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Near-optimal interpolation and quadrature in two variables: the Padua points

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Abstract

The Padua points, recently studied during an international collaboration at the University of Padua, are the first known near-optimal point set for bivariate polynomial interpolation of total degree. Also the associate algebraic cubature formulas are both, in some sense, near-optimal.

Which are the Padua points?

Chebyshev-Lobatto points in $[-1,1]$

$$C_{n+1} := \{\cos(j\pi/n),\ j=0,\dots,n\}$$

$$\mathrm{card}(C_{n+1}) = n+1 = \dim(\mathbb{P}^1_n)$$

Padua points in $[-1,1]\times[-1,1]$

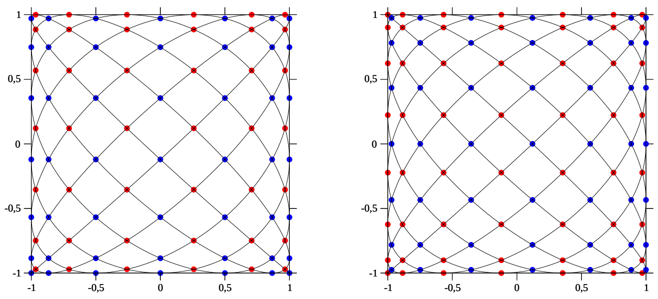
$$\mathrm{Pad}_n := (C^{\mathrm{odd}}_{n+1}\times C^{\mathrm{even}}_{n+2}) \cup (C^{\mathrm{even}}_{n+1}\times C^{\mathrm{odd}}_{n+2}) \subset C_{n+1}\times C_{n+2}$$

$$\mathrm{card}(\mathrm{Pad}_n) = \frac{(n+1)(n+2)}{2} = \dim(\mathbb{P}^2_n)$$

Alternative representation as self-intersections and boundary contacts of the generating curve

$$\boldsymbol{g}(t) := (-\cos((n+1)t), -\cos(nt))\ , \qquad t\in[0,\pi]$$

Figure 1. *The Padua points with their generating curve for $n=12$ (left, 91 points) and $n=13$ (right, 105 points), also as union of two Chebyshev-Lobatto subgrids (red and blue bullets).*



There are 4 families of such points, corresponding to successive rotations of 90 degrees.

Interpolation at the Padua points

trigonometric quadrature on the generating curve

$$\Downarrow$$

algebraic cubature at the Padua points $\boldsymbol{\xi}\in\mathrm{Pad}_n$ for the product Chebyshev measure

$$d\mu = \frac{dx_1\,dx_2}{\pi^2\sqrt{(1-x_1^2)(1-x_2^2)}}$$

with weights

$$w_{\boldsymbol{\xi}} := \frac{1}{n(n+1)}\cdot\begin{cases} 1/2 & \text{if } \boldsymbol{\xi} \text{ is a vertex point}\\ 1 & \text{if } \boldsymbol{\xi} \text{ is an edge point}\\ 2 & \text{if } \boldsymbol{\xi} \text{ is an interior point}\end{cases}$$

near-exactness in $\mathbb{P}^2_{2n}$
namely, exactness in $\mathbb{P}^2_{2n}\cap(\mathrm{span}\{T_{2n}(x_1)\})^{\perp_{d\mu}}$

$$\Downarrow$$

Lagrange interpolation formula

$$\mathcal{L}_{\mathrm{Pad}_n} f(\boldsymbol{x}) = \sum_{\boldsymbol{\xi}\in\mathrm{Pad}_n} f(\boldsymbol{\xi})\,L_{\boldsymbol{\xi}}(\boldsymbol{x})\ ,\ L_{\boldsymbol{\xi}}(\boldsymbol{\eta}) = \delta_{\boldsymbol{\xi}\boldsymbol{\eta}}$$

$$L_{\boldsymbol{\xi}}(\boldsymbol{x}) = w_{\boldsymbol{\xi}}(K_n(\boldsymbol{\xi},\boldsymbol{x}) - T_n(\xi_1)T_n(x_1))$$

$T_n(\cdot) = \cos(n\arccos(\cdot))$ and $K_n(\boldsymbol{x},\boldsymbol{y})$ reproducing kernel of the product Chebyshev orthonormal basis

$$\Theta_{\boldsymbol{j}}(\boldsymbol{x}) = T^*_{j_1}(x_1)\,T^*_{j_2}(x_2)\ ,\ \boldsymbol{j}=(j_1,j_2)\ ,\ 0\le|\boldsymbol{j}| = j_1+j_2\le n$$

$$K_n(\boldsymbol{x},\boldsymbol{y}) = \sum_{|\boldsymbol{j}|\le n}\Theta_{\boldsymbol{j}}(\boldsymbol{x})\,\Theta_{\boldsymbol{j}}(\boldsymbol{y})$$

The Lebesgue constant

Unisolvence gives the projection operator

$$\mathcal{L}_{\mathrm{Pad}_n} : C([-1,1]^2, \|\cdot\|_\infty) \to \mathbb{P}^2_n([-1,1]^2, \|\cdot\|_\infty)$$

Theorem (interpolation stability, cf. [1])

$$\|\mathcal{L}_{\mathrm{Pad}_n}\| = \max_{\boldsymbol{x}\in[-1,1]^2}\sum_{\boldsymbol{\xi}\in\mathrm{Pad}_n}|L_{\boldsymbol{\xi}}(\boldsymbol{x})| = \mathcal{O}(\log^2(n))$$

i.e., the Lebesgue constant has optimal order of growth [9].

