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Near optimal Tchakaloff meshes for compact sets with Markov exponent 2

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Abstract

By a discrete version of Tchakaloff Theorem on positive quadrature formulas, we prove that any real multidimensional compact set admitting a Markov polynomial inequality with exponent 2 possesses a near optimal polynomial mesh. This improves for example previous results on general convex bodies and starlike bodies with Lipschitz boundary, being applicable to any compact set satisfying a uniform interior cone condition. We also discuss two algorithmic approaches for the computation of near optimal Tchakaloff meshes in low dimension.

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1 Introduction

Let $K \subset \mathbb{R}^d$ (or $\mathbb{C}^d$) be a polynomial determing compact set (i.e., polynomials vanishing on K vanish everywhere). We recall that a polynomial mesh on K is a sequence of finite norming sets $X_n \subset K$, such that

$$\|p\|_K \leq C\, \|p\|_{X_n}\,,\ \ \forall p \in \mathbb{P}_n^d\,,\ \ card(X_n) = \mathcal{O}(n^s)\,, \tag{1}$$

for some constant $C \geq 1$ and $s \geq d$, where $\mathbb{P}_n^d$ denotes the subspace of polynomials of total-degree not exceeding n with dimension $N = N(n) = dim(\mathbb{P}_n^d) = \binom{n+d}{d}$, and $\|p\|_Y$ the uniform norm on a continuous or discrete compact set Y).

When $s = d$ the mesh is called "optimal" in the literature, since necessarily necessarily $card(X_n) \geq N \sim n^d/d!$, $n \to \infty$, so that it has the lowest possible order of growth with respect to n, whereas it is called near optimal when a logarithmic factor in n multiplies n^d, such as $\mathcal{O}(n^d \log^k n)$, $k \leq d$.

Polynomial meshes, that are ultimately good discrete models of compact sets when polynomials are involved, have been playing an important role in multivariate polynomial approximation during the last decade, from both the theoretical and the computational point of view. The latter is witnessed by the role of polynomial meshes in interpolation (Fekete-like subsets) and least squares, Bernstein-Markov measures and pluripotential numerics, and more recently in polynomial optimization. We may refer the reader for example to [2, 5, 8, 11, 15, 16, 20, 19], with the references therein.

We shall focus here on compact sets in $\mathbb{R}^d$. It is well-known by the fundamental construction of Calvi and Levenberg [5, Thm. 5] that any real compact set admitting a Markov polynomial inequality with exponent r, i.e. there exists a constant $\mathcal{M} > 0$ such that

$$\|\nabla p(x)\|_2 \leq \mathcal{M} n^r\, \|p\|_K\,,\ \ \forall p \in \mathbb{P}_n^d\,, \tag{2}$$

possesses a polynomial mesh with $\mathcal{O}(n^{rd})$ points.

On the other hand, optimal polynomial meshes have been constructed on several classes of compact sets, such as for example starlike and more general bodies with smooth boundary [11, 12, 15], bidimensional general convex bodies [13], general polytopes [11], and suitable sections of disk, sphere, ball and torus [8, 24]. Near optimal meshes are known on C^α starlike bodies with $\alpha = 2 - 2/d$ (in particular on planar Lipschitz starlike bodies, [12]), and on the general class of fat real subanalytic sets (essentially, finite unions of analytic images of boxes, cf. [20]). It should also be recalled that near optimal polynomial meshes are known to exist on any compact set in $\mathbb{C}^d$ (cf. [1, 3], and also [4]), but such results are essentially based

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on Fekete interpolation sets of suitable degree, that are explicitly available only in very few instances and are extremely hard to compute.

On the contrary, in this note we show that any real compact set satisfying a Markov polynomial inequality (2) with exponent $r = 2$ possesses a near optimal mesh, in view of a discrete version of *Tchakaloff Theorem* on positive quadrature, and that such a mesh can be computed by standard Linear and Quadratic Programming algorithms (at least in low dimension and for moderate degrees). Such a class includes for example any *convex body* [26], and more generally any compact body satisfying a *uniform interior cone condition* (that is with locally Lipschitz boundary), cf. [9].

We recall as a Lemma a discrete version of Tchakaloff Theorem on the existence of positive multivariate quadrature formulas exact on polynomial spaces. Originally proved by V. Tchakaloff in 1957 for absolutely continuous measures [23], it has then been extended to any measure with finite polynomial moments, cf. e.g. [7].

