Norming meshes by Bernstein-like inequalities *

M. Vianello[1]

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Abstract

We show that finite-dimensional univariate function spaces satisfying a Bernstein-like inequality admit norming meshes. In particular, we determine meshes with "optimal" cardinality for trigonometric polynomials on subintervals of the period. As an application we discuss the construction of optimal bivariate polynomial meshes by arc blending.

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1 From Bernstein inequalities to norming meshes

We begin with the following definition. Given a sequence $\{S_n\}$ of finite dimensional subspaces of $C(K)$ (the space of continuous functions on a real or complex d-dimensional compact subset K), a sequence $\{\mathcal{A}_n\}$ of finite subsets of K is called a $\{S_n\}-$*norming mesh* (in the sup-norm) if there exists a constant $C > 0$ such that

$$\|p\|_K \leq C\,\|p\|_{\mathcal{A}_n}\ ,\ \ \forall p \in S_n\ ; \tag{1}$$

here and below, $\|f\|_X$ denotes the sup-norm of a function bounded on the set X. Observe that necessarily $\mathrm{card}(\mathcal{A}_n) \geq \dim(S_n)$; then, we may term *"optimal"* (with respect to the cardinality) a norming mesh such that $\mathrm{card}(\mathcal{A}_n) = \mathcal{O}(\dim(S_n))$ as $n \to \infty$.

In recent years, several investigations have been devoted to the theory and applications of norming meshes in the polynomial case, $S_n = \mathbb{P}_n^d$, with

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[1]Dept. of Mathematics, University of Padova, Italy
e-mail: marcov@math.unipd.it

the additional requirement that $\mathrm{card}(\mathcal{A}_n) = \mathcal{O}(n^s)$ for some $s \geq d$; in such a case the sequence is called an *admissible mesh*. An admissible mesh with $s = d$ is called *optimal*; see [7, 12]. If in (1) we have a sequence C_n instead of C, increasing at most polynomially with n, the mesh is called *weakly admissible* [7]. Admissible and weakly admissible meshes are important structures in polynomial approximation theory: for example, in [7] it is shown that they are nearly optimal for least squares approximation, and contain Fekete-like interpolation sets with a slowly increasing Lebesgue constant. Algorithms for the approximate computation of such Fekete-like sets have been developed in [3, 4, 16]. For a recent survey on the state of the art in polynomial interpolation and approximation in $\mathbb{C}^d$, we refer the reader to [1].

In [7, Thm.5], it has been shown that any (real) compact set which satisfies a Markov polynomial inequality with exponent r has an admissible mesh with $\mathcal{O}(n^{rd})$ cardinality. On the other hand, existence of optimal (or near-optimal) admissible meshes has been proved constructively for several families of bidimensional and multidimensional compacts, such as for example polygons and polyhedra, euclidean spheres and balls, subanalytic sets, convex bodies and starlike domains with smooth boundary; cf., e.g., [5, 11, 12, 13, 14].

Most of these results exploit, in some way, the basic fact that *the one-dimensional interval posesses optimal admissible meshes*, see, e.g., [10, 15]. Recently [12], this result has been extended to *trigonometric polynomials* on *subintervals of the period*, where the construction of the optimal mesh has been obtained using together a Markov and a Bernstein inequality.

In this note, we focus on the univariate case, showing that the availability of a Bernstein-like inequality allows to construct directly a functional norming mesh. We then discuss the result in the framework of optimal meshes for polynomials, and for trigonometric polynomials on subintervals of the period. To this purpose, we first give the following general:

Proposition 1 *Let $\{S_n\}$ be a sequence of finite dimensional spaces of differentiable functions defined in the compact interval $[a,b]$. Assume that for any $p \in S_n$ and $x \in (a,b)$ the following Bernstein-like inequality holds*

$$|p'(x)| \leq \phi_n(x)\, \|p\|_{[a,b]} \;, \quad \phi_n \in L^1_+(a,b) \;, \tag{2}$$

and define

$$F_n(x) = \int_a^x \phi_n(t)\, dt \;. \tag{3}$$

Consider $m+1$ equally spaced points in $[0, F_n(b)]$, where $m > F_n(b)/2$, namely $y_j = jF_n(b)/m$, $j = 0, 1, \ldots, m$.

