Analytic transformations of admissible meshes *

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Abstract

We obtain good discrete sets for real or complex multivariate polynomial approximation (admissible meshes) on compact sets satysfying a Markov polynomial inequality, by analytic transformations. Then we apply the result to the construction of near optimal admissible meshes, and we discuss two examples concerning complex analytic curves and real analytic cylindroids.

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1 Introduction.

Polynomial inequalities involving sequences of discrete subsets $\{\mathcal{A}_n\}$ of a compact set $K \subset \mathbb{R}^d$ (or $K \subset \mathbb{C}^d$), such as

$$\|p\|_K \leq C \, \|p\|_{\mathcal{A}_n} \; , \;\; \forall p \in \mathbb{P}_n^d(K) \tag{1}$$

(with the notation $\|f\|_X = \sup_{x \in X} |f(x)|$ for f bounded function on the compact X), are known under different names in various contexts: admissible meshes, (L^∞) norming sets and Marcinkiewicz-Zygmund inequalities (especially for the sphere), stability inequalities (even in more general functional settings); cf., e.g., [11, 13, 18, 23]. A general theory of polynomial admissible meshes appears in a recent work by Calvi and Levenberg [11],

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where a key feature is that the cardinality of $\mathcal{A}_n$ grows at most polynomially with n,

$$\mathrm{card}(\mathcal{A}_n) = \mathcal{O}(n^s) \ , \quad s > 0 \tag{2}$$

Observe that necessarily $\mathrm{card}(\mathcal{A}_n) \geq \dim(\mathbb{P}_n^d(K))$, since A_n is $\mathbb{P}_n^d(K)$-determining. In the case when $C = C(\mathcal{A}_n)$ is not constant but grows at most polynomially with n, namely

$$C(\mathcal{A}_n) = \mathcal{O}(n^\alpha) \ , \quad \alpha > 0 \tag{3}$$

they speak of a *weakly* admissible mesh. In [11] it is shown that such meshes are nearly optimal for least squares approximation, and contain Fekete-like interpolation sets with a slowly increasing Lebesgue constant. Among their properties, it is worth to recall that (weakly) admissible meshes are preserved by affine mapping, and can be extended by finite union and product.

In some recent papers, the role of (weakly) admissible meshes in multivariate polynomial approximation has been deepened from both the theoretical and the computational points of view. It has been shown that discrete extremal sets of Fekete and Leja type can be extracted from such meshes working on the corresponding rectangular Vandermonde matrices, and using only basic procedures of numerical linear algebra, such as the QR and LU factorizations with pivoting; cf. [5, 6, 8, 24]. Moreover, resorting to a recent deep result on the asymptotics of Fekete points (cf. [2]), in [5, 6] it has been proved that such discrete extremal sets distribute asymptotically as the continuous Fekete points, i.e., the corresponding discrete measures converge weak-$*$ to the pluripotential equilibrium measure (cf. [14]). A survey of some recent results on weakly admissible meshes and discrete extremal sets can be found in [7].

In principle, following [11, Thm.5], it is always possible to construct an admissible mesh on a compact set which satisfies a Markov polynomial inequality (termed for brevity *Markov compacts*)

$$\|\nabla p\|_K \leq M n^r \|p\|_K \ , \quad \forall p \in \mathbb{P}_n^d(K) \tag{4}$$

where $\|\nabla p\|_K = \max_{\boldsymbol{z} \in K} \|\nabla p(\boldsymbol{z})\|_\infty$. This can be done essentially by a uniform discretization of the compact set (or even only of its boundary in complex instances) with $\mathcal{O}(n^{-r})$ spacing, but the resulting mesh has then $\mathcal{O}(n^{rd})$ cardinality for real compacts and, in general, $\mathcal{O}(n^{2rd})$ cardinality for complex compacts. Since $r = 2$ for many compacts, for example real convex compacts (cf. [15, 25]), the computational use of such admissible meshes becomes difficult or even impossible for $d = 2, 3$ already at moderate degrees.

