Jacobi norming meshes *

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Abstract

We prove by Bernstein inequality that Gauss-Jacobi(-Lobatto) nodes of suitable order are L^∞ norming meshes for algebraic polynomials, in a wide range of Jacobi parameters. A similar result holds for trigonometric polynomials on subintervals of the period, by a nonlinear transformation of such nodes and Videnskii inequality.

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1 Univariate L^∞ norming meshes

In this note we study L^∞ norming meshes for univariate function spaces, in particular Jacobi norming meshes for univariate algebraic polynomials, and for trigonometric polynomials on subintervals of the period.

Given a sequence $\{S_n\}$ of finite dimensional spaces of real-valued (or complex-valued) continuous functions on a compact interval $[a, b]$, $S_n \subset C([a, b])$, we term L^∞ norming mesh a sequence of sets $X_n \subset [a, b]$ such that

$$\|p\|_{[a,b]} \leq C\, \|p\|_{X_n}\ ,\ \ \forall p \in S_n\ , \tag{1}$$

for some constant $C > 0$ (that we term "norming constant"), where $\|f\|_D$ denotes the L^∞ norm of a bounded function f on a continuous or discrete set D. Since X_n turns out to be S_n-determining (i.e., a function $p \in S_n$ vanishing there vanishes everywhere in $[a, b]$), necessarily $\text{card}(X_n) \geq \dim(S_n)$.

We begin with the following Lemma, whose elementary proof follows the lines of [15] via Bernstein-like inequalities.

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Lemma 1 *Let $\{S_n\}$ be a sequence of finite dimensional spaces of differentiable functions defined in the compact interval $[a,b]$. Assume that for any $p \in S_n$ and $x \in (a,b)$ the following Bernstein-like inequality holds*

$$|p'(x)| \le \phi_n(x)\, \|p\|_{[a,b]} \ , \ \ \forall x \in (a,b) \ , \tag{2}$$

where $\phi_n \in L^1_+(a,b)$, and define

$$F_n(x) = \int_a^x \phi_n(s)\, ds \ . \tag{3}$$

Let $\{\xi_i = \xi_i^{(n)}\ ,\ i = 1,\ldots,N = N_n\}$ be a set of points such that $a \le \xi_1 < \xi_2 < \cdots < \xi_N \le b$, and for a fixed $\sigma \in (0,1)$

$$\max\left\{F_n(\xi_1)\,,\, F_n(b) - F_n(\xi_N)\,,\, \frac{1}{2}\max_i \delta_i\right\} \le \sigma \ , \tag{4}$$

where $\delta_i = F_n(\xi_{i+1}) - F_n(\xi_i)$, $i = 1,\ldots,N-1$.

Then, $X_n = \{\xi_i^{(n)}\}$, $n = 1,2,\ldots$, is a norming mesh for S_n on $[a,b]$, such that

$$\|p\|_{[a,b]} \le \frac{1}{1-\sigma}\, \|p\|_{X_n} \ , \ \ \forall p \in S_n \ . \tag{5}$$

Proof. Fix $p \in S_n$ and $x \in [a,b]$, and let $\xi_i \in X_n$. By the fundamental theorem of calculus and the Bernstein-like inequality

$$|p(x)| = \left|p(\xi_i) + \int_{\xi_i}^x p'(s)\, ds\right| \le |p(\xi_i)| + \int_{\min\{\xi_i,x\}}^{\max\{\xi_i,x\}} |p'(s)|\, ds$$

$$\le |p(\xi_i)| + \|p\|_{[a,b]} \int_{\min\{\xi_i,x\}}^{\max\{\xi_i,x\}} \phi_n(s)\, ds$$

