymptotic summation of power series

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Summary. We give an asymptotic expansion in powers of n^{-1} of the remainder $\sum_{j=n}^{\infty} f_j z^j$, when the sequence f_n has a similar expansion. Contrary to previous results, explicit formulas for the computation of the coefficients are presented. In the case of numerical series ($z = 1$), rigorous error estimates for the asymptotic approximations are also provided. We apply our results to the evaluation of $S(z; j_0, \nu, a, b, p) = \sum_{j=j_0}^{\infty} z^j (j+b)^{\nu-1} (j+a)^{-p}$, which generalizes various summation problems appeared in the recent literature on convergence acceleration of numerical and power series.

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1. Asymptotic summation formulas

Asymptotic integration, i.e., the procedure of integrating term by term a given asymptotic expansion, is a basic and powerful tool of asymptotic analysis (cf., eg., [15, Chap. 1]). The need for a discrete version of this procedure, which can be termed “asymptotic summation”, became of interest in connection with the convergence acceleration of power series (cf. [16] and references therein). Consider a power series

$$(1) \quad S(z) = \sum_{j=1}^{\infty} f_j z^j,$$

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where the sequence f_j possesses an asymptotic expansion in powers of j^{-1} ,

$$(2) \quad f_j \sim a_1 j^{-p_1} + a_2 j^{-p_2} + \dots, \quad j \rightarrow \infty, \quad 0 < p_1 < p_2 < \dots.$$

This clearly entails convergence of the series (1) for $|z| \leq 1$, $z \neq 1$, while convergence at $z = 1$ is guaranteed if $p_1 > 1$ (which then also ensures absolute convergence for $|z| \leq 1$).

In [16] Sidi proved that the remainder of (1), under assumption (2) with $p_{k+1} - p_k \equiv 1$, has an asymptotic expansion

$$(3) \quad R_n(z) := \sum_{j=n}^{\infty} f_j z^j \sim z^n n^{-q} \{b_1(z) n^{-1} + b_2(z) n^{-2} + \dots\}, \quad n \rightarrow \infty,$$

where $q = p_1 - 1$ if $z \neq 1$ and $q = p_1 - 2$ if $z = 1$, and used this expansion in connection with Levin's T-transformations. The coefficients $b_k(z)$, however, were not given in terms of computable formulas.

In Theorem 1.1, which together with Theorem 1.2 represents the main result of this paper, we provide such formulas under the slightly weaker assumption $p_{k+1} - p_k \in \mathbb{N}$, which are explicit for $z = 1$, and involve auxiliary recurrence formulas for $z \neq 1$. They are based essentially on known asymptotic expansions for the remainder of the power series defining the Riemann ζ -function and the generalized polylogarithm, respectively (cf. [15, Chap. 8]). The present results were partially developed in [5] ($z = 1$) and in [6].

Theorem 1.1 *Consider the power series (1), where the sequence f_j satisfies (2) with $p_{k+1} - p_k \in \mathbb{N}$ for every $k \geq 1$, and $p_1 > 1$ when $z = 1$. Then the remainder $R_n(z)$ possesses the asymptotic expansion (3), where*

$$(4) \quad q = \begin{cases} p_1 - 1 & \text{if } z \neq 1, \\ p_1 - 2 & \text{if } z = 1. \end{cases}$$

The coefficients $b_k(z)$, $k = p_s - p_1 + 1, \dots, p_{s+1} - p_1$, $s = 1, 2, \dots$, are given by

$$(5) \quad b_k(z) = \begin{cases} -\sum_{\ell=1}^s a_\ell \psi_{k+p_1-p_\ell-1}(-p_\ell, z) & \text{if } z \neq 1, \\ \sum_{\ell=1}^s \frac{a_\ell}{p_\ell-1} \binom{1-p_\ell}{k+p_\ell-p_1-1} B_{k+p_1-p_\ell-1} & \text{if } z = 1, \end{cases}$$

where the B_i are the Bernoulli numbers, and the ψ_i are recursively defined by

$$\psi_0(\alpha, z) = \frac{1}{z-1},$$

$$(6) \quad \psi_i(\alpha, z) = -\frac{z}{z-1} \sum_{r=0}^{i-1} \binom{\alpha}{i-r} \psi_r(r-i+\alpha, z), \quad i = 1, 2, \dots,$$

where α is an arbitrary negative number.

