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Quadrature of Quadratures: Compressed Sampling by Tchakaloff Points

F. Piazzon, A. Sommariva and M. Vianello
Department of Mathematics, University of Padova, ITALY (marcov@math.unipd.it)

Abstract

In view of a discrete version of Tchakaloff theorem on the existence of positive algebraic quadrature formulas, we can compute Tchakaloff points that are compressed sampling sets for multivariate quadrature and least squares.

Discrete Tchakaloff theorem

THEOREM 1 *Let μ be a positive measure with compact support in $\mathbb{R}^d$ and $\mathbb{P}_n^d$ be the space of real d-variate polynomials of total degree $\leq n$, restricted to supp(μ). Then there are $m \leq N = dim(\mathbb{P}_n^d)$ points $T_n = \{t_j\} \subseteq supp(\mu)$ and positive real numbers $\boldsymbol{w} = \{w_j\}$ such that*

$$\int_{\mathbb{R}^d} p(x)\,d\mu = \sum_{j=1}^{m} w_j\, p(t_j)\ , \quad \forall p \in \mathbb{P}_n^d \tag{1}$$

- originally proved for absolutely continuous measures by the Bulgarian mathematician V.L. Tchakaloff (1957), then extended to more general measures, cf. e.g. [5]
- we may term $T_n = \{t_j\} \subseteq supp(\mu)$ a set of Tchakaloff points (of degree n)
- key fact: the measure can be discrete, e.g., the integral can be a quadrature formula itself!

Compressed quadrature

let μ be a discrete measure with finite support $X = \{x_i\}$ and masses (weights) $\boldsymbol{\lambda} = \{\lambda_i\}$

$$\int_{\mathbb{R}^d} f(x)\,d\mu = \sum_{i=1}^{M} \lambda_i\, f(x_i) \tag{2}$$

if card$(X) = M > N$, by Tchakaloff theorem

$$\boxed{\sum_{i=1}^{M} \lambda_i\, p(x_i) = \sum_{j=1}^{m} w_j\, p(t_j)\ , \quad \forall p \in \mathbb{P}_n^d} \tag{3}$$

special cases: positive quadrature formulas on $\Omega \subset \mathbb{R}^d$ like (9)

- we are replacing a long sum by a shorter one where the re-weighted nodes are a subset $T_n = \{t_j\} \subset X = \{x_i\}$
- Compression Ratio: CR $= M/m \geq M/N > 1$ (even $\gg 1$)

but the proof of Tchakaloff theorem is not constructive: how can we select the nodes and compute the new weights?

Answer: quadratic programming!

Implementation by NonNegative Least Squares

let fix a polynomial basis span$\{p_1, \ldots, p_N\} = \mathbb{P}_n^d$ and the Vandermonde-like matrix

$$V = (v_{ij}) = (p_j(x_i))\ , \quad 1 \leq i \leq M\ ,\ 1 \leq j \leq N \tag{4}$$

- the underdetermined moment system

$$V^t \boldsymbol{u} = \boldsymbol{b} = V^t \boldsymbol{\lambda} \tag{5}$$

has a nonnegative sparse solution by Tchakaloff theorem

- sparsity cannot be recovered by ℓ^1 Compressed Sensing (min $\|\boldsymbol{u}\|_1$), since $\|\boldsymbol{u}\|_1 = \|\boldsymbol{\lambda}\|_1 = \mu(X)$ is constant
- sparse solution by NonNegative Least Squares (NNLS)

$$\boldsymbol{u}^*:\ \|V^t \boldsymbol{u}^* - \boldsymbol{b}\|_2^2 = \min \|V^t \boldsymbol{u} - \boldsymbol{b}\|_2^2\ , \quad \boldsymbol{u} \succeq \boldsymbol{0} \tag{6}$$

the nonzero components of $\boldsymbol{u}^*$ are the weights $\{w_j\}$ in (3) and identify the Tchakaloff points $T_n = \{t_j\}$; cf. [7] (univariate) and [10] (multivariate)

- we can use a standard NNLS solver like the Lawson-Hanson active set method, which recovers sparse solutions (optimized Matlab implementation in [9])

Error estimates

in practice we find a nonzero moment residual (6)

$$\|V^t \boldsymbol{u}^* - \boldsymbol{b}\|_2 = \varepsilon > 0 \tag{7}$$

(effect of rounding errors, tolerance in the NNLS solver, ...)