Lemma 1.1. *Let μ be a multivariate discrete measure supported at a finite set $X = \{x_i\} \subset \mathbb{R}^d$, with correspondent positive weights (masses) $\boldsymbol{\lambda} = \{\lambda_i\}$, $i = 1, \dots, M$.*

Then, there exist a quadrature formula with nodes $T_n = \{t_j\} \subseteq X$, that we may term the "Tchakaloff points" of (X, μ), and positive weights $\boldsymbol{w} = \{w_j\}$, $1 \le j \le m \le N = dim(\mathbb{P}_n^d)$, such that

$$\int_X p(x)\, d\mu = \sum_{i=1}^{M} \lambda_i f(p_i) = \sum_{j=1}^{m} w_j\, p(t_j)\,, \quad \forall p \in \mathbb{P}_n^d\,. \tag{3}$$

Proof. We recall also the proof (cf. e.g. [17]), since it gives the base for a numerical algorithm to compute Tchakaloff points and weights. Let $\{p_1, \dots, p_N\}$ be a basis of $\mathbb{P}_n^d$, and $V = (v_{ij}) = (p_j(x_i))$ the Vandermonde-like matrix of the basis computed at the support points. If $M > N$ (otherwise there is nothing to prove), existence of a positive quadrature formula for μ with cardinality not exceeding N can be immediately translated into existence of a nonnegative solution with at most N nonvanishing components to the underdetermined linear system

$$V^t \boldsymbol{u} = \boldsymbol{b}\,, \quad \boldsymbol{u} \ge \boldsymbol{0}\,, \tag{4}$$

where

$$\boldsymbol{b} = V^t \boldsymbol{\lambda} = \left\{ \int_X p_j(x)\, d\mu \right\}\,, \quad 1 \le j \le N\,, \tag{5}$$

is the column vector of μ-moments of the basis $\{p_j\}$.

Existence then holds by the well-known Caratheodory Theorem applied to the columns of V^t, which asserts that a conic (i.e., with positive coefficients) combination of any numer of vectors in $\mathbb{R}^N$ can be rewritten as a conic combination of at most N (linearly independent) of them; cf. [6]. □

We can now state and prove our main result.

Proposition 1.2. *Let $K \subset \mathbb{R}^d$ be a compact set admitting a Markov polynomial inequality like (2) with exponent $r = 2$.*

Then, K possesses a polynomial mesh Z_n with cardinality $\mathcal{O}(n^{2d})$. Moreover, the Tchakaloff points T_{2k_n} extracted from Z_{k_n} with unit mass measure, where $k_n = n\ell_n$, $\ell_n = [\log n] + 1$, form a near optimal polynomial mesh for K with $card(T_{2k_n}) = \mathcal{O}((n \log n)^d)$.

Proof. The first part is Calvi-Levenberg construction in [5, Thm. 5]. Let L be the maximal length of the convex hulls of the projections of K on the cartesian axes. Since polynomial meshes are affinely invariant, we may assume up to a translation that $K \subseteq [0, L]^d$. Fix $\theta \in (0, 1)$ and define $\nu = \left\lceil \frac{\sqrt{d}\mathcal{M}Ln^2}{\theta \exp(-\sqrt{d}\,\theta)} \right\rceil$. Consider in $[0, L]^d$ a uniform grid with stepsize $h = L/\nu$. For every box of the grid which intersects K choose a point in the intersection, and denote with Z_n the (finite) set of such points.

Observe that, by the estimate $|q(z)| \le \exp(d\mathcal{M}n^2\delta)\, \|q\|_K$, valid for every $q \in \mathbb{P}_n^d$ and for every $z \in \mathbb{R}^d$ such that $dist_\infty(z, K) \le \delta$ (cf. [5, Lemma 6]), applied to the components of $\nabla p = (\partial_1 p, \dots, \partial_d p)$, we get

$$\|\nabla p(z)\|_2 \le e^{d\mathcal{M}n^2\delta}\, \|\nabla p\|_K\,, \quad \forall z \in \mathbb{R}^d : dist_\infty(z, K) \le \delta\,. \tag{6}$$

Now, for every $x \in K$ we can choose $y \in Z_n$ such that $\delta = \|x - y\|_\infty \le h \le \theta/(\sqrt{d}\,\mathcal{M}n^2)\exp(-\sqrt{d}\,\theta) < \theta/(\sqrt{d}\,\mathcal{M}n^2)$. By the mean value theorem, for every $x, y \in K$ we have

$$|p(x) - p(y)| \le \|\nabla p(\xi)\|_2\, \|x - y\|_2$$

for a suitable ξ in the segment $[x, y]$. Then by (6) with $z = \xi$, together with $\|x - y\|_2/\sqrt{d} \le \|x - y\|_\infty \le h$, we get