On the other hand, weakly admissible meshes with approximately n^2 points and $C(\mathcal{A}_n) = \mathcal{O}(\log^2 n)$, and even (nonuniform) admissible meshes with $\mathcal{O}(n^2)$ points, can be constructed on some standard real bidimensional compacts like disks, triangles, quadrangles; cf. [9, 10]. Admissible and weakly admissible meshes with $\mathcal{O}(n^2)$ points can then be obtained on any convex or concave simple polygon (by polygon triangulation and finite union). These constructions are based on suitable algebraic or mixed algebraic-trigonometric transformations and one-dimensional Chebyshev-like points, and can be extended to higher dimension (balls, cylinders, tori, polyhedra), to obtain (weakly) admissible meshes with $\mathcal{O}(n^d)$ cardinality.

General results on the construction of admissible meshes in d-dimensional real compacts, have been recently proved in [16]. In particular, it is shown that "optimal" admissible meshes, i.e., meshes with $\mathcal{O}(n^d)$ cardinality, always exist in d-dimensional polynomial graph domains (domains bounded by graphs of polynomial functions), in convex polytopes, and in star-like domains with C^2 boundary. It is also conjectured that any real convex body possesses an optimal admissible mesh. Moreover, admissible meshes with $\mathcal{O}(n^d \log^{k(d)} n)$ cardinality, $k(d) = \mathcal{O}(d^2)$, are constructed in d-dimensional analytic graph domains (domains bounded by graphs of analytic functions).

In this paper, starting from the work in [19], we prove a quite general result on the existence of (weakly) admissible meshes, in multidimensional real or complex Markov compacts that are image of suitable analytic transformations. Then we apply the result to the construction of near optimal admissible meshes, and we discuss two examples concerning complex analytic curves and real analytic cylindroids (a class of graph domains).

2 Transformations of admissible meshes.

We begin by recalling the following definitions. Given a compact set $Q \subset \mathbb{C}^d$, its *polynomial convex hull* is

$$\hat{Q} = \{\boldsymbol{z} \in \mathbb{C}^d : |p(\boldsymbol{z})| \leq \|p\|_Q \,,\ \forall p \in \mathbb{P}_n^d\} \tag{5}$$

and Q is termed *polynomially convex* if $\hat{Q} = Q$. In one complex variable, this is equivalent to the fact that Q has a connected complement ($\hat{Q}$ being the union of Q with the bounded components of its complement). We refer the reader e.g. to [14, 17] for a discussion on this concept in the context of pluripotential theory and multivariate polynomial approximation.

We specify that by *analytic* function on a compact set we mean a function that is holomorphic in an open neighborhood of the set. Moreover, a compact set is termed *polynomial determining* if a polynomial that is null on it is identically null: clearly, any compact possessing a (weakly) admissible mesh is polynomial determining.

Theorem 1 (Analytic Transformations of (Weakly) Admissible Meshes) *Let $K \subset \mathbb{C}^d$ be a Markov compact, cf. (4). Let $Q \subset \mathbb{C}^d$ be a polynomial determining compact such that $K = \boldsymbol{\phi}(Q)$, where $\boldsymbol{\phi}$ is analytic on $\hat{Q}$ (the* *polynomial convex hull of Q, cf. (5)), and let $\mathcal{A}_n$ be a (weakly) admissible mesh for Q.*

Then, there exists a sequence of natural numbers $j(n) = \mathcal{O}(\log n)$ such that $\mathcal{A}'_n = \boldsymbol{\phi}(\mathcal{A}_{nj(n)})$ is a (weakly) admissible mesh for K, with $C(\mathcal{A}'_n) \sim C(\mathcal{A}_{nj(n)})$ as $n \to \infty$, and $\mathrm{card}(\mathcal{A}'_n) \le \mathrm{card}(\mathcal{A}_{nj(n)})$.