$$= |p(\xi_i)| + \|p\|_{[a,b]}\, |F_n(x) - F_n(\xi_i)| \ .$$

Now, $F_n(x)$ is a nondecreasing function with range $[0, F_n(b)]$, and $y_n = F_n(x)$ belongs to at least one (and at most two) of the (possibly degenerate) intervals $[0, F_n(\xi_1)]$, $[F_n(\xi_N), F_n(b)]$, $[F_n(\xi_i), F_n(\xi_{i+1})]$, $1 \le i \le N-1$. Thus there exists an index $i = i(x,n)$ such that $\left|y_n - F_n\left(\xi_{i(x,n)}\right)\right| \le \sigma$, from which we get

$$|p(x)| \le |p(\xi_{i(x,n)})| + \sigma\|p\|_{[a,b]} \le \|p\|_{X_n} + \sigma\|p\|_{[a,b]} \ ,$$

and hence (5). $\square$

Remark 1 We stress that the present formulation is more general than [15, Prop. 1]. In fact, there the range $[0, F_n(b)]$ is partitioned by equally spaced points (including the endpoints), whereas here the distribution of the points $\{F_n(\xi_i)\}$ has only to satisfy inequality (4) (and the endpoints are not necessarily included).

Remark 2 In the case when

$$\phi_n(x) = n\,\phi(x)\ ,\ \ \phi \in L^1_+(a,b)\ , \tag{6}$$

which is relevant to polynomial and trigonometric spaces in view of the classical Bernstein and Videnskii inequalities (see the next subsections), we have

$$F_n(x) = n\,F(x)\ ,\ \ F(x) = \int_a^x \phi(s)\,ds\ , \tag{7}$$

and condition (4) becomes

$$\max\left\{F(\xi_1)\,,\,F(b)-F(\xi_N)\,,\,\frac{1}{2}\max_i\,(F(\xi_{i+1})-F(\xi_i))\right\} \le \frac{\sigma}{n}\ . \tag{8}$$

1.1 Jacobi polynomial meshes

We focus now on norming meshes of Jacobi type for algebraic polynomials, namely for $S_n = \mathbb{P}_n = \text{span}\{1, x, x^2, \dots, x^n\}$. Let $P_N^{(\alpha,\beta)}$ be the Jacobi orthogonal polynomial of degree N, where $\alpha, \beta > -1$, let $\mathcal{G}_N^{(\alpha,\beta)}$ be the N Gauss-Jacobi points (the zeros of $P_N^{(\alpha,\beta)}$), and let $\mathcal{GL}_N^{(\alpha,\beta)}$ be the N Gauss-Jacobi-Lobatto points (the zeros of $P_{N-2}^{(\alpha,\beta)}$ together with the endpoints ± 1); cf., e.g., [14] and [12, §18].

Proposition 1 *The set sequence* $\left\{\mathcal{G}_{mn}^{(\alpha,\beta)}\right\}$*,* $n \ge 1$*, is a polynomial mesh on* $[-1,1]$*, namely*

$$\|p\|_{[-1,1]} \le C_1\,\|p\|_{\mathcal{G}_{mn}^{(\alpha,\beta)}}\ ,\ \ C_1 = \frac{m}{m-\nu\pi}\ ,\ \ \forall p \in \mathbb{P}_n\ , \tag{9}$$

provided that $m > \nu\pi$*, where*

$$\nu = \nu(\alpha,\beta) = \begin{cases} 1 & \alpha,\beta \in \left[-\frac{1}{2},\frac{1}{2}\right] \\ \max\left\{\frac{\alpha+2}{2},\frac{\beta+3}{2}\right\}\ , & \alpha > \frac{1}{2}\,,\ \textit{or}\ \ 0 < \alpha \le \frac{1}{2}\ \ \textit{and}\ \ |\beta| > \frac{1}{2} \\ \max\left\{\frac{\beta+2}{2},\frac{\alpha+3}{2}\right\}\ , & \beta > \frac{1}{2}\,,\ \textit{or}\ \ 0 < \beta \le \frac{1}{2}\ \ \textit{and}\ \ |\alpha| > \frac{1}{2} \end{cases} \tag{10}$$

