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DUBINER DISTANCE AND STABILITY OF LEBESGUE CONSTANTS

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Abstract. Some elementary inequalities, based on the non-elementary notion of Dubiner distance, show that norming sets for multivariate polynomials are stable under small perturbations in such a distance. As a corollary, we get a new result on the stability of Lebesgue constants.

1. Introduction

Some years ago, during an undergraduate course in numerical analysis, a clever student in mathematics asked me the following: we know that polynomial interpolation at the Chebyshev zeros or extrema is near-optimal concerning the Lebesgue constant; but what happens to such a constant if one fails to sample at those points by a small amount? A continuity argument (the Lebesgue constant is a continuous function of the nodes, see (1.3)-(1.4) below) gives an immediate qualitative answer. But can we give a quantitative estimate? A possible solution turns out to be nontrivial and comes from the theory of polynomial inequalities.

In this note we make a further step within this topic, that has been apparently and surprisingly overlooked for a long time in the framework of interpolation theory, while the study of interpolation sets and Lebesgue constants continues to be an open and active research field, especially in the multivariate case (cf., e.g., the quite recent papers [12, 14, 15, 19]).

As is well-known, the Lebesgue constant essentially measures the sensitivity of interpolation to function perturbation. Now, the question is: how much sensitive is the Lebesgue constant to *node perturbation*? This is a delicate matter, since as is known by the Runge phenomenon, the size of Lebesgue constants depends in a substantial way on the node distribution; cf., e.g., [24]. On the other hand, there is of course some applied interest in the problem, since in practice errors on the sampling locations are unavoidable. To our knowledge, only recently the question has attracted some interest and obtained some partial answers, in both the univariate and the multivariate frameworks, cf. [1, 2, 22].

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Following [22], here we pose the problem in the more general setting of the stability analysis of multivariate norming sets. Let $K \subset \mathbb{R}^d$ (or more generally $K \subset \mathbb{C}^d$) be a compact set, and $X \subset K$ a compact *norming set* for polynomials of degree at most n on K, namely

$$\|p\|_K \leq C\,\|p\|_X\;,\;\;\forall p \in \mathbb{P}_n^d(K)\;, \tag{1.1}$$

where $\|\cdot\|_I$ denotes the uniform norm of a bounded function on the (continuous or discrete) set I, and $p \in \mathbb{P}_n(K)$ the space of d-variate polynomials of (total) degree not exceeding n, restricted to K. Observe that necessarily

$$\mathrm{card}(X) \geq N = \dim(P_n^d(K))\;, \tag{1.2}$$

since X is determining for $\mathbb{P}_n^d(K)$, i.e. a polynomial in $\mathbb{P}_n^d(K)$ vanishing on X vanishes everywhere on K. This dimension is $N = \binom{n+d}{d}$ if K is polynomial determining (polynomials vanishing there vanish everywhere in $\mathbb{R}^d$), which happens for example when K has nonempty interior, but can be lower if K lies on an algebraic variety. For example we have that $N_n = \binom{n+3}{3} = (n+1)(n+2)(n+3)/6$ for the ball in $\mathbb{R}^3$ and $N_n = (n+1)^2$ for the sphere S^2; see, e.g., [13] for an algebraic geometry setting of the dimension problem.

In the discrete case, sequences of such norming sets are called *polynomial meshes* if $\mathrm{card}(X)$ grows algebraically with n and C is independent of n; if $C = C_n$ is not constant but grows subexponentially with n, then one speaks of a "weakly admissible" polynomial mesh. Polynomial meshes, formally defined in the seminal paper [11] by Calvi and Levenberg, are in practice good discrete models of compact sets, concerning for example polynomial fitting (least squares, interpolation on discrete extremal subsets), polynomial optimization, as well as other aspects of polynomial approximation theory; we refer the reader, e.g., to [5, 6, 17, 20, 21, 23, 25] and the references therein.