Then, the following functional inequality holds

$$\|p\|_{[a,b]} \leq \frac{1}{1 - F_n(b)/2m}\, \|p\|_{X_m} \;, \quad \forall p \in S_n \;, \tag{4}$$

where $X_m = \{F_n^{-1}(y_0), \dots, F_n^{-1}(y_m)\}$ *is constructed choosing one point in the inverse image of each* y_j.

Proof. Fix $x \in [a,b]$: by construction, there exists $x_k \in X_m$ such that $|F_n(x) - F_n(x_k)| \le F_n(b)/2m$. Now, for every $p \in S_n$ we can write the chain of inequalities

$$|p(x)| \le |p(x_k)| + |p(x) - p(x_k)| \le |p(x_k)| + \int_{\min\{x,x_k\}}^{\max\{x,x_k\}} |p'(t)|\, dt$$

$$\le |p(x_k)| + \|p\|_{[a,b]} \int_{\min\{x,x_k\}}^{\max\{x,x_k\}} \phi_n(t)\, dt \le |p(x_k)| + |F_n(x) - F_n(x_k)|\, \|p\|_{[a,b]} \ ,$$

which gives

$$|p(x)| \le |p(x_k)| + \frac{F_n(b)}{2m} \|p\|_{[a,b]} \ .$$

Taking the maximum for $x_k \in X_m$ on the right-hand side and then for $x \in [a,b]$ on the left hand-side, we get the inequality

$$\|p\|_{[a,b]} \le \|p\|_{X_m} + \frac{F_n(b)}{2m} \|p\|_{[a,b]} \ ,$$

that is (4). $\square$

Remark 1 Inequality (4) clearly implies the existence of $\{S_n\}$-norming meshes. Indeed, it is sufficient to take

$$m = \lceil \mu F_n(b)/2 \rceil \tag{5}$$

for a fixed $\mu > 1$, to get a norming mesh $\mathcal{A}_n = X_m$ with $\mathcal{O}(F_n(b))$ cardinality as $n \to \infty$, and constant $C = \mu/(\mu - 1)$.

Notice also that in the case when

$$\phi_n(x) = n\phi(x) \ , \ \ \phi \in L^1_+(a,b) \ , \tag{6}$$

as it happens for example with polynomials and trigonometric polynomials (see the discussion below), then in Proposition 1

$$X_m = \{F^{-1}(jF(b)/m)\} \ , \ j = 0, \dots, m \ , \ \ F(x) = \int_a^x \phi(t)\, dt \ , \tag{7}$$

and thus we have a norming mesh with $\mathcal{O}(n)$ cardinality, namely

$$\mathcal{A}_n = X_{\lceil \mu n F(b)/2 \rceil} \ , \ \ \mathrm{card}(\mathcal{A}_n) = \left\lceil \frac{\mu n F(b)}{2} \right\rceil + 1 \ , \ \ \mu > 1 \ , \tag{8}$$

and constant

$$C = \frac{\mu}{\mu - 1} \ . \tag{9}$$

Finally, it is worth observing that if ϕ_n in (2) is a.e. positive, then F_n is a strictly increasing and thus invertible function, and X_m in Proposition 1 is uniquely determined.

1.1 Polynomials

In the case of polynomials, $S_n = \mathbb{P}_n$, we can consider a reference interval, say $[a, b] = [-1, 1]$, since norming meshes, also called admissible meshes in the literature, are preserved by affine transformations. By the classical Bernstein inequality (cf., e.g., [2]), that is (2) with

$$\phi_n(x) = \frac{n}{\sqrt{1 - x^2}} \ , \ \ x \in (-1, 1) \ , \tag{10}$$

we get immediately in view of Remark 1 the existence of optimal admissible meshes, i.e., admissible meshes with $\mathcal{O}(n)$ cardinality.