- error estimate (with a μ-orthonormal basis $\{p_j\}$ in (4)-(7)):

$$\left| \sum_{i=1}^{M} \lambda_i\, f(x_i) - \sum_{j=1}^{m} w_j\, f(t_j) \right| \leq C_\varepsilon\, E_n(f) + \varepsilon\, \|f\|_{\ell^2_{\lambda}(X)}\ , \tag{8}$$

$$E_n(f) = \min_{p \in \mathbb{P}_n^d} \|f - p\|_{\ell^\infty(X)}\ , \quad C_\varepsilon = 2\left(\mu(X) + \varepsilon\sqrt{\mu(X)}\right)$$

$E_n(f)$ can be estimated by the regularity of f on a compact set $K \supset X$ satisfying a Jackson inequality

Algebraic quadrature for nonstandard regions

$$\int_\Omega p(x)\,\sigma(x)\,dx = \sum_{i=1}^{M} \lambda_i\, p(x_i) = \sum_{j=1}^{m} w_j\, p(t_j)\ , \quad \forall p \in \mathbb{P}_n^d \tag{9}$$

e.g.: high-cardinality formulas by domain splitting ($\sigma(x) \equiv 1$)

- regions bounded by circle arcs (ray tracing in optics [1, 6])
- polygons/polyhedra (applic. to FEM and CFD, cf. e.g. [11])

Figure 1: algebraic quadrature by splitting and compression by Tchakaloff points ($\circ$) for an obscured telescope lens ($n = 5$, CR ≈ 5) and a 14-side nonconvex polygon ($n = 20$, CR ≈ 7)

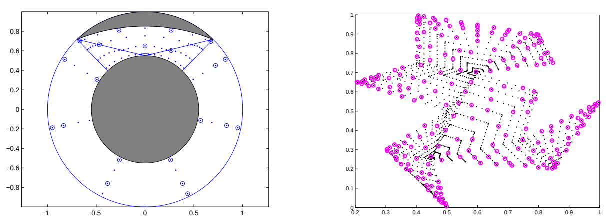


Table 1: Compression Ratio (CR) and moment residual on the regions above; CR tends to stabilize being asymptotically proportional to the number of integration elements

	deg n	5	10	15	20	25	30
	new card m	21	66	136	231	351	496
obscured telescope lens	old card M	108	282	603	957	1498	2032
	CR	5.1	4.3	4.4	4.1	4.3	4.1
	res ε	1e-15	3e-15	5e-15	1e-14	9e-15	1e-14
14-side polygon	old card M	208	468	1053	1573	2548	3328
	CR	9.9	7.1	7.7	6.8	7.3	6.7
	res ε	2e-17	2e-17	2e-17	4e-17	3e-16	1e-16
Athlon 64 2.4Ghz	cpu (sec)	0.04	0.14	0.39	1.37	4.01	13.30

note: Tchakaloff points of low-discrepancy sets can be used to compress 2d and 3d QMC integration, cf. [2]

Compression of least squares

basic ℓ^2 identity: let $X = \{x_i\}$ be a discrete sampling set and $\boldsymbol{\lambda} = (1, \ldots, 1)$; if card$(X) = M > \dim(\mathbb{P}_{2n}^d)$, replacing p by p^2 in (3) there are $m \leq \dim(\mathbb{P}_{2n}^d)$ Tchakaloff points such that

$$\boxed{\|p\|^2_{\ell^2(X)} = \sum_{i=1}^{M} p^2(x_i) = \sum_{j=1}^{m} w_j\, p^2(t_j) = \|p\|^2_{\ell^2_{\boldsymbol{w}}(T_{2n})}} \tag{10}$$

- Least Squares (LS) of degree n at X

$$\|f - L_X f\|_{\ell^2(X)} = \min_{p \in \mathbb{P}_n^d} \|f - p\|_{\ell^2(X)} \leq \sqrt{M}\, E_n(f) \tag{11}$$

- Compressed Least Squares (CLS) of degree n at $T_{2n} \subset X$

$$\|f - L^{\boldsymbol{w}}_{T_{2n}} f\|_{\ell^2_{\boldsymbol{w}}(T_{2n})} = \min_{p \in \mathbb{P}_n^d} \|f - p\|_{\ell^2_{\boldsymbol{w}}(T_{2n})} \tag{12}$$

- note: CLS are Weighted Least Squares of degree n at Tchakaloff points of degree $2n$

by (10) (and (8) for a nonzero moment residual), cf. [10, 12]

$$\boxed{\|f - L^{\boldsymbol{w}}_{T_{2n}} f\|_{\ell^2(X)} \lesssim \left(1 + \left(1 - \varepsilon\sqrt{M}\right)^{-1/2}\right)\sqrt{M}\, E_n(f)} \tag{13}$$

i.e., for $\varepsilon \ll 1/\sqrt{M}$ the LS and CLS reconstruction error estimates in $\ell^2(X)$ have substantially the same size!