In the case when $\mathrm{card}(X) = N$ it is interesting to see (1.1) in terms of the Lebesgue constant of the unisolvent interpolation set $X = \{\xi_1, \ldots, \xi_N\}$, namely

$$\Lambda_X = \|\mathcal{L}_X\| = \sup_{f \in C(K), f \neq 0} \frac{\|\mathcal{L}_X f\|_K}{\|f\|_K} = \max_{x \in K} \sum_{i=1}^{N} |\ell_i(x)|\;, \tag{1.3}$$

where

$$\ell_i(x) = \frac{det(V(\xi_1, \ldots, \xi_{i-1}, x, \xi_{i+1}, \ldots, \xi_N))}{det(V(\xi_1, \ldots, \xi_N))} \tag{1.4}$$

are the cardinal Lagrange polynomials, $\ell_i(\xi_j) = \delta_{ij}$, $V(\xi_1, \ldots, \xi_N)$ denotes the corresponding $N \times N$ Vandermonde matrix, and $\mathcal{L}_X f(x) = \sum_{i=1}^{N} f(\xi_i)\,\ell_i(x)$. Indeed it is not difficult to check that (1.1) holds with $C = \Lambda_X$, and conversely if (1.1) holds for some $C > 0$ then $\Lambda_X \leq C$ (see the arguments in the proof of Corollary 1 below).

In [22] it is proved that there is of stability property of norming set inequalities with respect to the Euclidean distance, whenever a Markov polynomial inequality

$$\|\nabla p(x)\|_2 \leq Mn^r\,\|p\|_K\;,\;\;\forall p \in \mathbb{P}_n^d(K)\;, \tag{1.5}$$

holds on K. For example, on *real convex bodies*, a perturbation of the nodes in the Euclidean distance not exceeding

$$\delta_{eu} = \frac{\alpha}{Mn^2C} = \frac{\alpha\,W(K)}{4n^2C}\;,\;\;\alpha \in [0,1)\;, \tag{1.6}$$

is still a norming set with $C/(1-\alpha)$ replacing C, where $W(K)$ is the width of K (the minimum distance between parallel supporting hyperplanes), and the factor 4 in the denominator can be replaced by 2 on centrally symmetric bodies (cf. [26]).

The role of the Euclidean distance, however, is only technical being motivated by the use of differential calculus (mean value theorem or Taylor formula) in the estimates. Other notions of distance on a compact set appear more suited for dealing with polynomials, such as the *Dubiner distance* (introduced in the seminal paper [16])

$$dub(x,y) = \sup_{deg(p)\geq 1,\, \|p\|_K\leq 1} \left\{ \frac{|\arccos(p(x)) - \arccos(p(y))|}{deg(p)} \right\} \ , \ \ x,y \in K \ . \qquad (1.7)$$

Among its properties, we recall that $dub(x,y) = dub_K(x,y)$ is determined modulo invertible affine transformations. Indeed, if T is such a transformation, then it is easily checked that $dub_{T(K)}(x,y) = dub_K(T^{-1}(x), T^{-1}(y))$.

The Dubiner distance plays a deep role in polynomial approximation. For example, it can be proved that good interpolation points for degree n on some standard real compact sets are spaced proportionally to $1/n$ in this distance. This happens with the Morrow-Patterson and the Padua interpolation points on the square [10], or the Fekete points on the cube, ball or simplex (in any dimension), cf. [7].

Moreover, it is easy to prove that a subset $X \subset K$ (not necessarily discrete) with covering radius in the Dubiner distance less than θ/n, $\theta \in (0, \pi/2)$, is a norming set for $\mathbb{P}_n^d(K)$ with $C = 1/\cos(\theta)$; cf. [3, 18]. This property has opened the way for an effective use of norming sets in the framework of polynomial optimization, cf. e.g. [23, 25].

Unfortunately, the Dubiner distance has been computed analytically only on the d-dimensional cube, ball and simplex, on the sphere S^{d-1}, and more recently in the case of univariate trigonometric polynomials (even on subintervals of the period); cf. [7, 25]. In particular, on the cube it turns out that

$$dub(x,y) = \max_{1\leq i\leq d} \{|\arccos(x_i) - \arccos(y_i)|\} \ , \ \ x,y \in [-1,1]^d \ , \qquad (1.8)$$

due to the fact that, by the Van der Corput-Schaake inequality, on $[-1,1]$ the Dubiner distance is simply the arccos distance, whereas on the sphere it turns out to be just the standard geodesic distance.