Conjecture

$$\max_{\boldsymbol{x}\in[-1,1]^2}\sum_{\boldsymbol{\xi}\in\mathrm{Pad}_n}|L_{\boldsymbol{\xi}}(\boldsymbol{x})| \lesssim \frac{4}{\pi^2}\log^2(n)$$

and the max is attained at one of the vertices of the square.

Classical convergence estimate

$$\|\mathcal{L}_{\mathrm{Pad}_n}f - f\|_\infty \le (1+\|\mathcal{L}_{\mathrm{Pad}_n}\|)\min_{p\in\mathbb{P}^2_n}\|f-p\|_\infty = o(n^{-p}\log^2(n))$$

for $f\in C^p([-1,1]^2)$, $p>0$

Numerical implementation

Representation of the interpolant at the Padua points in the product Chebyshev orthonormal basis

$$\mathcal{L}_{\mathrm{Pad}_n}f(\boldsymbol{x}) = \sum_{|\boldsymbol{j}|\le n} c'_{\boldsymbol{j}}\,\Theta_{\boldsymbol{j}}(\boldsymbol{x})$$

with coefficients

$$c'_{\boldsymbol{j}} = c_{\boldsymbol{j}} := \sum_{\boldsymbol{\xi}\in\mathrm{Pad}_n} w_{\boldsymbol{\xi}}\,f(\boldsymbol{\xi})\,\Theta_{\boldsymbol{j}}(\boldsymbol{\xi})\ ,\ \ c'_{(n,0)} = \frac{c_{(n,0)}}{2}$$

which are approximate Fourier-Chebyshev coefficients (with a correction for $\boldsymbol{j}=(n,0)$)

Two different fast algorithms (cf. [3, 5]) using

- optimized matrix subroutines
- 2-dimensional Fast Fourier Transform (competitive and more stable at large degrees)

Other features

- extension to interpolation over rectangles
- practical convergence estimate

$$\|\mathcal{L}_{\mathrm{Pad}_n}f - f\|_\infty \lessapprox \sum_{|\boldsymbol{j}|=n-2}^{n}|c'_{\boldsymbol{j}}|\,\|\Theta_{\boldsymbol{j}}\|_\infty \le 2\sum_{|\boldsymbol{j}|=n-2}^{n}|c'_{\boldsymbol{j}}|$$

Numerical results on interpolation

Figure 2. *Six test functions from the Franke-Renka-Brown test set [7];* top: F1, F2, F3; bottom: F5, F7, F8.

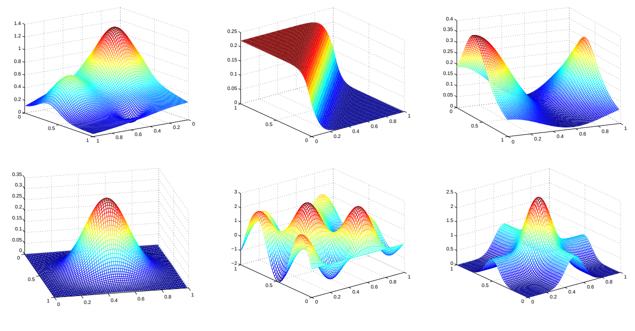


Table 1. *True and (estimated) interpolation errors for the test functions above.*

n	$\mathrm{card}(\mathrm{Pad}_n)$	F_1	F_2	F_3	F_5	F_7	F_8
10	66	9E-2 (2E-1)	4E-1 (6E-1)	8E-3 (6E-2)	4E-2 (2E-1)	3E-1 (1E+0)	1E-1 (4E-1)
20	231	7E-3 (2E-2)	6E-2 (8E-2)	1E-5 (8E-5)	6E-5 (8E-4)	8E-6 (2E-4)	3E-3 (1E-2)
30	496	1E-4 (8E-4)	1E-2 (1E-2)	2E-8 (1E-7)	1E-8 (2E-7)	7E-13 (2E-11)	2E-5 (1E-4)
40	861	3E-6 (1E-5)	2E-3 (2E-3)	2E-11 (2E-10)	4E-13 (2E-11)	4E-14 (8E-15)	6E-8 (6E-7)
50	1326	1E-8 (8E-8)	4E-4 (4E-4)	1E-13 (4E-13)	1E-15 (1E-15)	7E-14 (1E-14)	5E-11 (6E-10)