Before proving the theorem, some remarks are in order.

Remark 1 In the case when $\boldsymbol{\phi}$ is a polynomial mapping of degree m, say $\boldsymbol{\phi} = \boldsymbol{\psi}_m$, it is immediate to check that $\boldsymbol{\psi}_m(\mathcal{A}_{nm})$ is a (weakly) admissible mesh for $\boldsymbol{\psi}_m(Q)$ with constant $C(\mathcal{A}_{nm})$.

Remark 2 When Q itself is a Markov compact, then K is a Markov compact with at most the same exponent as soon as it is *not pluripolar* (a notion equivalent to the positivity of the transfinite diameter, cf. [14, 17]), and the transformation $\boldsymbol{\phi}$ is *regular* (i.e., its Jacobian matrix is everywhere nonsingular in Q), by a result of Baran and Plésniak [1]. Observe that a simple sufficient condition ensuring that a compact set in $\mathbb{R}^d$ or in $\mathbb{C}^d$ is not pluripolar, is that it has nonempty interior (in $\mathbb{R}^d$ or in $\mathbb{C}^d$, respectively), since a real or complex ball is not pluripolar [14]. At least for $d = 1$, it is in any case natural to assume that K in Theorem 1 is not polar, since it is known that a Markov compact in $\mathbb{C}$ cannot be polar [3].

Proof of Theorem 1. Let us term $\boldsymbol{\pi}_j$ the polynomial vector mapping of best uniform componentwise approximation of degree j to $\boldsymbol{\phi}$ on a compact Ω, and $E_j(\boldsymbol{\phi};\Omega) = \max_{\boldsymbol{w}\in\Omega} \|\boldsymbol{\phi}(\boldsymbol{w}) - \boldsymbol{\pi}_j(\boldsymbol{w})\|_\infty$ the corresponding error. In view of the Uniform Bernstein-Walsh-Siciak Theorem for analytic functions of several complex variables on polynomially convex sets (cf. [20, Lemma 1] and also [21, Lemma 3] for a simpler proof), applied componentwise to $\boldsymbol{\phi}$ on $\hat{Q} \supseteq Q$, it is immediate to prove that there exist two constants L and a such that

$$\varepsilon_j = E_j(\boldsymbol{\phi};Q) \le E_j(\boldsymbol{\phi};\hat{Q}) \le La^j \ , \quad L > 0 \ , \quad 0 < a < 1 \tag{6}$$

i.e., the convergence rate of $\boldsymbol{\pi}_j$ to $\boldsymbol{\phi}$ is at least geometric. Fix $p \in \mathbb{P}^d_n(K)$ and $\boldsymbol{z} \in K$, take $\boldsymbol{w} \in Q$ such that $\boldsymbol{z} = \boldsymbol{\phi}(\boldsymbol{w})$, and $\boldsymbol{z}_j = \boldsymbol{\pi}_j(\boldsymbol{w}) \in \boldsymbol{\pi}_j(Q)$. By the mean-value inequality we can write

$$|p(\boldsymbol{z})| \le |p(\boldsymbol{z}_j)| + |p(\boldsymbol{z}) - p(\boldsymbol{z}_j)| \le |p(\boldsymbol{z}_j)| + d \max_{\boldsymbol{s}\in[\boldsymbol{z},\boldsymbol{z}_j]} \|\nabla p(\boldsymbol{s})\|_\infty \, \|\boldsymbol{z} - \boldsymbol{z}_j\|_\infty$$

$$\le |p(\boldsymbol{z}_j)| + d\|\nabla p(\boldsymbol{\xi}_j)\|_\infty \, \varepsilon_j$$

for a suitable $\boldsymbol{\xi}_j$ in the segment $[\boldsymbol{z}, \boldsymbol{z}_j]$. Observe that $\mathrm{dist}(\boldsymbol{\xi}_j, K) \le \varepsilon_j$. Using the fact that K is a Markov compact, by the estimate $|q(\boldsymbol{\xi})| \le$