Proof. Let us rewrite the asymptotic expansion (2) as

$$(7) \quad f_j = \sum_{s=1}^{\nu} a_s j^{-p_s} + \rho_{\nu}(j), \quad \rho_{\nu}(j) = O(j^{-p_{\nu+1}}), \quad \nu = 1, 2, \dots,$$

and let $K_{\nu+1}$ be an estimate of the constant implied by the O -symbol above, valid for $j \geq n_0 \geq 1$. In view of (7), for each fixed ν the remainder in (3) can be expressed as

$$(8) \quad \sum_{j=n}^{\infty} f_j z^j = \sum_{s=1}^{\nu} a_s \sum_{j=n}^{\infty} j^{-p_s} z^j + \sum_{j=n}^{\infty} \rho_{\nu}(j) z^j.$$

First, we observe that for $n \geq n_0$ the second term on the right-hand side of (8) can be estimated by

$$(9) \quad \begin{aligned} \left| \sum_{j=n}^{\infty} \rho_{\nu}(j) z^j \right| &\leq \sum_{j=n}^{\infty} |\rho_{\nu}(j) z^j| \leq |z|^n K_{\nu+1} \sum_{j=n}^{\infty} j^{-p_{\nu+1}} \\ &= |z|^n O(n^{1-p_{\nu+1}}), \end{aligned}$$

cf. [15, Chap. 8, Example 3.2], where the fact that $|z| \leq 1$ has been used.

From the well-known asymptotic expansion for the remainder of the series defining the *Riemann ζ -function*, truncated at $p_{\nu+1} - p_s - 1$, one obtains

$$(10) \quad \sum_{j=n}^{\infty} j^{-p_s} = \frac{n^{1-p_s}}{p_s - 1} \left\{ \sum_{i=0}^{p_{\nu+1}-p_s-1} \binom{1-p_s}{i} B_i n^{-i} + O(n^{p_s-p_{\nu+1}}) \right\},$$

and analogously, for the *generalized polylogarithm* ($z \neq 1$), we get

$$(11) \quad \sum_{j=n}^{\infty} j^{-p_s} z^j = -z^n n^{-p_s} \left\{ \sum_{i=0}^{p_{\nu+1}-p_s-2} \psi_i(-p_s, z) n^{-i} + O(n^{1+p_s-p_{\nu+1}}) \right\},$$

for $s = 1, \dots, \nu$ (cf. [15, Chap. 8]). Substituting (10) and (11) into the double sum in (8), and summing up the coefficients corresponding to the same power of n , we easily recover the representation (5).

Hence, from (8)-(11) we obtain

$$(12) \quad \sum_{j=n}^{\infty} f_j = n^{-q} \sum_{k=1}^{p_{\nu+1}-p_1} b_k(1) n^{-k} + O(n^{1-p_{\nu+1}}),$$

and

$$(13) \quad \sum_{j=n}^{\infty} f_j z^j = z^n \left\{ n^{-q} \sum_{k=1}^{p_{\nu+1}-p_1-1} b_k(z) n^{-k} + O(n^{1-p_{\nu+1}}) \right\}, \quad z \neq 1.$$

Finally, setting for $m = 1, 2, \dots$, $\nu(m) := \min\{\nu \in \mathbb{N} : p_{\nu+1} \geq q + m + 2\}$, we get from (12) and (13)

$$(14) \quad \sum_{j=n}^{\infty} f_j z^j = z^n \left\{ n^{-q} \sum_{k=1}^m b_k(z) n^{-k} + \sum_{k=m+1}^{p_{\nu(m)+1}-q-2} b_k(z) n^{-k} + O(n^{1-p_{\nu(m)+1}+q}) \right\},$$

which, since the last two terms are $O(n^{-(m+1)})$, is the asymptotic expansion (3). $\square$

In the case of *numerical series* ($z = 1$), the remainder of the asymptotic expansion of R_n can be rigorously estimated as in the following theorem.