Moreover, observing that in (7) $F(x) = \arcsin(x) + \pi/2$, and thus $F(b) = F(1) = \pi$ and $F^{-1}(y) = \sin(y - \pi/2) = -\cos(y)$, the mesh points $X_m = \{-\cos(j\pi/m)\}$, $j = 0, \dots, m$, turn out to be nothing else than the classical $m+1$ Chebyshev-Lobatto points for degree m, where $m > n\pi/2$. This result, however, is already well-known, since Ehlich and Zeller proved a sharper inequality (see [10] and [5, Rem.1]), where the first factor on the right-hand side of (4) is $1/\cos(n\pi/2m) \leq 1/(1 - n\pi/2m)$ and the corresponding inequality is valid for every $m > n$.

On the other hand, the general formulation of Proposition 1 can still give some additional information on the generation of polynomial meshes. Indeed, observing that $\sqrt{1 - x^2} \geq \sqrt{1 - |x|}$, $x \in (-1, 1)$, and that $1/\sqrt{1 - |\cdot|} \in L^1_+(-1, 1)$, we have a Bernstein-like inequality in $\mathbb{P}_n$ with

$$\phi_n(x) = \frac{n}{\sqrt{1 - |x|}} \ , \ \ x \in (-1, 1) \ . \tag{11}$$

Using the notation of Remark 1 we get $F(x) = 2\sqrt{1 + x}$ for $x \in [-1, 0]$, and $F(x) = 4 - 2\sqrt{1 - x}$ for $x \in [0, 1]$. Since $F(1) = 4$, we have that the polynomial inequality (4) holds for every $m > 2n$, with $X_m = \{F^{-1}(4j/m)\}$, $j = 0, \dots, m$, where $F^{-1}(y) = y^2/4 - 1$ for $y \in [0, F(0)] = [0, 2]$, and $F^{-1}(y) = -y^2/4 + 2y - 3$ for $y \in [F(0), F(1)] = [2, 4]$. By Remark 1 we get a new family of optimal admissible meshes for the interval, $\mathcal{A}_n = X_{\lceil 2\mu n \rceil}$, with constant $C = \mu/(\mu - 1)$.

1.2 Trigonometric polynomials

In the case of trigonometric polynomials on *subintervals of the period* we can consider the reference interval $[-\omega, \omega]$, $0 < \omega \leq \pi$, that is $S_n = \mathbb{T}_n([-\omega, \omega)])$, since norming meshes are preserved by angle shifts. "Subperiodic" trigonometric approximation has received some attention in the recent literature, in view of the connection with multivariate polynomial approximation on arc based domains; cf. [6, 12].

In the paper [12] on optimal polynomial meshes, A. Kroó has shown that Chebyshev-Lobatto points for suitable $\mathcal{O}(n)$ degree are norming meshes for

$\mathbb{T}_n([-\omega,\omega)])$. The proof is based on the joint use of a Markov and of (a special formulation of) a Bernstein inequality which are known to hold in the subperiodic case, cf. [2, Ch.5]. These inequalities involve however constants that depend on ω and are not bounded as $\omega \to 0$. In particular, the optimal meshes $\mathcal{A}_n$, with constant $C = 2$, are $mn + 1$ Chebyshev-Lobatto points in $[-\omega,\omega]$, with $m > \pi \max\{\sqrt{A(\omega)}, 2B(\omega)\}$, where A and B appear in the Markov and Bernstein trigonometric inequality, respectively. In particular, $A(\omega) = 1/k + c(\pi-\omega)/\omega \to \infty$ as $\omega \to 0$; it appears in the trigonometric Markov inequality $\|t'\|_{[-\omega,\omega]} \leq A(\omega)n^2 \|t\|_{[-\omega,\omega]}$, valid for $n \geq k$ and for every $t \in \mathbb{T}_n([-\omega,\omega)])$, where $c \leq 16\pi$ is a suitable costant, cf. [2, E.15, p. 238].