$\exp(dMn^r\varepsilon)\|q\|_K$, valid for every $q \in \mathbb{P}_n^d$ and for every $\boldsymbol{\xi}$ such that $\mathrm{dist}(\boldsymbol{\xi}, K) \leq \varepsilon$ (cf. [11, Lemma 6]), applied to components of ∇p, we get

$$|p(\boldsymbol{z})| \leq |p(\boldsymbol{z}_j)| + d \exp(dMn^r\varepsilon_j)\|\nabla p\|_K \, \varepsilon_j \leq |p(\boldsymbol{z}_j)| + \sigma(n,j)\|p\|_K \tag{7}$$

where

$$\sigma(n,j) = \exp(dMn^r\varepsilon_j)dMn^r\varepsilon_j$$

Now, $\boldsymbol{\pi}_j(\mathcal{A}_{nj})$ is a (weakly) admissible mesh for $\boldsymbol{\pi}_j(Q)$ with constant $C(\mathcal{A}_{nj})$ (see Remark 1), from which we get the estimate $|p(\boldsymbol{z}_j)| \leq C(\mathcal{A}_{nj})\|p\|_{\boldsymbol{\pi}_j(\mathcal{A}_{nj})}$ and finally, taking the maximum in the left-hand side of (7),

$$\|p\|_K \leq C(\mathcal{A}_{nj})\|p\|_{\boldsymbol{\pi}_j(\mathcal{A}_{nj})} + \sigma(n,j)\|p\|_K \tag{8}$$

The next step is to bound $\|p\|_{\boldsymbol{\pi}_j(\mathcal{A}_{nj})}$ in a similar fashion. Fix $\hat{\boldsymbol{z}}_j \in \boldsymbol{\pi}_j(\mathcal{A}_{nj})$, take $\hat{\boldsymbol{w}} \in \mathcal{A}_{nj}$ such that $\hat{\boldsymbol{z}}_j = \boldsymbol{\pi}_j(\hat{\boldsymbol{w}})$, and $\hat{\boldsymbol{z}} = \boldsymbol{\phi}(\hat{\boldsymbol{w}}) \in \boldsymbol{\phi}(\mathcal{A}_{nj}) \subset K$. Exploiting the Markov inequality as above, we arrive to the estimate

$$|p(\hat{\boldsymbol{z}}_j)| \leq |p(\hat{\boldsymbol{z}})| + \sigma(n,j)\|p\|_K \leq \|p\|_{\boldsymbol{\phi}(\mathcal{A}_{nj})} + \sigma(n,j)\|p\|_K$$

Taking the maximum in the left-hand side and inserting the resulting bound for $\|p\|_{\boldsymbol{\pi}_j(\mathcal{A}_{nj})}$ into (8) we get finally

$$\|p\|_K \leq C(\mathcal{A}_{nj})\|p\|_{\boldsymbol{\phi}(\mathcal{A}_{nj})} + \beta(n,j)\|p\|_K \tag{9}$$

where

$$\beta(n,j) = (1 + C(\mathcal{A}_{nj}))\sigma(n,j) = (1 + C(\mathcal{A}_{nj}))\exp(dMn^r\varepsilon_j)dMn^r\varepsilon_j \tag{10}$$