Theorem 1.2 *Let $K_{\nu+1}$ be an estimate of the constant implied by the O -symbol in (7), for $j \geq n_0 \geq 1$. Then*

$$(15) \quad \left| \sum_{j=n}^{\infty} f_j - \sum_{k=1}^m b_k(1) n^{-(q+k)} \right| \leq C_{m+1}(n_0) n^{-(q+m+1)}, \quad n \geq n_0, \quad m = 1, 2, \dots,$$

where

$$(16) \quad C_{m+1}(n_0) = n_0^{q+m+2-p_{\nu(m)+1}} \left\{ K_{\nu(m)+1} \left(\frac{1}{p_{\nu(m)+1}-1} + \frac{1}{2} n_0^{-1} + \frac{p_{\nu(m)+1}}{12} n_0^{-2} \right) + \sum_{s=1}^{\nu(m)} |a_s| H_{\nu(m),s} \right\} + n_0^{1-p_{\nu(m)}} \sum_{i=1}^{p_{\nu(m)+1}-(q+m+2)} |b_{m+i}(1)| n_0^{1-i},$$

with $\nu(m) := \min \{ \nu \in \mathbb{N} : p_{\nu+1} \geq q + m + 2 \}$ and $(p_s - 1)H_{\nu,s}$ an estimate of the constant implied by the O -symbol in (10) for $n \geq n_0$, i.e. [15, Chap. 8, Example 3.2],

$$(17) \quad H_{\nu,s} = \frac{1}{p_s - 1} \left| \binom{p_s - 1}{\sigma_{\nu,s}} B_{\sigma_{\nu,s}} \right|, \quad s = 1, \dots, \nu - 1;$$

$$H_{\nu,\nu} = \begin{cases} \frac{1}{p_{\nu}-1} \left| \binom{p_{\nu}-1}{\sigma_{\nu,\nu}} B_{\sigma_{\nu,\nu}} \right| & \text{if } p_{\nu+1} - p_{\nu} > 1, \\ \frac{1}{2} + \frac{p_{\nu}}{12} n_0^{-1} & \text{if } p_{\nu+1} - p_{\nu} = 1, \end{cases}$$

where

$$(18) \quad \sigma_{\nu,s} = p_{\nu+1} - p_s + 1 - ((p_{\nu+1} - p_s - 1) \bmod 2).$$

Proof. Observe that, for $z = 1$, the O -symbol in (14) is obtained via the contribution of (9) for $\nu = \nu(m)$, and (10) for $s = 1, \dots, \nu(m)$. By resorting to a classical estimate concerning the asymptotic expansion (10) for the remainder of the series defining the Riemann ζ -function [15, Chap. 8, Example 3.2], we get for $n \geq n_0$

$$(19) \quad \sum_{j=n}^{\infty} j^{-p_{\nu+1}} \leq \frac{n^{1-p_{\nu+1}}}{p_{\nu+1} - 1} \left\{ |B_0| + \left| \binom{1-p_{\nu+1}}{1} B_1 \right| n^{-1} + \left| \binom{1-p_{\nu+1}}{2} B_2 \right| n^{-2} \right\},$$

and hence (9) can be rewritten as

$$(20) \quad \left| \sum_{j=n}^{\infty} \rho_{\nu}(j) \right| \leq K_{\nu+1} \left(\frac{1}{p_{\nu+1} - 1} + \frac{1}{2} n_0^{-1} + \frac{p_{\nu+1}}{12} n_0^{-2} \right) n^{1-p_{\nu+1}}, \quad n \geq n_0.$$

