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New Tools for Multivariate Polynomial Approximation

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Abstract

Weakly Admissible Meshes and their Discrete Extremal Sets (computed by basic numerical linear algebra) give new tools for multivariate polynomial least squares and interpolation.

Fekete Points

$K \subset \mathbb{R}^d$ (or $\mathbb{C}^d$) compact set or manifold

$\boldsymbol{p} = \{p_j\}_{1\le j\le N}$, $N = \dim(\mathbb{P}_n^d(K))$ polynomial basis

$\boldsymbol{\xi} = \{\xi_1,\dots,\xi_N\} \subset K$ interpolation points

$V(\boldsymbol{\xi},\boldsymbol{p}) = [p_j(\xi_i)]$ Vandermonde matrix, $\det(V) \neq 0$

$\Pi_n f(x) = \sum_{j=1}^N f(\xi_j)\,\ell_j(x)$ determinantal Lagrange formula

$$\ell_j(x) = \frac{\det(V(\xi_1,\dots,\xi_{j-1},x,\xi_{j+1},\dots,\xi_N))}{\det(V(\xi_1,\dots,\xi_{j-1},\xi_j,\xi_{j+1},\dots,\xi_N))}\ ,\quad \ell_j(\xi_i) = \delta_{ij}$$

Fekete points: $|\det(V(\xi_1,\dots,\xi_N))|$ is max in K^N

$$\Longrightarrow \text{Lebesgue constant} = \Lambda_n = \max_{x\in K}\sum_{j=1}^N |\ell_j(x)| \le N$$

Fekete points (and Lebesgue constants) are independent of the choice of the basis

Fekete points are analytically known only in a few cases:
interval: Gauss-Lobatto points, $\Lambda_n = \mathcal{O}(\log n)$
complex circle: equispaced points, $\Lambda_n = \mathcal{O}(\log n)$
cube: for tensor-product polynomials, $\Lambda_n = \mathcal{O}(\log^d n)$

recent important result:
Fekete points are asymptotically equidistributed with respect to the pluripotential equilibrium measure of K (cf. [1])

- open problem: efficient computation, even in the univariate complex case (large scale optimization problem in $N \times d$ variables [7])

- idea: extract Fekete points from a discretization of K: but which could be a suitable mesh?

Weakly Admissible Meshes

Weakly Admissible Mesh (WAM): sequence of discrete subsets

$$\mathcal{A}_n = \{a_1,\dots,a_{M_n}\} \subset K$$

satisfying the polynomial inequality

$$\|p\|_K \le C(\mathcal{A}_n)\|p\|_{\mathcal{A}_n}\ ,\quad \forall p \in \mathbb{P}_n^d(K)$$

with $\mathrm{card}(\mathcal{A}_n)$ and $C(\mathcal{A}_n)$ growing (at most) polinomially

Properties of WAMS:

- $C(\mathcal{A}_n)$ is invariant under affine mapping
- good interpolation sets are WAMs with $C(\mathcal{A}_n) =$ *Lebesgue constant* (e.g., Chebyshev points in one variable, *Padua points* [3] on the square)
- finite unions and products of WAMs are WAMs for the corresponding unions and products of compacts
- given a polynomial mapping π_m of degree m, then $\pi_m(\mathcal{A}_{nm})$ is a WAM for $\pi_m(K)$ with constants $C(\mathcal{A}_{nm})$
- Least Squares polynomial approximation of $f \in C(K)$ on a WAM is near optimal in the uniform norm

$$\|f - \mathcal{L}_{\mathcal{A}_n} f\|_K \approx C(A_n)\sqrt{\mathrm{card}(\mathcal{A}_n)}\ E_n(f;K)$$

- Fekete points extracted from a WAM have a Lebesgue constant theoretically bounded by $NC(\mathcal{A}_n)$ (but often much smaller!)

Discrete Extremal Sets

extracting a maximum determinant $N \times N$ submatrix from the $M \times N$ Vandermonde matrix $V = V(\boldsymbol{a},\boldsymbol{p}) = [p_j(a_i)]$

this is NP-hard: then we look for an approximate solution; surprisingly, this can be obtained by basic numerical linear algebra!