Choose now $m(n) = \lceil b \log n \rceil$ with $b > (r+\alpha)/|\log a|$, cf. (3), (4) and (6); observe that for admissible meshes we have $\alpha = 0$. Then

$$n^{r+\alpha}\varepsilon_{m(n)}(\boldsymbol{\phi}) = \mathcal{O}\left(n^{r+\alpha}a^{b\log n}\right) = \mathcal{O}\left(n^{r+\alpha-b|\log a|}\right) \to 0 \;, \quad n \to \infty$$

which implies that both, $\sigma(n, m(n))$ and $\beta(n, m(n))$, are infinitesimal as $n \to \infty$. Let n^* be the first index such that $\beta(n, m(n)) < 1$ for all $n \geq n^*$, and define $j(n) = m(n)$ for $n \geq n^*$, and $j(n) = \min\{j : \beta(n,j) < 1\}$ for $n < n^*$ (observe that $\beta(n,j) \to 0$ as $j \to \infty$ for any fixed n): clearly $j(n) = \mathcal{O}(\log n)$ and $\beta(n, j(n)) \to 0$ as $n \to \infty$. From (9) we get

$$\|p\|_K \leq \frac{C(\mathcal{A}_{nj(n)})}{1 - \beta(n, j(n))}\|p\|_{\mathcal{A}'_n} \;, \quad \forall p \in \mathbb{P}_n^d(K) \tag{11}$$

with $\mathcal{A}'_n = \boldsymbol{\phi}(\mathcal{A}_{nj(n)})$, i.e., $\mathcal{A}'_n$ is a (weakly) admissible mesh for K whose constant is asymptotic to $C(\mathcal{A}_{nj(n)})$. Clearly, $\mathrm{card}(\mathcal{A}'_n) \leq \mathrm{card}(\mathcal{A}_{nj(n)})$ since $\boldsymbol{\phi}$ is not injective, in general. $\square$

2.1 Near optimal meshes.

In a recent paper [16], Kroó studies the problem of existence of *"optimal"* polynomial meshes, namely admissible meshes on d-dimensional (real) compacts that have $\mathcal{O}(n^d)$ cardinality. He shows that such meshes can always be constructed on polynomial graph domains, that are compact domains bounded by graphs of polynomial functions (in the sense that each variable x_k is bounded by the graphs of two functions of the previous variables $x_1, \ldots, x_{k-1}$, cf. [16]). Moreover, he shows that convex polytopes and star-like domains with C^2 boundary possess an optimal admissible mesh, and conjectures that the same is true also for any convex body.

In [10], optimal admissible meshes with approximately n^d points have been constructed on standard real compacts for $d = 2, 3$, such as convex quadrangles (triangles) and disks, convex hexahedra (tetrahedra), balls, cylinders, solid tori, using basic univariate polynomial and trigonometric polynomial inequalities by Ehlich and Zeller [12] and suitable transformations.

In the case of analytic graph domains, that are compact domains bounded by graphs of analytic functions, Kroó shows that there exist *near optimal* admissible meshes, which have $\mathcal{O}(n^d \log^{d(d-1)} n)$ cardinality (see [16, Thm.1]). In the special case of rectangular analytic domains, that are compact domains of the form

$$K = \{(\boldsymbol{x}, y) : \, g_1(\boldsymbol{x}) \leq y \leq g_2(\boldsymbol{x}) \, , \; \boldsymbol{x} \in D \subset \mathbb{R}^{d-1}\} \tag{12}$$

where D is a (hyper-)rectangle, and g_1, g_2 are analytic functions such that $g_2 - g_1 > 0$ on D, it can be shown (by a slight modification of the arguments in [16]) that the cardinality becomes $\mathcal{O}(n^d \log^{d-1} n)$.

Also in context of the present paper we deal with compacts that are images of analytic transformations and meshes with logarithmic sub-optimality. The following corollary of Theorem 1, whose proof is immediate, will be useful below.

Corollary 1 *Let the assumptions of Theorem 1 be satisfied, and assume in addition that Q possesses an optimal admissible mesh (i.e., an admissible mesh with $\mathcal{O}(n^d)$ cardinality). Then $K = \boldsymbol{\phi}(Q)$ possesses a near optimal admissible mesh with $\mathcal{O}((nj(n))^d) = \mathcal{O}(n^d \log^d n)$ cardinality.*

We present now two examples. The first concerns *complex parametric curves* ($d = 1$).