As for the sum of the O -symbols in (10) for $s = 1, \dots, \nu(m)$, the estimate quoted above leads to

$$(21) \quad \left| \sum_{s=1}^{\nu(m)} \frac{a_s}{p_s - 1} n^{1-p_s} O(n^{p_s - p_{\nu(m)+1}}) \right| \leq n^{1-p_{\nu(m)+1}} \sum_{s=1}^{\nu(m)} |a_s| H_{\nu(m),s}, \quad n \geq n_0,$$

having used the fact that the coefficients with odd index in the expansion (10) vanish. We remark that when $p_{\nu+1} - p_\nu \equiv 1$, the estimate in [15, Example 3.2, Chap. 8] cannot be used directly, since the asymptotic expansion has to be truncated at the first term. Nevertheless, the second row in (17) is easily recovered by adding and subtracting the second term, i.e., $\binom{1-p_\nu}{1} B_1 n^{-1}$. Finally, recalling that $q + m + 2 - p_{\nu(m)+1} \leq 0$, we get from (14), (20) and (21) the estimate (15)–(16), valid for $n \geq n_0$. $\square$

Remark 1.3 Formulas (5)–(6) for the computation of the coefficients $b_k(z)$ when $z \neq 1$ appear quite involved and difficult to handle. On the other hand, in the case when $p_{k+1} - p_k \equiv 1$, which was that considered by Sidi in [16] to prove existence of the asymptotic expansion (and turns out to be the most common in the applications), the representation of coefficients is substantially simplified. Indeed, for $z \neq 1$ formulas (5)–(6) become, after a little algebra,

$$(22) \quad b_k(z) = - \sum_{i=0}^{k-1} a_{k-i} \varphi_i(k, z),$$

where the φ_i are recursively defined by

$$(23) \quad \varphi_0(k, z) = \frac{1}{z-1},$$

$$(23) \quad \varphi_i(k, z) = \frac{z}{1-z} \sum_{r=0}^{i-1} \binom{i+1-k-p_1}{i-r} \varphi_r(k, z), \quad i = 1, 2, \dots$$

In the case $z = 1$, (5) can be slightly simplified as

$$(24) \quad b_k(1) = \sum_{\ell=1}^k \frac{a_\ell}{p_1 + \ell - 2} \binom{2 - p_1 - \ell}{k - \ell} B_{k-\ell}.$$

We finally remark that, when $p_{k+1} - p_k \equiv 1$, we have $\nu(m) = m$ and $C_{m+1}(n_0)$ in (16) becomes

$$(25) \quad C_{m+1}(n_0) = K_{m+1} \left(\frac{1}{p_1 + m - 1} + \frac{1}{2} n_0^{-1} + \frac{p_1 + m}{12} n_0^{-2} \right) + \sum_{s=1}^m |a_s| H_{m,s}.$$

Remark 1.4 It is worth observing that the coefficients $b_k(z)$ are discontinuous at $z = 1$. In fact, it can be easily proved by induction on i that $\psi_i(-p_\ell, z) \sim \text{const.}/(z-1)^{i+1}$ as $z \rightarrow 1$. Then it follows that $b_k(z) \sim \text{const.}/(z-1)^{k+2}$, as $z \rightarrow 1$, and hence also that every remainder of the asymptotic expansion (2) tends to infinite as $z \rightarrow 1$. From the computational point of view, this means that the asymptotic summation formulas are difficult to use in the neighborhood of $z = 1$ (while they work quite satisfactorily at $z = 1$), as it will be shown in the numerical examples.

Remark 1.5 Bounds similar to those in Theorem 1.2 could be derived also in the case $z \neq 1$, provided that the constant implied by the O -symbol in (11) can be estimated. Unfortunately, this does not seem an easy task within the approach followed in [15, Chap. 8] to construct the asymptotic expansion.