similarly: compression of Weighted Least Squares, cf. [10]

Least square example

Figure 2: extraction of 153 Tchakaloff points ($\circ$) for CLS ($n = 8$) from $\approx$ 3700 Halton points on the union of 3 disks: CR $\approx$ 24

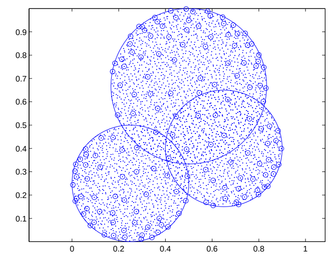


Table 2: CR, moment residual and $\ell^2(X)$ reconstruction error for the Gaussian $f_1(u,v) = \exp(-\rho^2)$ and the power function $f_2(u,v) = \rho^3$, $\rho = \sqrt{(u-1/2)^2 + (v-1/2)^2}$

	deg n	2	4	6	8	10	12
	m	15	45	91	153	231	325
	CR	246	82	40	24	16	11
	res ε	4.9e-14	1.2e-13	3.4e-13	4.3e-13	8.8e-13	2.5e-12
ℓ^2-err f_1	LS	3.0e-03	8.2e-05	1.7e-06	2.7e-08	3.5e-10	3.9e-12
	CLS	3.3e-03	8.6e-05	1.7e-06	2.7e-08	3.5e-10	3.9e-12
ℓ^2-err f_2	LS	1.2e-01	1.5e-02	4.2e-03	1.6e-03	7.2e-04	3.7e-04
	CLS	1.3e-01	1.6e-02	4.4e-03	1.7e-03	7.6e-04	3.8e-04

Compression of polynomial meshes

a polynomial mesh of a polynomial determining compact set $K \subset \mathbb{R}^d$ is a sequence of finite subsets $X_n \subset K$ such that

$$\|p\|_{L^\infty(K)} \leq C_n\, \|p\|_{\ell^\infty(X_n)}\ , \quad \forall p \in \mathbb{P}_n^d \tag{14}$$

where $C_n = \mathcal{O}(n^\alpha)$, $M_n = $ card$(X_n) = \mathcal{O}(n^\beta)$ ($\alpha \geq 0$, $\beta \geq d$)

polynomial meshes, introduced in the seminal paper [4] by Calvi and Levenberg, have good computational features, e.g.

- extension by algebraic transforms, finite union and product
- contain computable near optimal interpolation sets [3]
- are near optimal for uniform LS approximation, namely [4]

$$\|L_{X_n}\| = sup_{f \neq 0}\left\{\|L_{X_n} f\|_{L^\infty(K)} / \|f\|_{L^\infty(K)}\right\} \leq C_n\sqrt{M_n} \tag{15}$$

several known meshes have high-cardinality, even with $\beta = d$ e.g. on polygons/polyhedra or smooth convex sets, cf. [3, 8]

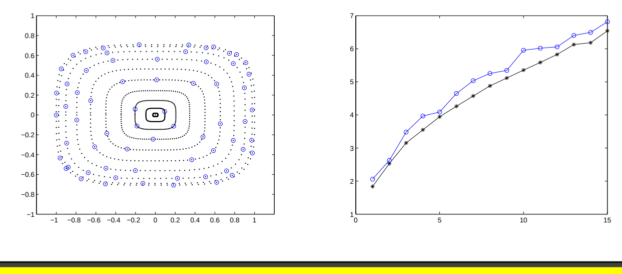
if $M_n > \dim(\mathbb{P}_{2n}^d)$, taking the CLS operator (12) at Tchakaloff points $T_{2n} \subset X_n$, we have

$$\boxed{\|L^{\boldsymbol{w}}_{T_{2n}}\| \lesssim C^*_n(\varepsilon) = C_n\sqrt{M_n}\left(1 - \varepsilon\sqrt{M_n}\right)^{-1/2}} \tag{16}$$

as long as $\sqrt{M_n} \ll \varepsilon^{-1}$ the estimates of LS and CLS operator norms almost coincide! and T_{2n} itself is a polynomial mesh

$$\|p\|_{L^\infty(K)} = \|L^{\boldsymbol{w}}_{T_{2n}} p\|_{L^\infty(K)} \lesssim C^*_n(\varepsilon)\, \|p\|_{\ell^\infty(T_{2n})}\ , \quad \forall p \in \mathbb{P}_n^d \tag{17}$$

Figure 3: polynomial mesh and compression by Tchakaloff points on a smooth convex set ($n = 5$, CR $= 971/66 \approx 15$); numerically evaluated LS ($*$) and CLS ($\circ$) operator norms



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