$$|p(x) - p(y)| \le e^{d\mathcal{M}n^2\delta}\, \mathcal{M}n^2\, \|x - y\|_2\, \|p\|_K < e^{\sqrt{d}\,\theta}\, \sqrt{d}\, h\, \|p\|_K \le \theta\, \|p\|_K\,,$$

and thus

$$|p(x)| \le |p(y)| + |p(x) - p(y)| \le \|p\|_{Z_n} + \theta\, \|p\|_K\,,$$

from which (1) follows with $C = 1/(1 - \theta)$. Notice that $card(Z_n)$ does not exceed the fraction of grid boxes intersecting K, and thus it is bounded by the overall number of grid boxes

$$card(Z_n) \le \nu^d \le c_d\, n^{2d}\,, \quad c_d = \left\lceil \frac{\sqrt{d}L\mathcal{M}}{\theta \exp(-\sqrt{d}\,\theta)} \right\rceil^d\,. \tag{7}$$

In the case of convex bodies, the proof can use simply the mean value theorem, so that the factor $\exp(-\sqrt{d}\,\theta)$ in the denominator is dropped, in both the definition of ν and (7) (we omit the details for brevity).

Concerning the second part, first observe that for every $p \in \mathbb{P}_n^d$ the polynomial p^{ℓ_n} is in $\mathbb{P}_{k_n}^d$ and thus

$$\|p\|_K^{\ell_n} = \|p^{\ell_n}\|_K \le C\,\|p^{\ell_n}\|_{Z_{k_n}}\ .$$

Now, consider on $X = Z_{k_n}$ the discrete measure μ with unit masses. By Lemma 1 we get for every $q \in \mathbb{P}_{k_n}^d$

$$\|q\|_{Z_{k_n}}^2 \le \|q\|_{\ell^2(Z_{k_n})}^2 = \|q\|_{\ell_w^2(T_{2k_n})}^2 = \sum_{j=1}^m w_j\, q^2(t_j)$$

$$\le \left(\sum_{j=1}^m w_j\right)\, \|q\|_{T_{2k_n}}^2 = \mu(Z_{k_n})\,\|q\|_{T_{2k_n}}^2 = card(Z_{k_n})\,\|q\|_{T_{2k_n}}^2\ .$$

Then we can write

$$\|p^{\ell_n}\|_K \le C\,\sqrt{card(Z_{k_n})}\,\|p^{\ell_n}\|_{T_{2k_n}} \le C\,\sqrt{c_d\,k_n^{2d}}\,\|p^{\ell_n}\|_{T_{2k_n}}$$

and thus

$$\|p\|_K \le \left(k_n^d\,C\,\sqrt{c_d}\right)^{1/\ell_n}\,\|p\|_{T_{2k_n}} = \mathcal{O}(1)\,\|p\|_{T_{2k_n}}$$

since

$$\left(k_n^d\,C\,\sqrt{c_d}\right)^{1/\ell_n} = \exp\left(\frac{\log k_n^d}{\ell_n}\right)\,\left(C\,\sqrt{c_d}\right)^{1/\ell_n}$$

$$= \exp\left(d\,\frac{\log n + \log \ell_n}{\ell_n}\right)\,\left(C\,\sqrt{c_d}\right)^{1/\ell_n}$$

$$\le \exp\left(d\,\left(1 + \frac{\log \ell_n}{\ell_n}\right)\right)\,\left(C\,\sqrt{c_d}\right)^{1/\ell_n} \sim e^d\,,\quad n \to \infty\,.$$

Notice finally that $card(T_{2k_n}) = \mathcal{O}((n\log n)^d)$ as $n \to \infty$, since

$$card(T_{2k_n}) \le dim(\mathbb{P}_{2k_n}^d) = \binom{2k_n + d}{d} \sim \frac{(2n\log n)^d}{d!}\,,\quad n \to \infty\,. \tag{8}$$

□

The class of compact sets covered by Proposition 1 is very wide. Indeed

Corollary 1.3. *Any compact domain (the closure of a bounded open set) in $\mathbb{R}^d$ satisfying a uniform interior cone condition (each point of K is the vertex of a suitably rotated fixed cone contained in K) possesses a near optimal polynomial mesh. This holds in particular for any convex body.*

In fact, such a property implies the fulfillement of a Markov inequality with exponent 2, which is inherited from the cone, cf. e.g. [25]. This is valid on any compact domain with (locally) *Lipschitz boundary*, the latter property implying the fulfillement of a uniform interior cone condition [9].