Here, we show that, in view of Proposition 1, an optimal trigonometric mesh $\mathcal{A}_n$ on subintervals of the period, such that $\mathrm{card}(\mathcal{A}_n)/n$ is bounded independently of ω, can be constructed directly by the trigonometric Bernstein inequality (also known as Videnskii's inequality in the literature, cf. [2, E.19, p. 243])

$$|t'(\theta)| \leq \frac{n}{\sqrt{1-\cos^2(\omega/2)/\cos^2(\theta/2)}} \|t\|_{[-\omega,\omega]} \ , \ \theta \in (-\omega,\omega) \ , \ \ \forall t \in \mathbb{T}_n \ . \tag{12}$$

Consider the primitive

$$G(\theta) = G(\theta;\omega) = \int \frac{1}{\sqrt{1-\cos^2(\omega/2)/\cos^2(\theta/2)}} \, d\theta$$

$$= 2 \arcsin\left(\frac{\sin(\theta/2)}{\sin(\omega/2)}\right) \ , \ \ 0 < \omega \leq \pi \ . \tag{13}$$

Observing that $G(\pm\omega) = \pm\pi$, and using the terminology of Remark 1 with $[a,b] = [-\omega,\omega]$, we get

$$F(\theta) = G(\theta) + \pi \ , \ \ F(b) = 2\pi \ , \tag{14}$$

which is a strictly increasing and thus invertible function, with inverse

$$F^{-1}(y) = \{2 \arcsin(-\cos(y/2)\sin(\omega/2)) \ , \ y \in [0,2\pi]\} \ . \tag{15}$$

Then, inequality (4) holds for every $m > \pi n$, with the $m+1$ angles

$$X_m = \{F^{-1}(2\pi j/m)\} = \{2 \arcsin(-\cos(j\pi/m)\sin(\omega/2)) \ , \ j = 0,\ldots,m\} \ , \tag{16}$$

that are the image of the classical $m+1$ Chebyshev-Lobatto points for degree m by the nonlinear transformation $f(u) = 2\arcsin(u\sin(\omega/2))$, $u \in [-1,1]$; we recall that this transformation plays a key role also in the construction of near optimal sets for subperiodic trigonometric interpolation, cf. [6, 9].

By Remark 1 we get a new family of optimal trigonometric norming meshes for subintervals of the period,

$$\mathcal{A}_n = X_{\lceil \pi\mu n \rceil} = \{2\arcsin(-\cos(\pi j/\lceil \pi\mu n \rceil)\sin(\omega/2))\,,\ j = 0,\ldots,\lceil \pi\mu n \rceil\}\,, \tag{17}$$

with constant $C = \mu/(\mu-1)$ and cardinality $\lceil \mu\pi n \rceil + 1$. For example, taking $\mu = 2$ we get an optimal subperiodic trigonometric mesh with $C = 2$, as in [12], but cardinality of order $\mathcal{O}(n)$ bounded independently of ω.

1.3 Optimal polynomial meshes by arc blending

We discuss now an application of optimal subperiodic trigonometric meshes, namely the construction of optimal polynomial meshes on bidimensional domains which can be parametrized by *linear blending* of elliptical arcs. Let

$$P(\theta) = A_1\cos(\theta) + B_1\sin(\theta) + C_1\ ,\quad Q(\theta) = A_2\cos(\theta) + B_2\sin(\theta) + C_2\ , \tag{18}$$

$\theta \in [\alpha,\beta]$, be two trigonometric planar curves of degree one,

$$A_i = (a_{i1}, a_{i2})\ ,\quad B_i = (b_{i1}, b_{i2})\ ,\quad C_i = (c_{i1}, c_{i2})\ ,\quad i = 1,2\ , \tag{19}$$

being suitable bidimensional vectors (with A_i, B_i not all zero), with the important property that the curves are both parametrized on the *same angular interval* $[\alpha,\beta]$, $0 < \beta - \alpha \leq 2\pi$. It is not difficult to show, by a possible riparametrization with a suitable angle shift when A_i and B_i are not orthogonal, that these curves are arcs of two ellipses centered at C_1 and C_2, respectively.