2. Stability of norming sets

Whenever the Dubiner distance is known or can be at least estimated, we can measure more precisely the admissible perturbation neighborhood in order to preserve a norming set inequality. This is indeed the meaning of the following elementary Proposition.

Proposition 2.1. *Let $X \subset K \subset \mathbb{C}^d$ be a finite norming set in the uniform norm for $\mathbb{P}_n^d(K)$, with constant $C > 0$; cf. (1.1). Let*

$$\delta_{dub} = \frac{\alpha}{n\,C} \ , \ \ \alpha \in [0,1) \ , \qquad (2.1)$$

and consider a perturbed set $\tilde{X} \subset K$ constructed by choosing for every $\xi \in X$ a point $\tilde{\xi}$ such that $dub(\xi, \tilde{\xi}) \leq \delta_{dub}$.

Then $\tilde{X}$ is a norming set, such that card$(\tilde{X}) \leq$ card(X) and

$$\|p\|_K \leq \frac{C}{1-\alpha} \|p\|_{\tilde{X}} \;, \;\; \forall p \in \mathbb{P}_n^d(K) \;. \tag{2.2}$$

Proof. First, we observe that for every $q \in \mathbb{P}_n^d(K)$, $\|q\|_K \leq 1$, and for every $x, y \in K$, in view of the definition of Dubiner distance and the mean value theorem for the cosine function

$$|q(x) - q(y)| = |\cos(\arccos(q(x))) - \cos(\arccos(q(y)))|$$
$$\leq |\arccos(q(x)) - \arccos(q(y))| \leq deg(q)\, dub(x, y) \;,$$

so that, for every $p \in \mathbb{P}_n^d(K)$,

$$|p(x) - p(y)| \leq deg(p)\, dub(x, y)\, \|p\|_K \;.$$

Hence, taking a point $\xi_* = \xi_*(p) \in X$ where the maximum modulus of p on X is attained, and $\tilde{\xi_*} \in \tilde{X}$ such that $dub(\xi_*, \tilde{\xi_*}) \leq \delta_{dub}$, we can write the chain of inequalities

$$\|p\|_K \leq C \|p\|_X = C |p(\xi_*)| \leq C |p(\tilde{\xi_*})| + C |p(\xi_*) - p(\tilde{\xi_*})|$$
$$\leq C \|p\|_{\tilde{X}} + \frac{deg(p)}{n} \alpha \|p\|_K \leq C \|p\|_{\tilde{X}} + \alpha \|p\|_K \;,$$

that is (2.2). $\square$

We can now state and prove a stability result for Lebesgue constants.

Corollary 2.2. *Let $X \subset K \subset \mathbb{C}^d$ be a unisolvent interpolation set for $\mathbb{P}_n^d(K)$, and denote by $\Lambda_X = \|\mathcal{L}_X\|$ its Lebesgue constant, cf. (1.3). Let*

$$\delta_{dub} = \frac{\alpha}{n\, \Lambda_X} \;, \;\; \alpha \in [0, 1) \;, \tag{2.3}$$

and consider a perturbed set $\tilde{X} \subset K$ constructed by choosing for every $\xi \in X$ a point $\tilde{\xi}$ such that $dub(\xi, \tilde{\xi}) \leq \delta_{dub}$.