2. Numerical examples

It is well known that convergence of the series (1) can be very slow (when p_1 is “small”) for z on (or close to) the boundary of the unit circle: in these cases, suitable acceleration techniques must be applied in order to evaluate the sum numerically. In this respect, existence of an asymptotic expansion like (3) is already extremely useful. In fact, it entails that for $z = 1$ the sequence of partial sums of (1) belongs to the class of *logarithmically* convergent sequences, while for $z \neq 1$ it belongs to the class of *linearly* convergent sequences. For such sequences various *transformation* and *extrapolation methods* have been studied; see the monograph [4] for a detailed analysis and an extensive bibliography, and [17, 18] for some of the recent developments. These algorithms require, at most, a knowledge of the sequence of exponents in (2), p_k , whereas neither the coefficients b_k in (3) nor the a_k in (2) are needed.

In the present paper we explore a different approach, namely, the *direct* use of the *asymptotic expansion* (3) for the numerical evaluation of the sum of the series, in cases where the sequence a_k is explicitly known, so that the coefficients b_k can be computed by means of (5), (6). We stress that, as already remarked in Sect. 1, our main improvement with respect to [16] consists exactly in the derivation of these *explicit* and *computable formulas*. In particular, we show that our approach can be used to compute the sum

$$S(z; j_0, \nu, a, b, p) = \sum_{j=j_0}^{\infty} z^j (j+b)^{\nu-1} (j+a)^{-p} ,$$

$$(26) \quad |z| \leq 1, \quad 0 < \nu \leq 1, \quad a, b \in \mathbb{C} \setminus \{-j_0, -j_0 - 1, -j_0 - 2, \dots\} ,$$

where $1 + p - \nu > 0$ if $z \neq 1$ and $1 + p - \nu > 1$ if $z = 1$, which generalizes various summation problems appeared in the recent literature on convergence acceleration of numerical and power series. Observe that (26) falls into the class characterized by (2), since we have

$$(27) \quad f_j := (j+b)^{\nu-1}(j+a)^{-p} \sim \sum_{k=1}^{\infty} a_k j^{-p_k}, \quad j \rightarrow \infty,$$

with

$$(28) \quad a_k = \sum_{i=1}^k \binom{-p}{i-1} \binom{\nu-1}{k-i} a^{i-1} b^{k-i}, \quad p_k = p + k - \nu, \quad k = 1, 2, \dots$$

Indeed, there is a renewed interest in the literature of the last decade for developing specialized summation methods, alternative to the general-purpose acceleration techniques recalled above, for various series of the form (26); cf., e.g., [1–3, 7, 8, 10–14], and [9, Sect. 3] for references and sources of the summation problems. Most of these papers deal with the case of numerical series, i.e., $S(\pm 1; j_0, \nu, a, b, p) = \sum_{j=j_0}^{\infty} (\pm 1)^j (j+b)^{\nu-1}(j+a)^{-p}$, and follow, essentially, two different approaches: Gaussian quadrature with respect to nonclassical weights [7, 8, 10, 13, 14], reducing the problem to the evaluation of a suitable integral on $(0, +\infty)$ (e.g., by Laplace transform), and of some special functions involved; a clever use of the *Euler-MacLaurin summation formula*, associated with the evaluation of Gauss hypergeometric function [11] or with computer algebra [12]. Observe, incidentally, that our derivation of the asymptotic expansion (3) for $z = 1$ is intimately related to the Euler-MacLaurin formula, cf. [15, Sect. 3]. In contrast, $z = -1$ (oscillatory case) is treated within the case $|z| = 1, z \neq 1$, the core of the procedure being, essentially, *summation by parts* [15, Sect. 5]. This difference between the procedures for $z = 1$ and $z \neq 1$ explains the appearance of a discontinuity of the $b_k(z)$ at $z = 1$ (cf. Remark 1.4).