Similarly, for the set sequence $\left\{\mathcal{GL}^{(\alpha,\beta)}\right\}$*,* $n \ge 1$*, we have that*

$$\|p\|_{[-1,1]} \le C_2\,\|p\|_{\mathcal{GL}_{mn+2}^{(\alpha,\beta)}}\ ,\ \ C_2 = \frac{m}{m-\nu\pi/2}\ ,\ \ \forall p \in \mathbb{P}_n\ , \tag{11}$$

provided that $m > \nu\pi/2$*.*

Remark 3 Observe that in Proposition 1 not all parameters $\alpha, \beta > -1$ are covered. There is a small region of excluded nonpositive parameters, that is $(\alpha, \beta) \in (-1, -1/2) \times (-1, 0] \cup [-1/2, 0] \times (-1, -1/2)$, where precise quantitative bounds such as (12), (13) and (14), were reported in [8] to be (and seem still to be) missing in the literature.

Proof of Proposition 1. Let $\theta_{N,i}$, $i = 1, \ldots, N$, denote the zeros of $P_N^{(\alpha,\beta)}(\cos(\theta))$ with $0 < \theta_{N,1} < \cdots < \theta_{N,N} < \pi$ (i.e., $\mathcal{G}_N^{(\alpha,\beta)} = \{\cos(\theta_{N,i})\}$). There is a vast literature on estimating the zeros of Jacobi polynomials, with bounds that are typically valid in (more or less) restricted ranges of parameters; cf., e.g., [6, 7, 8, 12, 14] with the references therein. For $\alpha, \beta \in [-\frac{1}{2}, \frac{1}{2}]$ the classical inequality

$$\frac{(i-1/2)\pi}{N+1/2} \leq \theta_{N,i} \leq \frac{i\pi}{N+1/2}\ , \quad i = 1, \ldots, N \tag{12}$$

holds, cf. [12, §18.16]. Now, by Bernstein polynomial inequality in (6) we have $\phi(s) = 1/\sqrt{1-s^2}$, $s \in (a, b) = (-1, 1)$, and $F(x) = \pi - \arccos(x)$ in (7). Setting $\xi_{N-i+1} = \cos(\theta_{N,i})$, $i = 1, \ldots, N$, and taking $N = mn$ in (12), we can write the estimate

$$\max\left\{F(\xi_1)\,,\, F(1) - F(\xi_N)\,,\, \frac{1}{2}\max_i\left(F(\xi_{i+1}) - F(\xi_i)\right)\right\}$$

$$= \max\left\{\pi - \theta_{N,N}\,,\, \theta_{N,1}\,,\, \frac{1}{2}\max_i\left(\theta_{N,i+1} - \theta_{N,i}\right)\right\}$$

$$\leq \max\left\{\frac{\pi/2}{N+1/2}, \frac{\pi}{N+1/2}, \frac{3\pi/4}{N+1/2}\right\} = \frac{\pi}{N+1/2} < \frac{\pi}{N} = \frac{\pi}{mn}\ ,$$

and thus by (8) and Lemma 1 we get (9) provided that $\sigma = \pi/m < 1$.

On the other hand, for $\alpha > 0$, $\beta > -1$ the estimate

$$\frac{(i-1+\alpha/2)\pi}{\tilde{N}} < \theta_{N,i} < \frac{(i+\alpha/2)\pi}{\tilde{N}}\ , \quad i = 1, \ldots, N\ , \tag{13}$$

where $\tilde{N} = N + (\alpha + \beta + 1)/2$, was obtained in [8, Lemma 4.1], along with

$$\frac{(i+(\alpha-1)/2)\pi}{\tilde{N}} < \theta_{N,i} < \frac{(i+(\alpha+1)/2)\pi}{\tilde{N}}\ , \quad i = 1, \ldots, N\ , \tag{14}$$

which is valid for $\alpha > -1$, $\beta > 0$. Then for $\alpha > 1/2$, or $0 < \alpha \leq 1/2$ and $|\beta| > 1/2$, and $N = mn$, we get