Then $\tilde{X}$ is a unisolvent interpolation set itself, with Lebesgue constant

$$\Lambda_{\tilde{X}} = \|\mathcal{L}_{\tilde{X}}\| \leq \frac{\Lambda_X}{1-\alpha} \;. \tag{2.4}$$

Proof. Clearly X is a norming set for $\mathbb{P}_n^d(K)$ with $C = \Lambda_X$ since

$$\|p\|_K = \|\mathcal{L}_X p\|_K \leq \Lambda_X \|p\|_X \;.$$

By Proposition 1 and assumption (2.3), we get

$$\|p\|_K \leq \frac{\Lambda_X}{1-\alpha} \|p\|_{\tilde{X}} \;,$$

which shows that $\tilde{X}$ is unisolvent itself, being $\mathbb{P}_n^d(K)$-determining. Concerning its Lebesgue constant, we can write for every $f \in C(K)$

$$\|\mathcal{L}_{\tilde{X}} f\|_K \leq \frac{\Lambda_X}{1-\alpha} \|\mathcal{L}_{\tilde{X}} f\|_{\tilde{X}} = \frac{\Lambda_X}{1-\alpha} \|f\|_{\tilde{X}} \leq \frac{\Lambda_X}{1-\alpha} \|f\|_K \;,$$

which implies (2.4). $\square$

Though the proofs of Proposition 1 and Corollary 1 are completely elementary, they are based on the non-elementary notion of Dubiner distance, and are able to give a new insight into the perturbation analysis of norming sets and the sensitivity of Lebesgue constants to node perturbation.

In order to illustrate the perturbation results, we treat two relevant bivariate cases, the square and the disk, by some examples.

The Dubiner distance on the square $K = [-1, 1]^2$ is given by (1.8) for $d = 2$. In Figure 1 we consider a norming grid of $(2n+1) \times (2n+1)$ Chebyshev-Lobatto points for degree $n = 3$ (whose constant is $C = \sqrt{2}$, cf. [9]), together with the Euclidean and the Dubiner perturbation neighborhoods (with $\alpha = 0.5$), namely the metric balls

$$B_2(\xi, \delta_{eu}) = \{x \in K : \|x - \xi\|_2 \le \delta_{eu}\} \ ,$$
$$B_{dub}(\xi, \delta_{dub}) = \{x \in K : dub(x, \xi) \le \delta_{dub}\} \ . \tag{2.5}$$

Since $W(K) = 2$ and the factor 4 in the denominator of (1.6) can be substituted by 2 due to the domain's central symmetry, we have that

$$\delta_{eu} = \frac{\alpha}{n^2 C} = \frac{1}{n^2\, 2\sqrt{2}} \ , \ \text{ whereas } \ \delta_{dub} = \frac{\alpha}{n\, C} = \frac{1}{n\, 2\sqrt{2}} \ . \tag{2.6}$$

Observe that the Dubiner neighborhoods are rectangles, that enclose the Euclidean neighborhoods except at the boundary. A similar situation (but with smaller neighborhoods) occurs for the Padua interpolation points in Figure 2, where $C = \Lambda_X$ is the Lebesgue constant, which are the only known near-optimal multivariate interpolation set, with a $\mathcal{O}(\log^2 n)$ Lebesgue constant. A remarkable feature of the Padua points, and ultimately the key for their interpolation properties, is that they are generated as self-intersection points and boundary contact points of the algebraic curve $T_n(x_1) + T_{n+1}(x_2) = 0$, where T_n is the n-th degree Chebyshev polynomial of the first kind, $T_n(t) = \cos(n \arccos(t))$, $t \in [-1, 1]$ (the curve is plotted in Figure 2 for $n = 3$); cf. [4, 10].

The fact that the Dubiner neighborhoods shrink at the boundary is due to the features of the univariate Dubiner distance in $[-1, 1]$ (the arccos distance), that behaves like the Euclidean distance at the center and like its square root at the boundary. More precisely, the width of the boundary Dubiner rectangles is in general

$$\begin{aligned} \text{length}\,(\{t : \arccos(t) \in [0, \delta_{dub}]\}) &= \text{length}\,(\{t : \arccos(t) \in [\pi - \delta_{dub}, \pi]\}) \\ &= 1 - \cos(\delta_{dub}) \le \frac{1}{2}\,\delta_{dub}^2 = \frac{\alpha}{2C}\,\delta_{eu} < \frac{\delta_{eu}}{2} \ , \end{aligned} \tag{2.7}$$

see Figure 3 for an example. We stress that a similar configuration of the Dubiner perturbation neighborhoods (in this case rectangular parallelepipeds) occurs for a Chebyshev-Lobatto norming grid in the cube $[-1, 1]^3$, with shrinking at the boundary in the direction of the coordinate axes, and maximal shrinking in one direction at the faces, in two directions at the edges and in all three directions at the vertices.