We recall that evaluation of $S(\pm 1; j_0, \nu, a, b, p)$, $p \in \mathbb{N}$, has been recognized in the literature as a basic tool for summation of series of the form $\sum_{j=j_0}^{\infty} (\pm 1)^j (j+b)^{\nu-1} r(j)$, where $r(j)$ is a *rational function* of j , by resorting to partial fraction decomposition [7]. The same argument clearly applies to the role of $S(z; j_0, \nu, a, b, p)$ in connection with power series like $\sum_{j=j_0}^{\infty} z^j (j+b)^{\nu-1} r(j)$.

As for power series, the methods proposed are closely tied to specific values of the parameters in (26), the aim being, essentially, to compute $\sum_{j=0}^{\infty} z^{2j+1} (2j+1)^{-p}$, for $p = 2$ (cf. [3, 8]), or $p = 3$ (cf. [8]), or more generally $p \in \mathbb{N}, p \geq 2$ (cf. [2]). It is worth noting that this last series, which arises for example in plate contact problems, can be rewritten in

terms of the well-known *Lerch ζ -function*, $\Phi(z, p, a) = S(z; 0, 1, a, 0, p)$, as $\sum_{j=0}^{\infty} z^{2j+1} (2j+1)^{-p} = (z/2^p) S(z^2; 0, 1, 1/2, 0, p)$. Evaluation of the Riemann ζ -function and of the Lerch ζ -function is accomplished in [1] by means of the trapezoidal method on suitable integral representations in the complex plane, together with Euler Mac-Laurin summation.

Our method for the *evaluation* of $S(z; j_0, \nu, a, b, p)$ via *asymptotic summation* rests on the representation

$$S = S_n + R_n = \sum_{j=j_0}^{n-1} z^j f_j + \sigma_{n,m} + \varepsilon_{n,m} ,$$

$$(29) \quad \sigma_{n,m} = z^n n^{-q} \sum_{k=1}^m b_k(z) n^{-k} , \quad \varepsilon_{n,m} = Z^n O\left(n^{-(m+q+1)}\right) ,$$

cf. Theorem 1.1, where for brevity the dependence on the parameters z, j_0, ν, a, b, p , has not been displayed. In this case the coefficients $b_k(z)$ are given by the simplified formulas (23)-(24), since $p_{k+1} - p_k \equiv 1$ (cf. (28)). The goal clearly should be to find a way to choose n and m , such that the relative error $|\varepsilon_{n,m}/S|$ is below a prescribed tolerance, minimizing at the same time the computational effort. This is not simple, however, especially for $z \neq 1$, where explicit bounds for the asymptotic error term are not available (cf. Remark 1.5).

It is worth recalling that the series (3) is asymptotic; thus its remainder $\varepsilon_{n,m}$ tends to 0 as $n \rightarrow \infty$, but is not infinitesimal (in general) as $m \rightarrow \infty$, exhibiting a minimum in m for each fixed value of n (clearly this minimum is infinitesimal as $n \rightarrow \infty$). According to this typical behavior, confirmed also by our numerical experiments, we first tried the following approach. Given a *tolerance*, say ε , and an *increasing sequence* of indexes $\{n_s\}$, $s = 0, 1, 2, \dots$, obtained, for example, by doubling iteratively a starting index n_0 , or by adding iteratively a fixed integer step, *minimize* (heuristically) with respect to m the error term $\varepsilon_{n_s, m}$, that is, obtaining a sequence $\{m_s\}$ such that $|b_{m_s-1}| n_s^{-(m_s-1)} \geq |b_{m_s}| n_s^{-m_s}$ and $|b_{m_s}| n_s^{-m_s} < |b_{m_s+1}| n_s^{-(m_s+1)}$ for every s . The iteration is stopped at the first value of s such that

$$(30) \quad \left| \frac{\varepsilon_{n_s, m_s}}{S} \right| \cong \left| \frac{S_{n_s} - S_{n_{s-1}} + \sigma_{n_s, m_s} - \sigma_{n_{s-1}, m_{s-1}}}{S_{n_s} + \sigma_{n_s, m_s}} \right| \leq \varepsilon ,$$

cf. (29). This procedure, however, turned out to be very costly, since, as it can be easily seen from (22)-(24) and (29), the leading term in the *complexity* of the computation of $\sigma_{n,m}$ (measured in terms of basic arithmetic operations, multiplications and additions) increases *quadratically* ($z = 1$) or *cubically* ($z \neq 1$) with m .