Key asymptotic result (cf. [4]): Discrete Extremal Sets extracted from a WAM by the greedy algorithms below have the same asymptotic behavior of the true Fekete points

$$\frac{1}{N}\sum_{j=1}^N \delta_{\xi_j} \xrightarrow{N\to\infty} \int_K f(x)\,d\mu_K\ ,\quad \forall f \in C(K)$$

where μ_K is the equilibrium measure of K

Approximate Fekete Points

algorithm greedy 1 (AFP, Approximate Fekete Points)
method: greedy maximization of submatrix volumes

iteration core: subtract from each row its orthogonal projection onto the largest norm one (preserves volumes like with parallelograms)

implementation: QR factorization with column pivoting (Businger and Golub, 1965) applied to V^t

Matlab script:
$\boldsymbol{w} = V' \backslash \mathtt{ones}(1:N);\ ind = \mathtt{find}(\boldsymbol{w} \neq \boldsymbol{0});\ \boldsymbol{\xi} = \boldsymbol{a}(ind)$

Discrete Leja Points

algorithm greedy 2 (DLP, Discrete Leja Points)
method: greedy maximization of nested subdeterminants

iteration core: one column step of Gaussian elimination with row pivoting (preserves the relevant subdeterminants)

implementation: LU factorization with row pivoting

feature: DLP form a sequence

in one variable DLP correspond to the usual notion: $\xi_k = \mathrm{argmax}_{z\in\mathcal{A}_n}\prod_{j=1}^k |z-\xi_j|\ ,\quad k = 2,\dots,N$

Matlab script:
$[L,U,\boldsymbol{\sigma}] = \mathtt{LU}(V,\text{“}\mathtt{vector}\text{”});\ ind = \boldsymbol{\sigma}(1:N);\ \boldsymbol{\xi} = \boldsymbol{a}(ind)$

Example: AFP in one variable

FIGURE 1. $N = 31$ Approximate Fekete points (deg $n = 30$) from Admissible Meshes in: one interval, two and three disjoint intervals

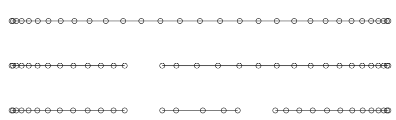


FIGURE 2. As above for some compacts in the complex plane

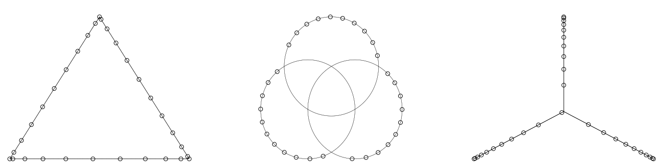


TABLE 1. Numerically estimated Lebesgue constants of interpolation points in some 1-dimensional real and complex compacts (Figs. 1-2)

points	$n = 10$ $N = 11$	20 21	30 31	40 41	50 51	60 61
equisp intv	29.9	1e+4	6e+6	4e+8	7e+9	1e+10
Fekete intv	2.2	2.6	2.9	3.0	3.2	3.3
AFP intv	2.3	2.8	3.1	3.4	3.6	3.8
AFP 2intvs	3.1	6.3	7.1	7.6	7.5	7.2
AFP 3intvs	4.2	7.9	12.6	6.3	5.8	5.3
AFP disk	2.7	3.0	3.3	3.4	3.5	3.7
AFP triangle	3.2	6.2	5.2	4.8	9.6	6.1
AFP 3disks	5.1	3.0	7.6	10.6	3.8	8.3
AFP 3branches	4.7	3.5	3.8	8.3	5.0	4.8

Multivariate examples

- Polygon WAMs: by triangulation/quadrangulation

$$\Longrightarrow \mathrm{card}(\mathcal{A}_n) = \mathcal{O}(n^2) \text{ and } C(\mathcal{A}_n) = \mathcal{O}(\log^2 n)$$

FIGURE 3. Left: $N = 45$ AFP ($\circ$) and DLP ($*$) of an hexagon for $n = 8$ from the WAM (dots) obtained by bilinear transformation of a 9×9 product Chebyshev grid on two quadrangle elements ($M = 153$ pts);
Right: $N = 136$ AFP ($\circ$) and DLP ($*$) for degree $n = 15$ in a hand shaped polygon with 37 sides and a 23 element quadrangulation ($M \approx 5500$).

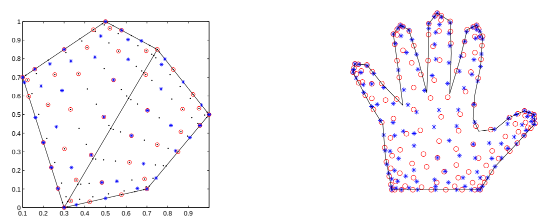


TABLE 2. Hexagon: interpolation and Least Squares operator norm (Lebesgue const.) and approximation errors for the Franke test function.