In Figures 4-7 we illustrate the case of the disk. We recall that the Dubiner distance on the unit disk $K = B_2(0, 1)$ is

$$dub(x, y) = \arccos\left(\langle x, y\rangle + \sqrt{1 - \|x\|_2^2}\,\sqrt{1 - \|y\|_2^2}\right) \ , \tag{2.8}$$

$\langle x, y\rangle$ denoting the Euclidean scalar product, cf. [8]. Geometrically, it is the geodesic distance of the points obtained by lifting x, y to the upper hemisphere. The Dubiner balls are then projections of spherical caps of the hemisphere onto the equatorial plane. If the Dubiner distance of the ball center from the boundary does

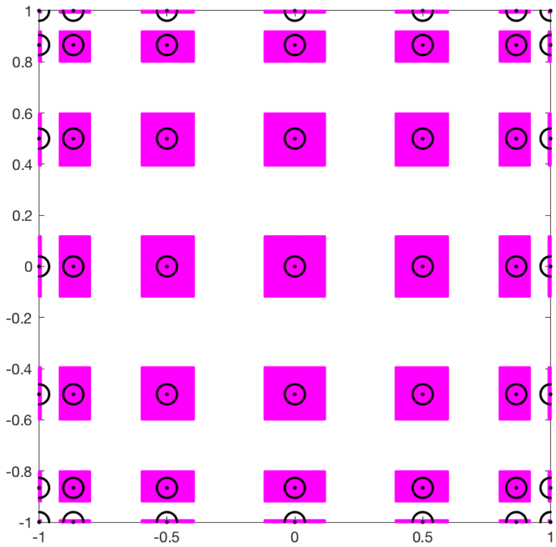

FIGURE 1. Chebyshev norming grid on the square and perturbation neighborhoods with $\alpha = 0.5$ in the Euclidean distance (black circles) and in the Dubiner distance (purple rectangles), for degree $n = 3$ (49 points, $C = \sqrt{2}$).

not exceed the ball radius, then such a Dubiner ball is an ellipse, since it corresponds to an affine transformation of a circle (the cap boundary). In the limit case where the ball center lies on the disk boundary, the ball is a circular segment; see Figure 5.

In Figure 4 we display two polar norming grids for the disk with constant $C = 2$ and their perturbation neighborhoods, corresponding to $(2n + 1)$ Chebyshev-Lobatto diameter points and $4n$ equally spaced angles, for degree $n = 3$ and degree $n = 6$ (37 and 145 points, respectively); cf. [9]. In Figure 5 we see a detail near the boundary. In Figure 6 we show 15 quasi-Lebesgue interpolation points of the disk for degree $n = 4$ and their perturbation neighborhoods. The points are taken from [19], where the best known interpolation sets for the disk up to degree 25 have been computed, by approximate minimization of the Lebesgue constant.

The situation is similar to that of the square, with the Dubiner neighborhoods enclosing the Euclidean neighborhoods except near the boundary (see Figure 7, and Figure 5 for the shape of a boundary point neighborhood). Observe in Figure 4-bottom that the Dubiner neighborhoods can overlap. This however can happen only for a norming set that is not an interpolation set (otherwise, we could eliminate some points keeping a norming set but violating condition (1.2)).

The examples and considerations above suggest that the "right" perturbation neighborhoods can be obtained as the union of the Euclidean and the Dubiner ones, namely

$$U(\xi; n, \alpha, K) = B_2(\xi, \delta_{eu}) \cup B_{dub}(\xi, \delta_{dub}) \ , \tag{2.9}$$