This last observation suggests to adopt a different strategy, namely, of trying to *balance* the computational cost of the sum $\sigma_{n,m}$ with that of S_n in (29), at least in order of magnitude. By observing that, in general, the sequence $f_j = (j+b)^{\nu-1}(j+a)^{-p}$ in (26) is computed as $\exp[(\nu-1)\ln(j+b) - p\ln(j+a)]$, we can accomplish this simply by estimating the average cost of the numerical computation of the functions $\exp(\cdot)$ and $\ln(\cdot)$ in the ranges of interest (in terms of elementary arithmetic operations), and then by counting the total number of operations involved in the evaluation of $\sigma_{n,m}$ and S_n . The cost of function evaluations depends on implementation, namely on the compiler and/or on the specific computer used; we did the balancing on the assumption that the computation of $\ln(\cdot)$, in terms of machine time, corresponds to 10, and the computation of $\exp(\cdot)$ to 20, elementary operations. These are the average values obtained on a personal computer based on a 486 processor, with the Turbo Pascal compiling system (in double-precision arithmetic). Other experiments with different programming languages and different compilers on various computers showed that the quoted numbers (namely 10 and 20) are quite reasonable, giving an underestimation by a factor two or three at worst.

The resulting algorithm can then be briefly described as follows. *Compute iteratively* the coefficients $b_s(z)$, $s = 1, 2, \dots$, by (22)-(24) and (28), and correspondingly the sums S_{n_s} and $\sigma_{n_s,i}$, $1 \leq i \leq s$ (cf. (29)), where n_s is chosen in such a way as to *balance* the costs of S_{n_s} and $\sigma_{n_s,s}$ as just described, *until*

$$(31) \quad \left| \frac{\varepsilon_{n_s,i}}{S} \right| \cong \left| \frac{S_{n_s} - S_{n_{s-1}} + \sigma_{n_s,i} - \sigma_{n_{s-1},i}}{S_{n_s} + \sigma_{n_s,i}} \right| \leq \varepsilon,$$

for some i , $1 \leq i \leq s-1$. Observe that convergence of the scheme is ensured since $n_s \rightarrow \infty$ as $s \rightarrow \infty$. The algorithm has been tested on various series of the form (26), choosing a *relative tolerance* $\varepsilon = 10^{-14}$ in the test (31). The results are summarized in Tables 1–9 below, where for some values of the parameters involved we reported $(n, m) = (n_s, i)$, the first couple for which (31) is satisfied. In all of the examples the “true” relative errors turned out to be below the prescribed tolerance, so they are not reported in the tables. From this point of view the performance of the algorithm is quite satisfactory, as it is also with respect to the complexity, except for power series with z close to 1. The reader should concentrate on the values of n , since $n-1$ terms (when $j_0 = 1$) are to be summed in the original series to obtain a relative error of 10^{-14} , when m terms of the asymptotic expansion of the remainder are used. Consider, for example, the effectiveness of asymptotic summation on the extremely slowly convergent series in Table 5. In this case, prescribing a relative tolerance of $\varepsilon = 10^{-3}$ (for $a = 1$) we obtained the couple $(n, m) = (5, 2)$, while direct summation of the series should require more than 10^{30} terms to get the same accuracy.

Tables 1–5 refer to certain *slowly convergent numerical series*, frequently used as computational tests in previous papers on acceleration; see [7, 11, 13, 14]. The exact values have been taken from these papers, or computed with Mathematica [19]. Note that convergence slows down as the parameter a (or α) increases: this phenomenon has been observed also with other methods [7, 14]. The method proposed in [11] for the special series $S(1; 1, \nu, a, b, p)$, however, does not suffer from this disadvantage.