Consider the domain

$$\Omega = \{(x,y) = \sigma(t,\theta) = tP(\theta) + (1-t)Q(\theta)\ ,\quad (t,\theta) \in [0,1]\times[\alpha,\beta]\}\ , \tag{20}$$

which is the transformation of the rectangle $[0,1]\times[\alpha,\beta]$ obtained by convex combination (linear blending) of the arcs $P(\theta)$ and $Q(\theta)$. This transformation can describe directly or by finite union several types of domain obtained as section of a disk (ellipse) by straight lines, such as for example circular (elliptical) segments, sectors (even asymmetric), zones, lenses. For an overview we refer the reader to [8], where arc blending is considered in the framework of numerical cubature (differently from [8], we do not need here that the transformation σ be also injective, but only surjective).

In the present context, the key observation is that a bivariate polynomial $p \in \mathbb{P}_n^2$ becomes by the change of variables (20) a mixed algebraic-trigonometric polynomial in a tensor product-space,

$$p(\sigma(t,\theta)) \in \mathbb{P}_n([0,1]) \bigotimes \mathbb{T}_n([\alpha,\beta])\ . \tag{21}$$

Consider the optimal polynomial mesh in $[0,1]$, given by $\nu n + 1$ Chebyshev-Lobatto points ($\nu > 1$)

$$\mathcal{T}_n = \{(\cos(j\pi/\nu n) + 1)/2\,,\ j = 0,\ldots,\nu n\}\ , \tag{22}$$

with constant $C_1 = 1/\cos(\pi/2\nu)$; cf. [5, 10]. Moreover, consider also the optimal trigonometric mesh in $[\alpha, \beta]$, given by the $\lceil \pi\mu n \rceil + 1$ angles ($\mu > 1$)

$$\Theta_n = \{2 \arcsin(-\cos(\pi j/\lceil \pi\mu n \rceil) \sin((\beta-\alpha)/2)) - (\alpha+\beta)/2\,,\ j = 0, \ldots, \lceil \pi\mu n \rceil\}\ , \tag{23}$$

with constant $C_2 = \mu/(\mu - 1)$; cf. (17). Then we can write

$$\|p\|_\Omega = \|p \circ \sigma\|_{[0,1]\times[\alpha,\beta]} \leq C_1 C_2 \, \|p \circ \sigma\|_{\mathcal{T}_n \times \Theta_n} = \|p\|_{\sigma(\mathcal{T}_n \times \Theta_n)}\ , \tag{24}$$

i.e., $\sigma(\mathcal{T}_n \times \Theta_n)$ is an optimal polynomial mesh for Ω, since it has cardinality $\mathcal{O}(n^2)$, with constant $C = C_1 C_2$. For example, taking $\mu = \nu = 2$, we get an optimal polynomial mesh for Ω with constant $C = 2\sqrt{2}$ and cardinality not greater than $(2n + 1)(\lceil 2\pi n \rceil + 1)$.

In Figure 1 below we show an example of such *optimal polynomial meshes* in two different sections of the unit disk, an *annular sector* and a *circular segment*, at degree $n = 4$; the blending transformation (20) is $\sigma(t, \theta) = (tr_1 + (1-t)r_2)(\cos(\theta), \sin(\theta))$, $r_1 = 0.3$, $r_2 = 1$, $[\alpha, \beta] = [-\pi/4, \pi/4]$ for the annular sector, and $\sigma(t, \theta) = (\cos(\theta), (2t - 1)\sin(\theta))$, $[\alpha, \beta] = [0, 3\pi/4]$ for the circular segment (notice that in the latter case the right cardinality is obtained by subtracting the $2n$ repetitions of the point $(1, 0)$).

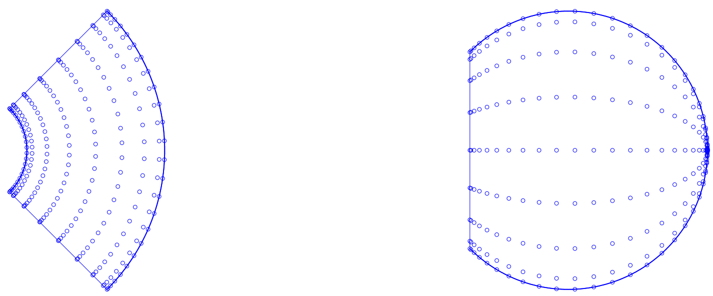

Figure 1: Optimal polynomial meshes at degree $n = 4$ for an annular sector (left, $9 \times 27 = 243$ points), and a circular segment (right, $9 \times 27 - 8 = 235$ points).