Proposition 1 *Any compact curve in $\mathbb{C}$, having an analytic parametrization, possesses a near optimal admissible mesh with $\mathcal{O}(n \log n)$ cardinality.*

Proof. The set of points of a complex curve with an analytic parametrization, $K = \phi([a, b]) \subset \mathbb{C}$, satisfies a Markov inequality with exponent at most 2.

Indeed, being a connected compact set, it satisfies in any case the Markov inequality

$$\|p'\|_K \leq \frac{2e}{\text{cap}(K)} n^2 \|p\|_K \ , \ \ \forall p \in \mathbb{P}_n^d(K)$$

where $\text{cap}(K)$ is the capacity of K, cf. [22]. On the other hand, it could have a Markov exponent $r < 2$, think for example to the case of the circle, $\phi(\theta) = e^{i\theta}$, $0 \leq \theta \leq 2\pi$, where $\|p'\|_K \leq n \|p\|_K$.

Moreover, $Q = [a, b]$ is polynomially convex, and by a classical result of Ehlich and Zeller (cf. [10, 12]) possesses an optimal admissible mesh, formed for example by its $2n + 1$ Chebyshev-Lobatto points $\mathcal{A}_n = X_{[a,b]}(2n)$, with $C(\mathcal{A}_n) \equiv 2$. Here and below

$$X_{[a,b]}(m) = \left\{ \frac{b-a}{2} \cos(k\pi/m) + \frac{b+a}{2} \ , \ 0 \leq k \leq m \right\} \tag{13}$$

denotes the $m + 1$ Chebyshev-Lobatto points of an interval $[a, b]$. The constant of the mesh is 2 and not $\sqrt{2}$ as in [10] since we deal with polynomials having complex coefficients, and thus we have to estimate through the real and complex part.

Being a C^1 parametric curve, K has an admissible mesh with $\mathcal{O}(n^r)$ cardinality, $r \in [1, 2]$, which is the image of a suitable set of equally spaced parameters, as it has been shown in [4, Prop.17] following the construction of [11, Thm.5]. On the other hand, Corollary 1 shows that, due to analyticity, even when $r > 1$ the curve K indeed possesses a near optimal admissible mesh, $\mathcal{A}'_n = \phi(X_{[a,b]}(2nj(n)))$, with $\mathcal{O}(nj(n)) = \mathcal{O}(n \log n)$ cardinality. $\square$

Notice that we did not need that the parametrization be regular (in the sense that the tangent vector is everywhere different from zero).

The second example concerns *real graph domains* like (12), where D is more general than a rectangular domain, that we call analytic "cylindroids" thinking to their shape in three variables. We notice that the notion of graph domain used here is an extension of that in [16].

Proposition 2 *Any analytic graph domain in $\mathbb{R}^d$ like (12), where $D \subset \mathbb{R}^{d-1}$ is a Markov compact having nonempty interior and an optimal admissible mesh, possesses a near optimal admissible mesh with $\mathcal{O}(n^d \log^{d-1} n)$ cardinality.*

Proof. Let D be any $(d - 1)$-dimensional real Markov compact which possesses an optimal admissible mesh (and nonempty interior). This is certainly the case of a rectangular domain, with no loss of generality the unit cube $D = [0, 1]^{d-1}$, where such a mesh is for example $(X_{[0,1]}(2n))^{d-1}$ (cf. (13)), in view of [10, 12]. Take K as in (12): observe that $K = \boldsymbol{\phi}(Q)$ with $Q = D \times [0, 1]$, $\boldsymbol{\phi}(\boldsymbol{x}, t) = (\boldsymbol{x}, tg_1(\boldsymbol{x}) + (1 - t)g_2(\boldsymbol{x}))$ which is analytic, and that Q is a Markov compact (as product of Markov compacts), and is polynomially convex, being a real compact (cf. [14, 17]).