Tables 6–9 refer, essentially, to three “slowly convergent” instances of the *power series* defining the *Lerch ζ -function*. The exact sums were found in the literature [2, 8], or computed with Mathematica. Observe that convergence is again quite satisfactory, also at $z = 1$ ($\omega = 2$ in Tables 7 and 9), but slows down as ω decreases, especially in the immediate neighborhood of $z = 1$ (cf. Remark 1.4). Note also that the behavior near $z = 1$ seems somewhat better for real values of z (compare Table 8 with Table 7).

Table 1. Computation of $S(1; 1, 1/2, a, 0, 1) = \sum_{j=1}^{\infty} j^{-1/2}(j+a)^{-1}$

a	1	2	4	8	16	32	64
n	15	18	27	43	62	97	147
m	10	11	14	17	21	26	34

Table 2. Computation of $S(-1; 1, 1/2, a, 0, 1) = \sum_{j=1}^{\infty} (-1)^j j^{-1/2}(j+a)^{-1}$

a	1	2	4	8	16	32	64
n	32	32	54	83	121	198	301
m	10	10	11	13	15	17	20

Table 3. Computation of $S(1; 1, 1/2, i\alpha, 0, 1) = \sum_{j=1}^{\infty} j^{-1/2}(j+i\alpha)^{-1}$

α	1	2	4	8	16	32	64
n	20	23	30	41	64	92	148
m	11	13	15	18	21	28	35

Table 4. Computation of $S(-1; 1, 1/2, i\alpha, 0, 1) = \sum_{j=1}^{\infty} (-1)^j j^{-1/2}(j+i\alpha)^{-1}$

α	1	2	4	8	16	32	64
n	39	49	62	76	112	182	277
m	10	10	12	14	16	18	21

Table 5. Computation of $S(1; 1, 1, 9/10, a, 1/2, 1) = \sum_{j=1}^{\infty} (j+1/2)^{-1/10}(j+a)^{-1}$

a	1	8	64	$3/2 + i$	$3/2 + 8i$	$3/2 + 64i$
n	12	42	143	19	39	142
m	9	17	34	12	18	35

Table 6. Computation of the Lerch ζ -function
 $S(z; 0, 1, 1, 0, 1/2) = \sum_{j=0}^{\infty} z^j (j+1)^{-1/2}$, $z = e^{i\omega\pi/2}$

ω	2	1	0.8	0.6	0.4	0.2	0.1	0.05	0.01	0.005
n	36	60	60	75	93	161	256	487	2056	3904
m	10	11	12	12	14	15	19	20	31	22

Table 7. Computation of $\frac{z}{4}S(z^2; 0, 1, 1/2, 0, 2) = \sum_{j=0}^{\infty} z^{2j+1}(2j+1)^{-2}$, $z = e^{i\omega\pi/2}$

ω	2	1	0.8	0.6	0.4	0.2	0.1	0.05	0.01	0.001
n	15	27	36	47	60	93	136	221	997	7984
m	9	9	10	9	10	12	15	18	17	20

Table 8. Computation of $\frac{x}{4}S(x^2; 0, 1, 1/2, 0, 2) = \sum_{j=0}^{\infty} x^{2j+1}(2j+1)^{-2}$, $x \in (0, 1]$

x	0.5	0.6	0.7	0.8	0.9	0.95	0.99	0.995	0.999	1
n	20	27	36	47	93	161	749	1406	6303	15
m	3	2	3	6	6	9	7	7	7	9

Table 9. Computation of $\frac{z}{8}S(z^2; 0, 1, 1/2, 0, 3) = \sum_{j=0}^{\infty} z^{2j+1}(2j+1)^{-3}$, $z = e^{i\omega\pi/2}$

ω	2	1	0.8	0.6	0.4	0.2	0.1	0.05	0.01	0.001
n	15	27	36	36	47	75	113	190	826	5703
m	9	9	8	10	10	11	15	18	13	13

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