$$\max\left\{\pi - \theta_{N,N}\,,\, \theta_{N,1}\,,\, \frac{1}{2}\max_i\left(\theta_{N,i+1} - \theta_{N,i}\right)\right\} < \pi\ \max\left\{\frac{\beta+3}{2\tilde{N}}, \frac{\alpha+2}{2\tilde{N}}, \frac{1}{\tilde{N}}\right\}$$

$$= \pi\ \max\left\{\frac{\beta+3}{2\tilde{N}}, \frac{\alpha+2}{2\tilde{N}}\right\} = \frac{\nu\pi}{\tilde{N}} < \frac{\nu\pi}{N} = \frac{\nu\pi}{mn}\ ,$$

from which (9) follows, provided that $\sigma = \nu\pi/m < 1$. A chain of estimates of the same kind gives the third instance in (10) when $\beta > 1/2$, or $0 < \beta \leq 1/2$ and $|\alpha| > 1/2$, in view of (14).

The proof in the case of the Gauss-Jacobi-Lobatto points is similar since the same zeros are involved, with the difference that the interval endpoints belong to the family, so that $F(\xi_1) = 0 = F(1) - F(\xi_{mn+2})$ and hence a factor $1/2$ appears in the final estimate, which gives the condition $\sigma = \nu\pi/(2m) < 1$ when applying Lemma 1. $\square$

Remark 4 The fact that $n+1$ Gauss-Jacobi(-Lobatto) points form good interpolation meshes (i.e., have a slowly increasing Lebesgue constant Λ_n) is a well-known result by Szegö (cf., e.g., [11]), who showed that $\Lambda_n = \mathcal{O}(\log n)$ for $-1 < \alpha, \beta \leq 1/2$ and $\Lambda_n = \mathcal{O}(n^{\gamma+1/2})$, $\gamma = \max(\alpha, \beta)$, otherwise. Clearly, the polynomial inequality $\|p\|_{[-1,1]} \leq C_n \|p\|_{X_n}$, $C_n = \Lambda_n$, holds for every $p \in \mathbb{P}_n$, with $X_n = \mathcal{G}_{n+1}^{(\alpha,\beta)}$ or $X_n = \mathcal{GL}_{n+1}^{(\alpha,\beta)}$. Here we have shown that, for the displayed ranges of α, β, a suitable number of Gauss-Jacobi(-Lobatto) points gives instead a polynomial inequality with $C_n \equiv C$.

Indeed, estimate (11) was already obtained by Bernstein inequality in [15] in the special case $\alpha = \beta = -1/2$ (the Chebyshev-Lobatto points of the first kind). As recalled there, in this instance there is a slightly tighter classical estimate by Ehlich and Zeller [5] (see also [2]). On the other hand, even though they are not fully tight, the present estimates (9)-(11) hold for a wide range of parameters α, β, that includes all nonnegative couples, for example the ultraspherical instances of Gauss-Legendre(-Lobatto) points ($\alpha = \beta = 0$), and Chebyshev(-Lobatto) points of the second kind ($\alpha = \beta = 1/2$).

1.2 Jacobi trigonometric meshes

"Subperiodic" trigonometric approximation, that is approximation by trigonometric polynomials on subintervals of the period, has received some attention recently, especially for its connections with multivariate polynomial approximation and cubature on domains defined by circular arcs, such as sections of disk, sphere, torus; cf., e.g., [3, 4, 10, 13] and the references therein.

In the subperiodic setting, L^∞ norming meshes for trigonometric polynomials have been studied for example in [9, 15]. Here, with reference to Lemma 1 we have that

$$S_n = \mathbb{T}_n([-\omega, \omega]) = \text{span}\{1, \cos(u), \sin(u), \ldots, \cos(nu), \sin(nu),\ u \in [-\omega, \omega]\},$$

the space of trigonometric polynomials of degree not exceeding n restricted to an angular interval $[-\omega, \omega]$ with $0 < \omega \leq \pi$. The role of (2) is played by the following Videnskii inequality, valid for any $t \in \mathbb{T}_n([-\omega, \omega])$