	$n = 5$	$n = 10$	$n = 15$	$n = 20$	$n = 25$	$n = 30$
intp AFP	8.1	14.5	35.7	56.0	97.7	87.0
intp DLP	8.0	32.0	48.5	92.8	107.4	198.7
LS WAM	3.4	4.5	5.3	6.0	6.6	7.3
intp AFP	2E-01	2E-02	6E-03	6E-04	4E-05	1E-06
intp DLP	2E-01	6E-02	6E-03	5E-04	1E-04	4E-06
LS WAM	7E-01	2E-01	3E-02	4E-03	3E-04	1E-05

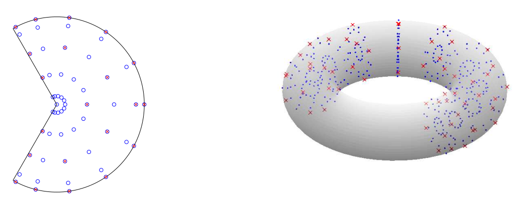
Developments and Applications

- algebraic cubature [9]
- smooth transformations of WAMs [2, 10, 11]
- surface/solid WAMs [5, 10]: polyhedra, cones, toroids, ...
- computing ”true” Fekete and Lebesgue points [7, 12]
- numerical PDEs: spectral elements [12], collocation [13], ...

Highlight [6]: the $2n+1$ angles $\theta_j = 2\arcsin(t_j \sin(\omega/2))$, $T_{2n+1}(t_j) = 0$, are near optimal for trigonometric interpolation in $[-\omega,\omega] \subseteq [-\pi,\pi]$ (and have *positive* quadrature weights!)

$\Longrightarrow$ subarc-based product WAMs on sections of *disk, sphere, ball, torus*, such as circular sectors and lenses, spherical caps, quadrangles, lunes, slices, ..., with $\mathrm{card}(\mathcal{A}_n) = \mathcal{O}(n^k)$ and $C(\mathcal{A}_n) = \mathcal{O}(\log^k n)$ ($k = 2$ for surfaces and $k = 3$ for solids)

FIGURE 5. Left: WAM of a sector ($\omega = 2\pi/3$) for $n = 5$ (blue, $M = 61$) and $N = 21$ AFP (red); Right: WAM of a solid torus section ($\omega = 2\pi/3$) for $n = 5$ (union of 11 disk WAMs, blue, $M = 396$) and $N = 56$ AFP (red).



Note: publications/preprints of the *Padova-Verona research group on Constructive Approximation and Applications* can be downloaded at http://www.math.unipd.it/~marcov/CAA.html

References

[1] R. Berman, S. Boucksom and D. Witt Nyström, *Fekete points and convergence towards equilibrium measures on complex manifolds*, arXiv:0907.2820 (Acta Math., to appear).

[2] L. Bos, J.-P. Calvi, N. Levenberg, A. Sommariva and M. Vianello, *Geometric Weakly Admissible Meshes, Discrete Least Squares Approximations and Approximate Fekete Points*, Math. Comp. 80 (2011).

[3] L. Bos, M. Caliari, S. De Marchi, M. Vianello and Y. Xu, *Bivariate Lagrange interpolation at the Padua points: the generating curve approach*, J. Approx. Theory 143 (2006).

[4] L. Bos, S. De Marchi, A. Sommariva and M. Vianello, *Computing multivariate Fekete and Leja points by numerical linear algebra*, SIAM J. Numer. Anal. 48 (2010).

[5] L. Bos, S. De Marchi, A. Sommariva and M. Vianello, *3-dimensional Weakly Admissible Meshes*, Approximation Theory Workshop - FoCM 2011, Budapest, July 8-10 2011.

[6] L. Bos and M. Vianello, *Subperiodic trigonometric interpolation and quadrature*, preprint.

[7] M. Briani. A. Sommariva and M. Vianello, *Computing Fekete and Lebesgue points: simplex, square, disk*, preprint.

[8] J.-P. Calvi and N. Levenberg, *Uniform approximation by discrete least squares polynomials*, J. Approx. Theory 152 (2008).

[9] M. Gentile, A. Sommariva and M. Vianello, *Polynomial interpolation and cubature over polygons*, J. Comput. Appl. Math. 235 (2011).

[10] A. Kroó, *On Optimal Polynomial Meshes*, J. Approx. Theory, to appear.

[11] F. Piazzon and M. Vianello, *Analytic transformations of admissible meshes*, East J. Approx. 16 (2010).

[12] F. Rapetti, A. Sommariva and M. Vianello, *Symmetric Lebesgue points for triangle-based spectral elements*, in preparation.

[13] P. Zitnan, *The collocation solution of Poisson problems based on approximate Fekete points*, Eng. Anal. with Bound. Elem. 35 (2011).