On the other hand, K is itself a Markov compact, since the transformation $\boldsymbol{\phi}$ is regular, in the sense that its Jacobian matrix $J\boldsymbol{\phi}$ is everywhere nonsingular in Q. In fact, it is the triangular matrix

$$J\boldsymbol{\phi}(\boldsymbol{x},t) = \begin{bmatrix} \mathbb{I}_{d-1} & \mathbf{0} \\ & \\ t\nabla g_1(\boldsymbol{x}) + (1-t)\nabla g_2(\boldsymbol{x}) & g_1(\boldsymbol{x}) - g_2(\boldsymbol{x}) \end{bmatrix}$$

with diagonal $\{1,\ldots,1,g_1(\boldsymbol{x}) - g_2(\boldsymbol{x})\}$ and determinant $\det(J\boldsymbol{\phi}(\boldsymbol{x},t)) = g_1(\boldsymbol{x}) - g_2(\boldsymbol{x}) < 0$. Moreover, K is not pluripolar, since $\boldsymbol{\phi}$ maps interior points of Q to interior points of K. By a result of Baran and Plésniak [1] (see Remark 2 above), $K = \boldsymbol{\phi}(Q)$ is a Markov compact with at most the same exponent of Q.

Now, let $\mathcal{B}_n$ be an optimal admissible mesh for D: it is easy to check that $\mathcal{B}_n \times X_{[0,1]}(2n)$ is an optimal admissible mesh for Q. If we applied directly Corollary 1, we would conclude that $\mathcal{A}'_n = \boldsymbol{\phi}(\mathcal{B}_{nj(n)} \times X_{[0,1]}(2nj(n)))$ is a near optimal admissible mesh for K, with $\mathcal{O}(n^d \log^d n)$ cardinality.

We can reduce the cardinality, by using the peculiar structure of the transformation. If in the proof of Theorem 1 we use the polynomial approximation $\boldsymbol{\psi}_j(\boldsymbol{x},t) = (\boldsymbol{x}, tu_j(\boldsymbol{x}) + (1-t)v_j(\boldsymbol{x}))$ instead of $\boldsymbol{\pi}_j(\boldsymbol{x},t)$, where u_j and v_j are the best polynomial approximations of degree j to g_1 and g_2, respectively, the rate is still geometric and all the reasoning remains valid, taking the mesh $\mathcal{B}_{nj(n)} \times X_{[0,1]}(2n)$ in Q. Indeed, for every $p \in \mathbb{P}^j_n(\boldsymbol{\psi}_j(Q))$ we have that $p \circ \boldsymbol{\psi}_j \in \mathbb{P}^{d-1}_{nj}(D) \bigotimes \mathbb{P}^1_n([0,1])$, and $\mathcal{B}_{nj} \times X_{[0,1]}(2n)$ is an admissible mesh for the tensor-product polynomial space $\mathbb{P}^{d-1}_{nj}(D) \bigotimes \mathbb{P}^1_n([0,1])$. Thus $\boldsymbol{\psi}_j(\mathcal{B}_{nj} \times X_{[0,1]}(2n))$ is an admissible mesh for $\mathbb{P}^d_n(\boldsymbol{\psi}_j(Q))$. We conclude observing that the resulting admissible mesh $\boldsymbol{\phi}(\mathcal{B}_{nj(n)} \times X_{[0,1]}(2n)) \subset K$ has cardinality $\mathcal{O}(n^{d-1} \log^{d-1} n)\, \mathcal{O}(n) = \mathcal{O}(n^d \log^{d-1} n)$. $\square$

It is interesting to notice that D (the domain of g) can be taken in the class closed under finite union and algebraic transformation (cf. Remark 1), starting from the (Markov) compacts which in [16] have been shown to possess an optimal mesh, such as convex polytopes and star-like domains with C^2 boundary.

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