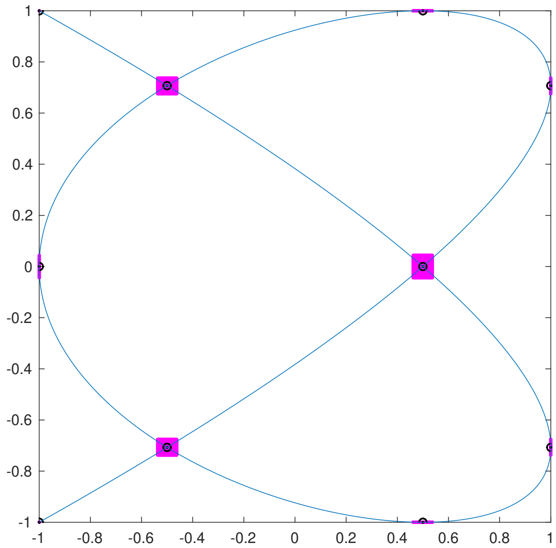

FIGURE 2. Ten Padua points on the square (with their generating curve) and perturbation neighborhoods with $\alpha = 0.5$ in the Euclidean distance (black circles) and in the Dubiner distance (purple rectangles), for degree $n = 3$ ($\Lambda_X \approx 3.78$).

cf. (1.6), (2.1), (2.5). Indeed, for K convex body the key inequality $|p(\xi_*) - p(\tilde{\xi_*})| \leq \alpha \, \|p\|_K$ in the proof of Proposition 1 above is true if either $\tilde{\xi_*}$ is in the Dubiner or in the Euclidean neighborhood (cf. also the proof of [22, Prop. 1]).

Such "union" neighborhoods can be nonconvex in some cases, but they are the largest known regions where perturbed sampling still ensures an increase at most by a factor $1/(1-\alpha)$ of the Lebesgue constant (or in general of the norming set constant).

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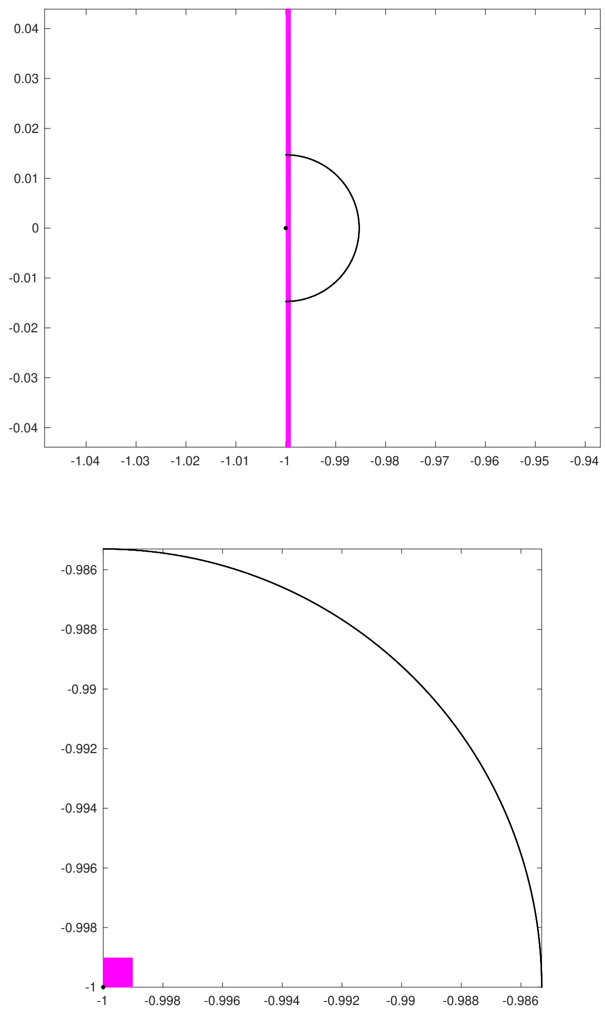

FIGURE 3. Details of Figure 2 (side point and corner point).