Remark 2 As already observed in [12], the availability of optimal subperiodic trigonometric meshes allows to construct optimal multivariate polynomial meshes, whenever algebraic polynomials on a multidimensional domain or manifold, by a suitable change of variables, belong to tensor-product spaces involving univariate trigonometric (and possibly algebraic) polynomials on suitable intervals.

For example, it is clear that a polynomial on the 2-sphere, in spherical coordinates, belongs to the tensor-product space of univariate trigonometric polynomials with respect to the longitude and the (co)latitude. Then, with the same reasoning developed above we can construct, by cartesian product of the angular meshes, *optimal polynomial meshes* on standard subregions of the sphere which are lat-long rectangles (up to rotations), such as for example *spherical caps* and *spherical zones*.

References

[1] T. Bloom, L. Bos, J.-P. Calvi and N. Levenberg, Polynomial interpolation and approximation in $\mathbb{C}^d$, Ann. Polon. Math. 106 (2012), 53–81.

[2] P. Borwein and T. Erdélyi, Polynomials and Polynomial Inequalities, Springer, New York, 1995.

[3] L. Bos, J.-P. Calvi, N. Levenberg, A. Sommariva and M. Vianello, Geometric Weakly Admissible Meshes, Discrete Least Squares Approximation and Approximate Fekete Points, Math. Comp. 80 (2011), 1601–1621.

[4] L. Bos, S. De Marchi, A. Sommariva and M. Vianello, Computing multivariate Fekete and Leja points by numerical linear algebra, SIAM J. Numer. Anal. 48 (2010), 1984–1999.

[5] L. Bos and M. Vianello, Low cardinality admissible meshes on quadrangles, triangles and disks, Math. Inequal. Appl. 15 (2012), 229–235.

[6] L. Bos and M. Vianello, Subperiodic trigonometric interpolation and quadrature, Appl. Math. Comput. 218 (2012), 10630–10638.

[7] J.P. Calvi and N. Levenberg, Uniform approximation by discrete least squares polynomials, J. Approx. Theory 152 (2008), 82–100.

[8] G. Da Fies, A. Sommariva and M. Vianello, Algebraic cubature by linear blending of elliptical arcs, preprint (available online at: http://www.math.unipd.it/∼marcov/publications.html).

[9] G. Da Fies and M. Vianello, On the Lebesgue constant of subperiodic trigonometric interpolation, J. Approx. Theory 167 (2013), 59–64.

[10] H. Ehlich and K. Zeller, Schwankung von Polynomen zwischen Gitterpunkten, Math. Z. 86 (1964), 41–44.

[11] K. Jetter, J. Stöckler and J.D. Ward, Norming sets and spherical cubature formulas, in: Advances in computational mathematics (Guangzhou, 1997), 237–244, Lecture Notes in Pure and Appl. Math. 202, Dekker, New York, 1999.

[12] A. Kroó, On optimal polynomial meshes, J. Approx. Theory 163 (2011), 1107–1124.

[13] F. Piazzon and M. Vianello, Analytic transformations of admissible meshes, East J. Approx. 16 (2010), 389–398.

[14] W. Plésniak, Nearly optimal meshes in subanalytic sets, Numer. Algorithms 60 (2012), 545-553.

[15] T.J. Rivlin and E.W. Cheney, A comparison of uniform approximations on an interval and a finite subset thereof, SIAM J. Numer. Anal. 3 (1966), 311–320.

[16] A. Sommariva and M. Vianello, Computing approximate Fekete points by QR factorizations of Vandermonde matrices, Comput. Math. Appl. 57 (2009), 1324–1336.