In particular, Proposition 1 is valid on any *convex body*, where one can prove that a Markov inequality holds with $r = 2$ and $\mathcal{M}$ proportional to the reciprocal of the body width (the minimum distance between parallel supporting hyperplanes) by a factor 4 (or 2 on centrally symmetric bodies), cf. [26]. This improves the previous results for general convex bodies and starlike Lipschitz bodies in dimension $d > 2$, where the best known cardinality for polynomial meshes was $\mathcal{O}(n^{2d-2})$, cf. [11, 12]. It can also be seen as a further step towards the proof of the Conjecture: "Every convex body in $\mathbb{R}^d$ possesses an optimal polynomial mesh", cf. [11].

The proof of Proposition 1 is completely constructive, and easily implementable, at least in low dimension. In particular, differently from other relevant families of points in multivariate polynomial approximation, such as Fekete points or Lebesgue points, Tchakaloff points can in principle be computed by basic algorithms of Linear and Quadratic Programming.

In fact, the discrete version of Tchakaloff Theorem in Lemma 1, requires ultimately to compute a *sparse* nonnegative solution to the *underdetermined linear system* (4)-(5). In the literature on quadrature compression, essentially two approaches have been used.

The Linear Programming (LP) approach consists in minimizing the linear functional $\boldsymbol{c}^t\boldsymbol{u}$ for a suitable choice of the vector $\boldsymbol{c}$, subject to the constraints $V^t\boldsymbol{u} = \boldsymbol{b}$ and $\boldsymbol{u} \ge \boldsymbol{0}$. In fact, the solution is a vertex of the polytope defined by the constraints, which has (at least) $M - N$ null components, cf. e.g. [21]. Observe that a usual choice of the popular compressed sensing field (Basis Pursuit, cf. [10]), namely $\boldsymbol{c} = (1,\dots,1)$ that is minimizing $\|\boldsymbol{u}\|_1$ subject to the constraints, is not feasible in the present context, since $\|\boldsymbol{u}\|_1 = \mu(X)$ for any $\boldsymbol{u}$ satisfying (4) by exactness of the quadrature formula on the constants.

As an alternative, the Quadratic Programming approach consists in solving the *NonNegative Least Squares* (NNLS) problem

$$\text{compute } \boldsymbol{u}^* :\ \|V^t\boldsymbol{u}^* - \boldsymbol{b}\|_2 = \min \|V^t\boldsymbol{u} - \boldsymbol{b}\|_2\,,\ \ \boldsymbol{u} \ge \boldsymbol{0}\,, \tag{9}$$

that can be done by the well-known *Lawson-Hanson active set optimization method* [14], which automatically seeks a *sparse solution* and is implemented for example by the `lsqnonneg` native algorithm of Matlab. Our limited computational experience, in low dimension ($d = 2, 3$) and with moderate degrees (polynomial spaces of dimension up to the hundreds), has shown that in such setting Lawson-Hanson NNLS is more efficient than the most common implementations of LP. A Matlab code for the computation of Tchakaloff points based on NNLS is provided in the software packages quoted in [17, 22], where the reader can find a more detailed discussion.

In order to make an illustrative example, in Figure 1 we display for degree $n = 4$ the grid-based mesh $Z_n = Z_4$ with $\theta = 1/2$ (approximately 8800 points) and the near optimal Tchakaloff mesh $T_{2k_n} = T_{16}$ (153 points extracted from the approximately 140000 points of $Z_{k_n} = Z_8$) on a quarter of a Cassini oval, that is

$$K = \{x = (x_1, x_2) \in \mathbb{R}^2 : ((x_1 - a)^2 + x_2^2)((x_1 + a)^2 + x_2^2) \le b^4,\ x_1, x_2 \ge 0\},$$

with $a = 1$, $b = 2$ (the Cassini ovals are convex for $b/a \ge \sqrt{2}$); the Tchakaloff points have been computed by Lawson-Hanson NNLS algorithm.

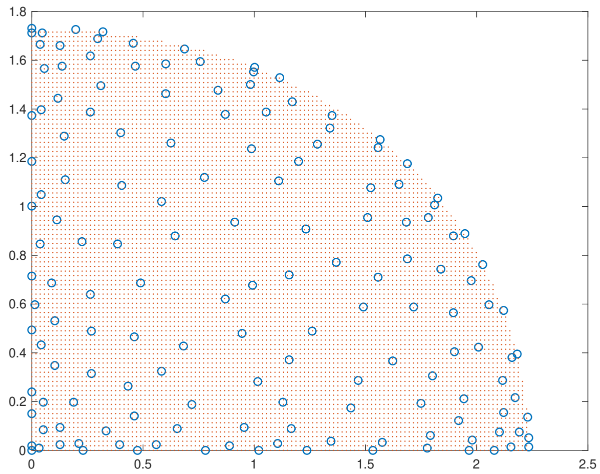

Figure 1: Grid-based polynomial mesh (around 8800 points) and Tchakaloff near optimal mesh (153 points) for degree $n = 4$ on a quarter of a Cassini oval.