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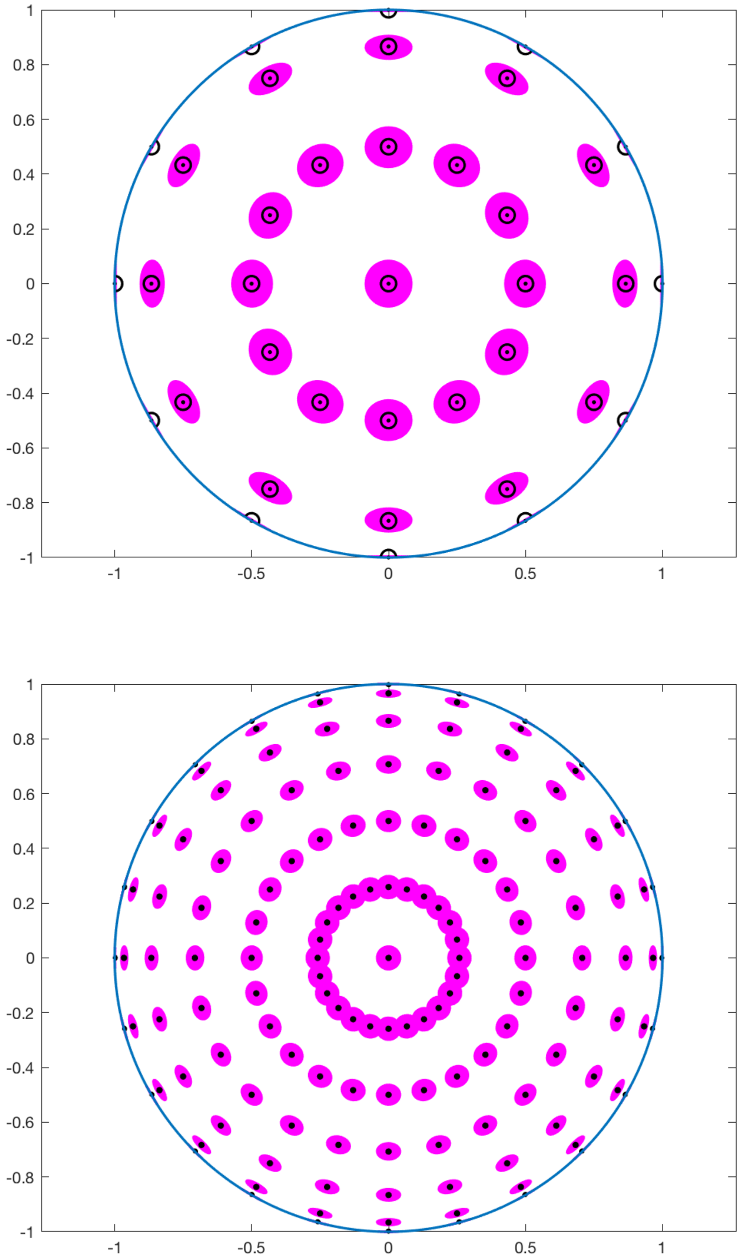

FIGURE 4. Polar norming grids on the disk with $C = 2$ and perturbation neighborhoods with $\alpha = 0.5$ in the Euclidean distance (black circles) and in the Dubiner distance (purple regions), for degree $n = 3$ (top, 37 points) and $n = 6$ (bottom, 145 points).

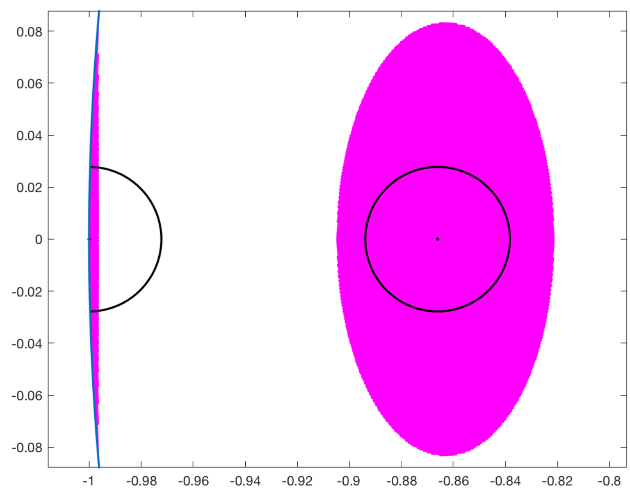

FIGURE 5. Detail of Figure 4-top at the boundary.