Nontensorial Clenshaw-Curtis cubature

- integration of the interpolant at the Chebyshev-Lobatto points gives the 1D Clenshaw-Curtis quadrature formula
- integration of the interpolant at the Padua points gives a 2D nontensorial Clenshaw-Curtis cubature formula

$$I(f) := \iint_{[-1,1]^2} f(\boldsymbol{x})\,d\boldsymbol{x} \approx I_{\mathrm{Pad}_n}(f) := \iint_{[-1,1]^2}\mathcal{L}_{\mathrm{Pad}_n}f(\boldsymbol{x})\,d\boldsymbol{x}$$

$$= \sum_{|\boldsymbol{j}|\le n} c'_{\boldsymbol{j}}\,m_{\boldsymbol{j}} = \sum_{\boldsymbol{\xi}\in\mathrm{Pad}_n}\lambda_{\boldsymbol{\xi}}\,f(\boldsymbol{\xi})\ ,\ \ \lambda_{\boldsymbol{\xi}} := w_{\boldsymbol{\xi}}\sum_{|\boldsymbol{j}|\le n} m'_{\boldsymbol{j}}\,\Theta_{\boldsymbol{j}}(\boldsymbol{\xi})$$

(exact for $f\in\mathbb{P}^2_n$), via the Chebyshev moments

$$m_{\boldsymbol{j}} := \iint_{[-1,1]^2}\Theta_{\boldsymbol{j}}(\boldsymbol{x})\,d\boldsymbol{x} = \int_{-1}^{1}T^*_{j_1}(t)\,dt\int_{-1}^{1}T^*_{j_2}(t)\,dt$$

The cubature weights $\{\lambda_{\boldsymbol{\xi}}\}$ are not all positive, nevertheless

Theorem (cubature stability and convergence, cf. [8])

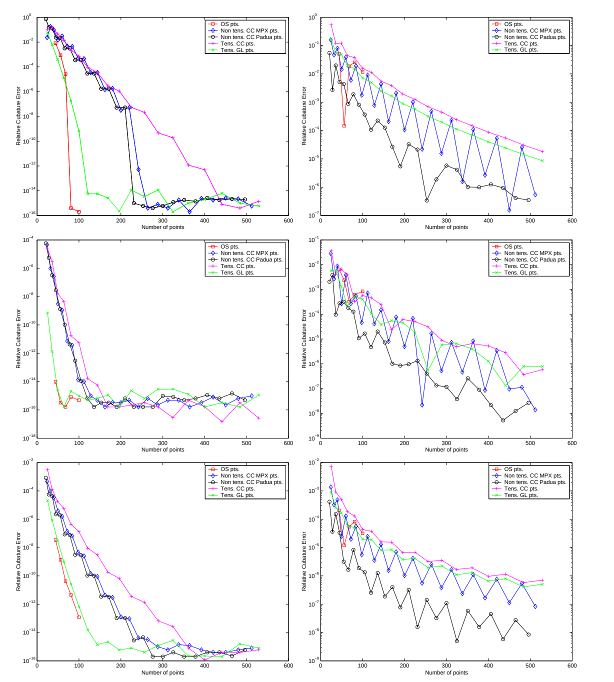
$$\lim_{n\to\infty}\|I_{\mathrm{Pad}_n}\| = \lim_{n\to\infty}\sum_{\boldsymbol{\xi}\in\mathrm{Pad}_n}|\lambda_{\boldsymbol{\xi}}| = \textit{area of the square} = 4$$

and $I(f) = I_{\mathrm{Pad}_n}(f) + o(n^{-p})$ for $f\in C^p([-1,1]^2)$, $p\ge 0$.

Numerical results on cubature

- near-optimality: on non-entire integrands, I_{Pad_n} performs better than tensor-product Gauss-Legendre quadrature and even than the few known minimal formulas, see Fig. 3 right (similar to the well-known 1D phenomenon studied in [10])
- fast algorithm: computing the weights by a suitable matrix formulation (cf. [3]) is very fast in Matlab/Octave (e.g. less than 0.01 sec at $n=100$ on a 2.4Ghz PC)

Figure 3. *Cubature errors of tensorial and nontensorial cubature formulas versus the number of evaluations: CC=Clenshaw-Curtis, GL=Gauss-Legendre, MPX=Morrow-Patterson-Xu [11], OS=Omelyan-Solovyan (minimal) [6].*
Left: $f=(x_1+x_2)^{20}$; $f=\exp(x_1+x_2)$; $f=\exp(-(x_1^2+x_2^2))$
Right: $f=1/(1+16(x_1^2+x_2^2))$; $f=\exp(-1/(x_1^2+x_2^2))$; $f=(x_1^2+x_2^2)^{3/2}$



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