$$|t'(u)| \leq \frac{n}{\sqrt{1 - \cos^2(\omega/2)/\cos^2(u/2)}} \|t\|_{[-\omega,\omega]},\ u \in (-\omega, \omega), \qquad (15)$$

cf. [1, E.19, p. 243]. Moreover, we shall resort to the following nonlinear transformation $\tau:\ [-1,1]\to[-\omega,\omega]$,

$$\tau(x)=2\arcsin\left(\sin\left(\frac{\omega}{2}\right)\,x\right)\ ,\quad x\in[-1,1]\ , \tag{16}$$

that plays a key role in the theory of subperiodic interpolation and quadrature, cf. [3, 4].

We are now ready to state and prove the following

Proposition 2 *Let $\nu=\nu(\alpha,\beta)$ be defined as in (10), and the transformation τ as in (16). Then, the set sequences $\left\{\tau\left(\mathcal{G}_{mn}^{(\alpha,\beta)}\right)\right\}$ and $\left\{\tau\left(\mathcal{GL}_{mn+2}^{(\alpha,\beta)}\right)\right\}$, $n\geq 1$, are norming meshes for trigonometric polynomials on $[-\omega,\omega]$, $0<\omega\leq\pi$, namely*

$$\|t\|_{[-\omega,\omega]}\leq C_1\,\|t\|_{\tau\left(\mathcal{G}_{mn}^{(\alpha,\beta)}\right)}\ ,\quad C_1=\frac{m}{m-2\nu\pi}\ ,\quad \forall t\in\mathbb{T}_n([-\omega,\omega])\ , \tag{17}$$

provided that $m>2\nu\pi$, and

$$\|t\|_{[-\omega,\omega]}\leq C_2\,\|t\|_{\tau\left(\mathcal{GL}_{mn+2}^{(\alpha,\beta)}\right)}\ ,\quad C_2=\frac{m}{m-\nu\pi}\ ,\quad \forall t\in\mathbb{T}_n([-\omega,\omega])\ , \tag{18}$$

provided that $m>\nu\pi$.

Proof. By Videnskii inequality (15), in (7) we have

$$F(u)=\int_{-\omega}^{u}\frac{1}{\sqrt{1-\cos^2(\omega/2)/\cos^2(v/2)}}\,dv=\pi+\arcsin\left(\frac{\sin(u/2)}{\sin(\omega/2)}\right)\ .$$

Let $\theta_{N,i}$, $i=1,\dots,N$, denote the zeros of $P_N^{(\alpha,\beta)}(\cos(\theta))$ with $0<\theta_{N,1}<\cdots<\theta_{N,N}<\pi$, and set

$$\xi_{N-i+1}=\tau(\cos(\theta_{N,i}))=2\arcsin\left(\sin\left(\frac{\omega}{2}\right)\,\cos(\theta_{N,i})\right)\ ,$$

$i=1,\dots,N$. Then

$$F(\xi_i)=\pi+2\arcsin\left(\cos(\theta_{N,N-i+1})\right)$$
$$=\pi+2\left(\frac{\pi}{2}-\arccos(\cos(\theta_{N,N-i+1}))\right)=2\left(\pi-\theta_{N,N-i+1}\right)\ ,$$

from which follow

$$F(\xi_1)=2(\pi-\theta_{N,N})\ ,\quad F(\omega)-F(\xi_N)=2\pi-2(\pi-\theta_{N,1})=2\theta_{N,1}\ ,$$

and

$$\max_i\left\{F(\xi_{i+1}-F(\xi_i)\right\}=2\max_i\left\{\theta_{N,i+1}-\theta_{N,i}\right\}\ .$$

The proof now proceeds like that of Proposition 1, taking into account a new factor equal to 2 appearing in the estimates. □

We conclude by observing that, as already obtained in [15] for the τ-image of the Gauss-Chebyshev-Lobatto points (improving the estimates of [9]), the norming constants C_1 and C_2 are independent of ω.

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