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References

[1] T. Bloom, L. Bos, J.P. Calvi and N. Levenberg. Polynomial Interpolation and Approximation in $\mathbb{C}^d$. *Ann. Polon. Math.*, 106: 53–81, 2012.

[2] T. Bloom, N. Levenberg, F. Piazzon and F. Wielonsky. Bernstein-Markov: a survey. *Dolomites Res. Notes Approx. DRNA*, 8: 75–91, 2015.

[3] A. Brudnyi. On near-optimal admissible meshes. arXiv preprint 1402.2303, 2014.

[4] A. Brudnyi. Banach spaces of polynomials as "large" subspaces of ℓ^∞-spaces. *J. Funct. Anal.*, 267: 1285–1290, 2014.

[5] J.P. Calvi and N. Levenberg. Uniform approximation by discrete least squares polynomials. *J. Approx. Theory*, 152: 82–100, 2008.

[6] C. Caratheodory. Über den Variabilittsbereich der Fourierschen Konstanten von positiven harmonischen Funktionen. *Rend. Circ. Mat. Palermo*, 32: 193–217, 1911.

[7] R.E. Curto and L.A. Fialkow. A duality proof of Tchakaloff's theorem. *J. Math. Anal. Appl.*, 269: 519–532, 2002.

[8] S. De Marchi, F. Piazzon, A. Sommariva and M. Vianello. Polynomial Meshes: Computation and Approximation. Proceeedings of CMMSE 2015, 414–425, ISBN 978-84-617-2230-3, preprint online at: http://www.math.unipd.it/~marcov/pdf/wams.pdf.

[9] M.C. Delfour and J.P. Zolesio. Shapes and Geometries. SIAM, Philadelphia, 2011.

[10] S. Foucart and H. Rauhut. A Mathematical Introduction to Compressive Sensing. Birkhäuser, 2013.

[11] A. Kroó. On optimal polynomial meshes. *J. Approx. Theory*, 163: 1107–1124, 2011.

[12] A. Kroó. Bernstein type inequalities on star-like domains in $\mathbb{R}^d$ with application to norming sets. *Bull. Math. Sci.*, 3: 349–361, 2013.

[13] A. Kroó. On the existence of optimal meshes in every convex domain on the plane. *J. Approx. Theory*, Published online 15 March 2017.

[14] C.L. Lawson and R.J. Hanson. Solving least squares problems. Revised reprint of the 1974 original. Classics in Applied Mathematics 15, SIAM, Philadelphia, 1995.

[15] F. Piazzon. Optimal polynomial admissible meshes on some classes of compact subsets of $\mathbb{R}^d$. *J. Approx. Theory*, 207: 241–264, 2016.

[16] F. Piazzon. Pluripotential Numerics. *Constr. Approx.*, published online 21 June 2018.

[17] F. Piazzon, A. Sommariva and M. Vianello. Caratheodory-Tchakaloff Subsampling. *Dolomites Res. Notes Approx. DRNA*, 10: 5–14, 2017.

[18] F. Piazzon and M. Vianello. Constructing optimal polynomial meshes on planar starlike domains. *Dolomites Res. Notes Approx. DRNA*, 7: 22–25, 2014.

[19] F. Piazzon and M. Vianello. A note on total degree polynomial optimization by Chebyshev grids. *Optim. Lett.*, 12: 63–71, 2018.

[20] W. Pleśniak. Nearly optimal meshes in subanalytic sets. *Numer. Algorithms*, 60: 545–553, 2012.

[21] E.K. Ryu and S.P. Boyd. Extensions of Gauss quadrature via linear programming. *Found. Comput. Math.*, 15: 953–971, 2015.

[22] A. Sommariva and M. Vianello. Compression of multivariate discrete measures and applications. *Numer. Funct. Anal. Optim.*, 36: 1198–1223, 2015.

[23] V. Tchakaloff. Formules de cubatures mécaniques à coefficients non négatifs. (French) *Bull. Sci. Math.*, 81: 123–134, 1957.

[24] M. Vianello. Subperiodic Dubiner distance, norming meshes and trigonometric polynomial optimization. *Optim. Lett.*, 12: 1659–1667, 2018.

[25] H. Wendland. Scattered Data Approximation. Cambridge University Press, 2005.

[26] D. R. Wilhelmsen. A Markov inequality in several dimensions. *J. Approx. Theory*, 11: 216–220, 1974.