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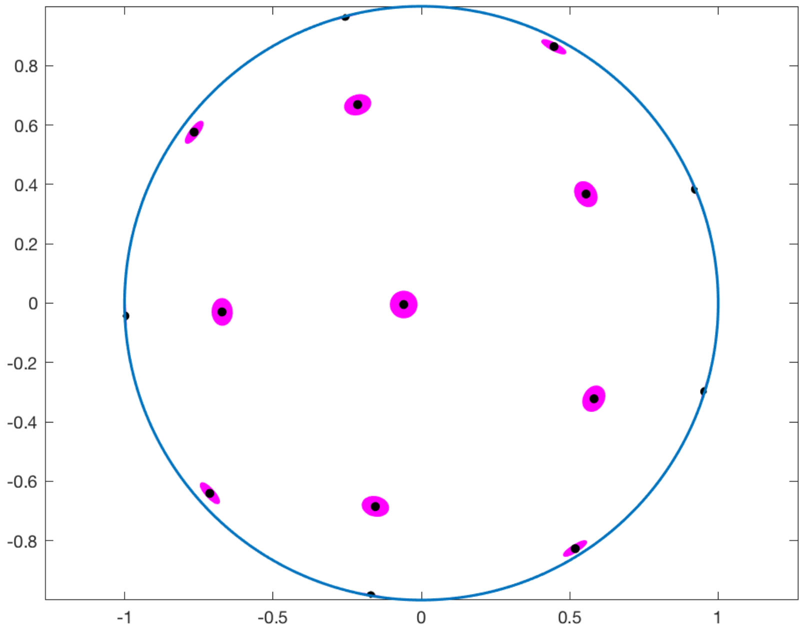

Figure 6. Fifteen quasi-Lebesgue points on the disk and their perturbation neighborhoods with $\alpha = 0.5$ in the Euclidean distance (black circles) and in the Dubiner distance (purple regions), for degree $n = 4$ ($\Lambda_X \approx 2.96$).

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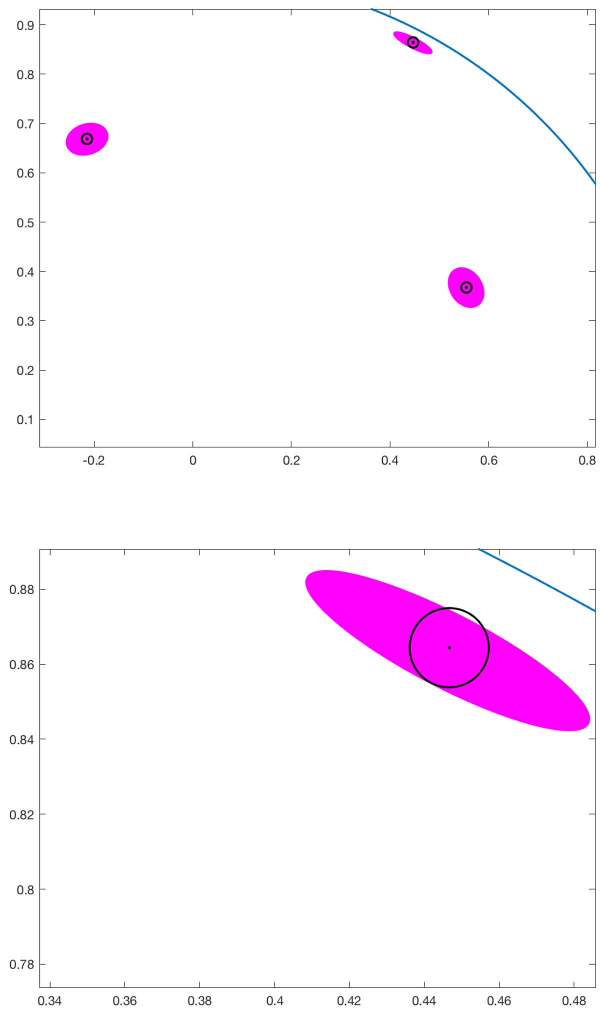

FIGURE 7. Details of Figure 6.